HOMOMORPHIC IMAGE OF A FUZZY IDEAL OF A BCH-ALGEBRA

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Abstract. In 1991 Professor Xi Ougen put forward the proposition that the homomorphic image of a fuzzy ideal with sup property of a BCK-algebra is also a fuzzy ideal.(cf. Proposition 5 of [1].) It is pointed out that there is a mistake in its proof presented in [1], although the proposition itself is correct. In this paper the proposition is generalized to BCH-algebras and a new correct proof is given. Moreover, the condition of sup property is removed.

§1. Introduction.

In 1991, Xi Ougen put forward the following propositon:

Proposition 5 of [1]. *Suppose that f is a homomorphism from BCK-algebra $X = (X; *, 0)$ onto BCK-algebra $(X'; *', 0')$, μ is a fuzzy ideal of X with sup property, then the homomorphic image ν of μ under f is a fuzzy ideal of X'.*

This proposition itself is correct, but there is a mistake in its proof presented in [1].

In order to show that $\nu = f(\mu)$ is a fuzzy ideal of X', one must prove that $\nu(x') \geq \min\{\nu(x' *' y'), \nu(y')\}, \forall x', y' \in X'$. It is proved in [1] in this way:

For any $x', y' \in X$, there exist $x, y \in X$ such that $x' = f(x), y' = f(y)$ since $X' = f(X)$. Noticing that μ has the sup property, there is a $x_0 \in f^{-1}(f(x))$ such that $\mu(x_0) = \sup\limits_{z \in f^{-1}(f(x))} \mu(z)$. Then

$$\begin{aligned}
\nu(x') = \nu(f(x)) &= \sup_{z \in f^{-1}(f(x))} \mu(z) \\
&= \mu(x_0) \geq \min(\mu(x_0 * y), \mu(y)) \text{ } (\mu \text{ is a fuzzy ideal}) \\
&\stackrel{?}{=} \min(\nu(f(x_0 * y)), \nu(f(y))) \\
&= \min(\nu(f(x_0) *' f(y)), \nu(f(y))) \\
&= \min(\nu(f(x) *' f(y)), \nu(f(y))) \\
&= \min(\nu(x' *' y'), \nu(y')).
\end{aligned}$$

The problem is : How can one get the equation

$\min(\mu(x_0 * y), \mu(y)) = \min(\nu(f(x_0 * y)), \nu(f(y)))$?

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In fact

$\nu(f(x_0 * y)) = \sup_{z\in f^{-1}(f(x_0*y))} \mu(z) \geq \mu(x_0 * y),$

$\nu(f(y)) = \sup_{t\in f^{-1}(f(y))} \mu(t) \geq \mu(y),$

therefore we can only obtain

$\min(\mu(x_0 * y), \mu(y)) \leq \min(\nu(f(x_0 * y)), \nu(f(y))).$

The equation may not hold. However, the proposition itself holds for BCK- algebras. In this paper we generalize this proposition for BCH-algebras and give a new proof. Moreover, the conditon of sup property of μ in Prposition 5 of [1] is removed.

§2. Preliminaries.

Definition 1[2]**.** A (2, 0) type algebra $(X; *, 0)$ is called a BCH- algebra if it satisfies the following conditions:$\forall x, y, z \in X$

(H-1) $x * x = 0,$

(H-2) $x * y = y * x = 0 \Rightarrow x = y,$

(H-3) $(x * y) * z = (x * z) * y.$

A binary relation $\leq$ is defined on X as follows: for any $x, y \in X,\ x \leq y \Leftrightarrow x * y = 0.$

A BCH-algebra $(X; *, 0)$ has the following properties:

(P-1)[3] $x * 0 = x,\ \forall x \in X.$

(P-2)[3] $x * (x * y) \leq y,\ \forall x, y \in X.$

Definition 2[3]**.** Let $(X; *, 0)$ be a BCH-algebra. A nonempty subset $I \subseteq X$ is called an ideal of X if

(i) $0 \in I,$

(ii) $x * y \in I, y \in I$ imply $x \in I,\ \forall x, y \in X.$

Definition 3[4],[5]**.** A (2, 0) type algebra $(X; *, 0)$ is called a BCI-algebra if it satisfies the following conditions: for any $x, y, z \in X$

