SOLID VARIETIES, TRANSITION SEMIGROUPS AND UNARY CLONES

K. Denecke, P. Jampachon*, S.L. Wismath†

Received February 13, 2003; revised February 24, 2003

Abstract. There is a well-known connection between hyperidentities of an algebra and identities satisfied by the clone of the algebra. The clone of an algebra is a heterogeneous algebra, and the correspondence between hyperidentities and clone identities is rather complicated to work with. It is also of interest to study this correspondence in a restricted setting, that of hyperidentities of unary algebras (of arbitrary unary type) and identities in the unary clone of unary term operations. This unary clone is just a monoid, usually called the transition monoid of the unary algebra. This correspondence is an important one in automata theory, since any finite unary algebra $\mathcal{A}$ can be regarded as an automaton: the set A is regarded as a set of states, with one state chosen as an initial state and a subset of A chosen as the set of final states, and the operations of the algebra regarded as inputs. Then terms and identities over the unary algebra $\mathcal{A}$ correspond to monoid words from the free monoid generated by the operations of $\mathcal{A}$. In this paper we study the correspondence between hyperidentities and clone identities in this special case of unary algebras and transition monoids. We also look at generalizations of this approach to algebras of type $(n, n, \dots)$, and the corresponding n-clones, which correspond to tree automata of type $(n, n, \dots)$.

1 Preliminaries

There is a well-known connection, described by Taylor in [Tay;81], between hyperidentities of an algebra and identities satisfied by the clone of the algebra. The clone of an algebra is a heterogeneous algebra, and the correspondence between hyperidentities and clone identities is rather complicated to work with. In this paper we study this correspondence in a restricted setting, that of hyperidentities of unary algebras (of arbitrary unary type) and identities in the unary clone of unary term operations. This unary clone is just a monoid, usually called the transition monoid of the unary algebra. Since unary algebras correspond to finite automata, this correspondence is an important one in automata theory. In 1965 Schützenberger ([Sch;65]) found an interesting connection between star-free languages and aperiodic monoids. The precise formulation of the correspondence between certain sets of formal languages and sets of finite semigroups is due to Eilenberg (see [Eil;74]) and uses the concept of a pseudovariety. The connection between finite deterministic automata and regular languages is given by Kleene's famous theorem ((Kle;50]). In this paper we consider the third side, the connection between finite deterministic automata and their transition monoids. This connection allows for instance to apply the decomposition methods of finite semigroups using a wreath product in which the factors are, alternately, finite groups and finite aperiodic semigroups in automata theory ([Kro-R;65]). If $s \approx t$ is an identity in a finite unary algebra, then the word $t^{\mathcal{A}}$ is recognized by the automaton corresponding to the unary algebra iff the word $s^{\mathcal{A}}$ is recognized. If $s \approx t$

2000 *Mathematics Subject Classification.* 08A40, 08A60, 08A02, 20M35.

Key words and phrases. hyperidentity, unary clone, transition semigroup, n-clone.

*Research of the second author supported by the Ministry of University Affairs, Thailand

†Research of the third author supported by NSERC of Canada.

is even an identity in the transition monoid, and as we will prove, a hyperidentity in the algebra $\mathcal{A}$, then all words which arise from $s^{\mathcal{A}}$ by replacing the input letters by arbitrary words are recognized iff the same holds for $t^{\mathcal{A}}$. Therefore one may expect more applications of our results in Automata Theory. We also look at generalizations of this approach to algebras of type $(n, n, \ldots)$, and the corresponding n-clones, which correspond to tree automata of type $(n, n, \ldots)$.

We begin by defining a type τ of algebras to be unary if all the operation symbols of the type have arity 1. We will assume throughout that τ_0 is such a unary type (possibly with an infinite number of operation symbols). We let $(f_i)_{i\in I}$ be an indexed set of unary operation symbols, corresponding to the type τ_0. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of individual variables, and for each $n \geq 1$ let $X_n = \{x_1, \ldots, x_n\}$. We shall denote by $W_{\tau_0}(X_n)$ the set of all n-ary terms of type τ_0, and by $W_{\tau_0}(X) = \cup_{n\geq 1} W_{\tau_0}(X_n)$ the set of all (finitary) terms of type τ_0. This set is the universe of the algebra $\mathcal{F}_{\tau_0}(X) := (W_{\tau_0}(X); (\bar{f}_i)_{i\in I})$, with operations defined by $\bar{f}_i(t) := f_i(t)$, for $t \in W_{\tau_0}(X)$; the algebra $\mathcal{F}_{\tau_0}(X)$ is well known to be the absolutely free algebra of type τ_0 on the set X. In the special case $X_1 = \{x_1\}$, the elements of $W_{\tau_0}(\{x_1\})$ are called *unary terms.* An identity $s \approx t$ of terms of unary type, (or more formally the pair of terms (s, t)), contains at most two different variables. A unary identity containing only one variable is called a *regular* identity. A unary algebra is a pair $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ consisting of a set A and a set of operations defined on A which are all unary. Thus a unary algebra is an algebra of type τ_0 for some unary type τ_0. Let $\mathcal{A}$ be a unary algebra. Then every unary term $t \in W_{\tau_0}(\{x_1\})$ induces a unary term operation $t^{\mathcal{A}}$ on $\mathcal{A}$, which is inductively defined by the following steps:

(i) The variable $x_1 \in X_1$ induces the identity mapping $id_A : A \to A$, that is, $x_1^{\mathcal{A}} := id_A$,

(ii) If $t'^{\mathcal{A}}$ is the unary term operation induced by the unary term t' and if $t = f_i(t')$ is a compound term, then $t^{\mathcal{A}} := f_i^{\mathcal{A}} \circ t'^{\mathcal{A}}$, where $\circ$ denotes the composition of unary operations defined on A.

We will use $T^{(1)}(\mathcal{A})$ for the set of all unary term operations of the unary algebra $\mathcal{A}$. This set is the universe of a monoid generated by $\{f_i^{\mathcal{A}} \mid i \in I\} \cup \{id_A\}$, which is called the transformation or transition monoid of $\mathcal{A}$ or *the unary clone of* $\mathcal{A}$. We shall also use the name $T^{(1)}(\mathcal{A})$ for this monoid, so

$$T^{(1)}(\mathcal{A}) = (\langle \{f_i^{\mathcal{A}} \mid i \in I\} \cup \{id_A\}\rangle; \circ, id_A).$$

The corresponding semigroup $S(\mathcal{A})$ generated by $\{f_i^{\mathcal{A}} \mid i \in I\}$ is called the *transition semigroup of* $\mathcal{A}$. Unary algebras are sometimes called *X-algebras* (see e.g. [Pet-C-B;02]).

2 Hypersubstitutions and Substitutions To consider identities in unary clones or transition semigroups, we need to build up a language of type (2,0) or (2), respectively, consisting of monoid or semigroup words. Let $\{F_i \mid i \in I\}$ be a new set of variables, also indexed by I. Then we denote by $\mathcal{F}_S(\{F_i \mid i \in I\})$ the free semigroup generated by $\{F_i \mid i \in I\}$, and similarly we denote by $\mathcal{F}_M(\{F_i \mid i \in I\})$ the free monoid generated by $\{F_i \mid i \in I\} \cup \{\lambda\}$, where λ is the empty word. We shall denote the binary operation in both the free semigroup and monoid by $\circ$. The basis of our work will be the following definition of a mapping φ, which will effect the translation between unary terms of our type τ_0 and the monoid words on our new meta-level alphabet $\{F_i \mid i \in I\}$. Specifically, we want to define a mapping φ between the set $W_{\tau_0}(\{x_1\})$ of unary terms of type τ_0 and the monoid generated by $\{F_i \mid i \in I\} \cup \{\lambda\}$.

Definition 2.1 *The mapping* $\varphi : W_{\tau_0}(\{x_1\}) \longrightarrow \langle\{F_i \mid i \in I\} \cup \{\lambda\}\rangle$ *is inductively defined as follows:*

(i) $\varphi(x_1) := \lambda$,

(ii) $\varphi(f_i(t)) := F_i \circ \varphi(t)$.