(I1) $((x * y) * (x * z)) * (z * y) = 0;$

(I2) $(x * (x * y)) * y = 0;$

(I3) $x * x = 0;$

(I4) $x * y = y * x = 0 \Rightarrow x = y.$

Proposition 1[6]**.** *A BCI-algebra is a BCH-algebra, but a BCH-algebra need not be a BCI-algebra.*

Definition 4[7]**.** A BCI-algebra is called a BCK-algebra if it satisfies:

(K) $0 * x = 0,\ \forall x \in X.$

Proposition 2[3]**.** *A BCK-algebra is a BCI-algebra, but a BCI-algebra need not be a BCK-algebra.*

An ideal in a BCI(BCK)-algebra is defined in the same way as in Definition 2.

Definition 5[8]**.** Let X be a set, a fuzzy subset of X is a function μ from X into [0, 1].

Definition 6[10]**.** Let μ be a fuzzy subset of X. For $\lambda \in [0,\ 1]$, define
$\mu_\lambda = \{x \in X|\mu(x) \geq \lambda\}$. μ_λ is called the λ-level-set of μ.
$\mu'_\lambda = \{x \in X|\mu(x) > \lambda\}$. μ'_λ is called the strong λ-level-set (or strong λ-cut) of μ.

In this paper we use μ'_λ instead of μ_λ, because we find that μ'_λ has better properties than μ_λ. (cf. Theorem 3 of [1])

Definition 7[9]**.** A fuzzy subset μ of a BCH-algebra $(X;*,0)$ is called a fuzzy ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x),\ \forall x \in X$;
(ii) $\mu(x) \geq \mu(x*y) \wedge \mu(y),\ \forall x, y \in X$.

(Here, $\mu(x*y) \wedge \mu(y) = \min\{\mu(x*y), \mu(y)\}$.)

The concept of a fuzzy ideal in a BCI(BCK)-algebra is defined in the same way. (cf. Definition 3 of [1].)

Definition 8[10]**.** Let $X,\ Y$ be sets and $f : X \to Y$ be a surjection. Suppose that μ is a fuzzy subset in X. Define a fuzzy subset ν in Y as follows:

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x),\ \forall y \in Y.$$

ν is called the image of μ under f and is denoted by $\nu = f(\mu)$.

Here $f^{-1}(y) = \{x \in X|f(x) = y\}$.

Definition 9[3]**.** Let $X = (X;*,0)$ and $X' = (X';*',0')$ be BCH- algebras and $f : X \to X'$ be a mapping from X to X'. f is called a homomorphism if

$$f(x*y) = f(x) *' f(y),\ \forall x, y \in X.$$

Homomorphism for BCI(BCK)-algebras is defined in the same way.

For brief of symbols, we adopt the convection : the operation $*'$ in X' is also denoted by $*$.

A homomorphism f is called an epimorphism if it is a surjection, f is called a monomorphism if it is an injection. f is called an isomorphism if it is both an epimorphism and also a monomorphism.

Definition 10[9]**.** Suppose that $X,\ Y$ are BCH-algebras and $f : X \to Y$ is an epimorphism. If μ is a fuzzy ideal of X, then $\nu = f(\mu)$ is called the homomorphic image of μ under f.

Proposition 3[2]**.** *Let $X = (X;*,0)$ and $X' = (X';*,0')$ be BCH-algebras and $f : X \to X'$ be a homomorphism. Then (i) $f(0) = 0'$; (ii) $x \leq y \Rightarrow f(x) \leq f(y)$.*

Proposition 4[3]**.** *An ideal I of a BCH-algebra X has the property: for any $x, y \in X$*

(P-3) $x \leq y,\ y \in I \Rightarrow x \in I$.

As to basic theory of BCH(BCI,BCK)-algebras, the reader is referred to [2]-[7], [11] and [12].

§3. Homomorphic image of a fuzzy ideal of a BCH-algebra.