On both the set $W_{\tau_0}(\{x_1\})$ of unary terms and the monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ we are using the operation of superposition of terms. The superposition of unary terms is defined inductively by $x_1(t) := t$, and if $t' = f_i(t'')$, then $t'(t) := f_i(t'')(t) := f_i(t''(t))$. It is sometimes convenient to make this explicit with a new operation symbol S_1^1 for the superposition of unary terms; in this case, we have $S_1^1(x_1, t) := t$ and $S_1^1(f_i(t''), t) := f_i(S_1^1(t'', t))$. This approach will be used again in Section 6. This operation S_1^1 is a binary operation, and is clearly associative; the clone axioms for the unary clone $T^{(1)}(\mathcal{A})$ make this algebra a monoid. The binary composition $\circ$ in $\mathcal{F}_M(\{F_i \mid i \in I\})$ is also an instance of the general superposition operation S_1^1. Moreover, the set $W_{\tau_0}(\{x_1\})$ is the universe of the unary clone $Clone^{(1)}(\tau_0) = (W_{\tau_0}(\{x_1\}); S_1^1, x_1)$, which is also a monoid.

We will use the mapping φ to establish an isomorphism between these two monoids, the unary clone $Clone^{(1)}(\tau_0)$ and the free monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$. The next Proposition shows that φ is a homomorphism, with respect to the superposition operation S_1^1.

Proposition 2.2 *For any two terms* $t', t'' \in W_{\tau_0}(\{x_1\})$ *we have*

$$\varphi(t'(t'')) = \varphi(t') \circ \varphi(t'').$$

Proof. We give a proof by induction on the complexity of the term t'. If $t' = x_1$ then $\varphi(x_1(t'')) = \varphi(t'') = \lambda \circ \varphi(t'') = \varphi(x_1) \circ \varphi(t'')$. If $t' = f_i(t)$ and if we assume inductively that $\varphi(t(t'')) = \varphi(t) \circ \varphi(t'')$, then we have $\varphi(f_i(t)(t'')) = \varphi(f_i(t(t''))) = F_i \circ \varphi(t(t'')) = (F_i \circ \varphi(t)) \circ \varphi(t'') = \varphi(f_i(t)) \circ \varphi(t'')$. ■

Proposition 2.3 *The mapping* $\varphi : W_{\tau_0}(\{x_1\}) \longrightarrow \langle\{F_i \mid i \in I\} \cup \{\lambda\}\rangle$ *is bijective.*

Proof. By definition, φ maps terms in $W_{\tau_0}(\{x_1\})$ to words in $\mathcal{F}_M(\{F_i \mid i \in I\})$, and it is clearly surjective. Assuming that $\varphi(t_1) = \varphi(t_2)$, we prove the injectivity of φ by induction on the complexity (the number of occurrences of operation symbols) of t_1. If t_1 is the variable $x_1 \in X_1$, then $\varphi(t_1) = \lambda = \varphi(t_2)$. Since x_1 is the only unary term of type τ_0 which is mapped to λ, we then have $t_2 = t_1 = x_1$. Now suppose that $\varphi(t_1) = \varphi(t_2) \Longrightarrow t_1 = t_2$, and consider the term $t_1' = f_i(t_1)$. We may assume that t_2' is also not a variable, since otherwise $\varphi(t_2') = \lambda = \varphi(t_1')$ and $t_1' = x_1$. Hence there is a unary operation symbol f_j and a unary term t_2'' such that $t_2' = f_j(t_2'')$. Then we have $\varphi(t_2') = \varphi(f_i(t_2'')) = F_i \circ \varphi(t_2'')$, while $\varphi(t_1') = \varphi(f_i(t_1)) = F_i \circ \varphi(t_1)$. Thus

$$\varphi(t_1') = \varphi(t_2') \Rightarrow F_i \circ \varphi(t_1) = F_j \circ \varphi(t_2) \Rightarrow F_i = F_j \text{ and } \varphi(t_1) = \varphi(t_2) \Rightarrow f_i = f_j \text{ and}$$

$$t_1 = t_2 \Rightarrow t_1' = t_2'$$

using the fact that $\mathcal{F}_M(\{F_i \mid i \in I\})$ is free. ■

In the remainder of this section we examine the connection between certain special identities of unary algebras and identities in their unary clones. Hyperidentities are identities which hold in a particular stronger sense, defined using the concept of a hypersubstitution. For more detailed information on hyperidentities we refer the reader to [Den-W;00]. A hypersubstitution is a mapping which assigns to each operation symbol of a type a term, of the same arity as the operation symbol.