The main theorem of this paper is

Theorem 1. *If $f : X \to Y$ is a BCH-epimorphism, μ is a fuzzy ideal of X, then $\nu = f(\mu)$ is also a fuzzy ideal of Y.*

(Here, by BCH-epimorphism we mean that both X and Y are BCH-algebras and $f : X \to Y$ is an epimorphism. Similarly for BCI or BCK-algebras.)

To prove the main theorem we need first to prove three lemmas.

Lemma 1. *A fuzzy subset μ of a BCH-algebra X is a fuzzy ideal if and only if $\forall \lambda \in [0,\ 1]$, μ'_λ is an ideal of X when $\mu'_\lambda \neq \emptyset$.*

Proof. "$\Rightarrow$": Suppose that μ is a fuzzy ideal of X. Then by Definition 7 (i) we have $\mu(0) \geq \mu(x),\ \forall x \in X$. For any $\lambda \in [0,\ 1]$, if $\mu'_\lambda \neq \emptyset$, then there exists $x_0 \in \mu'_\lambda$, so $\mu(x_0) > \lambda$. It follows that $\mu(0) \geq \mu(x_0) > \lambda \Rightarrow 0 \in \mu'_\lambda$.

Moreover, suppose $x * y \in \mu'_\lambda,\ y \in \mu'_\lambda$, then $\mu(x * y) > \lambda,\ \mu(y) > \lambda$. By Definition 7(ii) we have $\mu(x) \geq \mu(x * y) \wedge \mu(y) > \lambda$, so $x \in \mu'_\lambda$. Hence μ'_λ is an ideal of X by Definition 2.

"$\Leftarrow$": We only need to prove that (i) and (ii) of Definition 7 are true.

If (i) isn't true, then there exists $x_0 \in X$ such that $\mu(0) < \mu(x_0)$. Because of the density of real numbers, there is a real number r such that $0 \leq \mu(0) < r < \mu(x_0) \leq 1$. It follows that $x_0 \in \mu'_r \Rightarrow \mu'_r \neq \emptyset$. Hence μ'_r is an ideal of X, so $0 \in \mu'_r$, this means $\mu(0) > r$, a contradiction.

Moreover, if (ii) isn't true, then there exist $x_0, y_0 \in X$ such that

$$\mu(x_0) < \mu(x_0 * y_0) \wedge \mu(y_0).$$

Once more using the density of real numbers, we have r such that

$$\mu(x_0) < r < \mu(x_0 * y_0) \wedge \mu(y_0). \quad (1)$$

It is clear that $$\mu(x_0 * y_0) \wedge \mu(y_0) \leq \mu(x_0 * y_0), \quad (2)$$

and $$\mu(x_0 * y_0) \wedge \mu(y_0) \leq \mu(y_0) \quad (3)$$

From (1) and (2) it follows that $\mu(x_0 * y_0) > r \Rightarrow x_0 * y_0 \in \mu'_r$. (4)

From (1) and (3) it follows that $\mu(y_0) > r \Rightarrow y_0 \in \mu'_r$. (5)

Since μ'_r is an ideal of X, so by (4) and (5) we get $x_0 \in \mu'_r \Rightarrow \mu(x_0) > r$, this contradicts (1). $\square$

Lemma 2. *Let $(X;*,0)$, $(X';*,0')$ be BCH-algebras, $f: X \to X'$ be an epimorphism. If I is an ideal of X, then $f(I)$ is an ideal of X'.*

Proof. Since I is an ideal of X, we have $0 \in I$. So $0' = f(0) \in f(I)$ by Proposition 3 (i).

If $x' * y', y' \in f(I)$, where $x', y' \in X'$, then there exists $x \in X$ such that $x' = f(x)$ since f is surjective. From $y' \in f(I)$ there exists $y \in I$ such that $y' = f(y)$.

Then

$$x' * y' = f(x) * f(y) = f(x * y).$$

Note that $x' * y' \in f(I)$, so there exists $z \in I$ such that

$$f(z) = x' * y' = f(x * y).$$

Hence by (H-1) we have

$$f(x * y) * f(z) = 0'. \tag{6}$$

Because X is a BCH-algebra, using (P-2) we get

$$(x * z) * ((x * z) * y) \leq y.$$

Noticing $y \in I$, I is an ideal of X, by (P-3) and (H-3) it follows that

$$(x * ((x * z) * y)) * z = (x * z) * ((x * z) * y) \in I.$$

Now $z \in I$ and I is an ideal of X, making use of (H-3) and Definition 2 we have

$$x * ((x * y) * z) = x * ((x * z) * y) \in I.$$

Therefore

$$x' \overset{\text{(P-1)}}{=} f(x) * 0' \overset{(6)}{=} f(x) * (f(x * y) * f(z)) = f(x * ((x * y) * z)) \in f(I).$$

This shows that $x' * y' \in f(I), y' \in f(I) \Rightarrow x' \in f(I)$. Combining $0' \in f(I)$ it is clear that $f(I)$ is an ideal of X' by Definition 2. $\square$

Lemma 3. *Let X, Y be sets. If $f: X \to Y$ is a surjection, μ is a fuzzy subset of X. Let $\nu = f(\mu)$, then $\nu'_\lambda = f(\mu'_\lambda)$, $\forall \lambda \in [0,\ 1]$.*

(For $A \subseteq X$, we define $f(A) = \{f(a) | a \in A\}$.)

Proof. Let λ be any fixed element in $[0,\ 1]$. For any $y \in f(\mu'_\lambda)$ there exists $b \in \mu'_\lambda$ such that $y = f(b)$. So $b \in f^{-1}(y)$ and $\mu(b) > \lambda$. It follows that $\nu(y) \overset{\text{Def.8}}{=} \sup\limits_{t \in f^{-1}(y)} \mu(t) \geq \mu(b) > \lambda \Rightarrow y \in \nu'_\lambda$, therefore

$$f(\mu'_\lambda) \subseteq \nu'_\lambda. \tag{7}$$

On the other hand, for any $y \in \nu'_\lambda \Rightarrow \nu(y) > \lambda$. By Definition 8 $\nu(y) = \sup\limits_{t \in f^{-1}(y)} \mu(t)$, so we obtain $\sup\limits_{t \in f^{-1}(y)} \mu(t) > \lambda$. This means that λ is not an upper bound of the set $\{\mu(t) | t \in f^{-1}(y)\}$, hence there exists $t_0 \in f^{-1}(y)$ such that $\mu(t_0) > \lambda$. It follows that $y = f(t_0)$ and $t_0 \in \mu'_\lambda \Rightarrow y \in f(\mu'_\lambda)$, therefore

$$\nu'_\lambda \subseteq f(\mu'_\lambda). \tag{8}$$

From (7) and (8) we get $\nu'_\lambda = f(\mu'_\lambda)$. □

Now we are ready to prove the main theorem.

Proof of Theorem 1. By the known conditions and Lemma 3 we have $\nu'_\lambda = f(\mu'_\lambda)$, $\forall\lambda \in [0,\ 1]$. If $\nu'_\lambda \neq \emptyset$, then $\mu'_\lambda \neq \emptyset$ since $\nu'_\lambda = f(\mu'_\lambda)$. As μ is a fuzzy ideal of X, from Lemma 1 we know that μ'_λ is an ideal of X. Making use of Lemma 2 we have $f(\mu'_\lambda)$ is an ideal of Y. Since $\nu'_\lambda = f(\mu'_\lambda)$, we get that ν'_λ is an ideal of Y when $\nu'_\lambda \neq \emptyset$, $\forall\lambda \in [0,\ 1]$. Thus by Lemma 1 ν is a fuzzy ideal of Y. □

Theorem 2. *If $f : X \to Y$ is a BCI-epimorphism, μ is a fuzzy ideal of X and $\nu = f(\mu)$, then ν is a fuzzy ideal of Y.*

Proof. By Proposition 1 and Theorem 1. □

Theorem 3. *If $f : X \to Y$ is a BCK-epimorphism, μ is a fuzzy ideal of X and $\nu = f(\mu)$, then ν is a fuzzy ideal of Y.*

Proof. By Proposition 2 and Theorem 2. □

Remark. Theorem 3 is just the Proposition 5 of [1], but here we have removed the condition of sup property of μ.

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