Definition 2.4 *A hypersubstitution of unary type is a mapping*

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_0}(\{x_1\}).$$

Any hypersubstitution σ of unary type induces a mapping $\hat{\sigma}$ defined on the set of all unary terms of the type, as follows.

Definition 2.5 *Let σ be a hypersubstitution of unary type. Then σ induces a mapping $\hat{\sigma} : W_{\tau_0}(\{x_1\}) \longrightarrow W_{\tau_0}(\{x_1\})$, by setting*

(i) $\hat{\sigma}[x_1] := x_1$,

(ii) $\hat{\sigma}[f_i(t)] := \sigma(f_i)(\hat{\sigma}[t]) \qquad (= S_1^1(\sigma(f_i), \hat{\sigma}[t]))$.

Let $Hyp(\tau_0)$ be the set of all hypersubstitutions of the unary type τ_0. Then a binary operation $\circ_h$ on $Hyp(\tau_0)$ can be defined, by setting $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$. We also have an identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) = f_i(x_1)$. Altogether this gives a monoid $(Hyp(\tau_0); \circ_h,\ \sigma_{id})$. We shall show that hypersubstitutions of unary type are closely related to substitutions on the monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$, that is, substitutions of $\{F_i \mid i \in I\}$ into $\langle\{F_i \mid i \in I\} \cup \{\lambda\}\rangle$. For the analogous semigroup case, when we do not use the empty word λ, we need the concept of a pre-hypersubstitution. A hypersubstitution σ of unary type is called a pre-hypersubstitution if for every $i \in I$ the term $\sigma(f_i)$ is not a variable. We denote by $Pre(\tau_0)$ the set of all pre-hypersubstitutions of type τ_0. It is easy to see that this set is closed under the composition $\circ_h$, and that $Pre(\tau_0)$ is a submonoid of $Hyp(\tau_0)$.

Any mapping $\eta : \{F_i \mid i \in I\} \longrightarrow \langle\{F_i \mid i \in I\} \cup \{\lambda\}\rangle$ is called a substitution. The extension $\overline{\eta}$ of the substitution η is an endomorphism of the free monoid $\mathcal{F}_M(\{F_i \mid i \in I\} \cup \{\lambda\})$ and is uniquely determined by η. Now we consider the set $Subst_M(2)$ of all such substitutions η. We can define a composition $\odot$ on this set, by defining $\eta_1 \odot \eta_2 := \overline{\eta_1} \circ \eta_2$, and we have an identity for this operation which is the identity mapping id on $\mathcal{F}_M(\{F_i \mid i \in I\} \cup \{\lambda\})$. Thus here too we obtain a monoid $(Subst_M(2); \odot, id)$. Since the sets $\{f_i \mid i \in I\}$ of operation symbols of type τ_0 and $\{F_i \mid i \in I\}$ of new variable symbols are both indexed by I, there is a bijection π between them, with $\pi(f_i) = F_i$ for $i \in I$. Using π we define a mapping $\overline{\pi} : W_{\tau_0}(\{x_1\}) \longrightarrow \langle\{F_i \mid i \in I\} \cup \{\lambda\}\rangle$, by setting

(i) $\overline{\pi}(x_1) = \lambda$

(ii) $\overline{\pi}(f_i(t)) = \pi(f_i) \circ \overline{\pi}(t)$.

Using the inverse mapping π^{-1} of the bijection π we define a mapping $\overline{\pi^{-1}}$ by

(i) $\overline{\pi^{-1}}(\lambda) = x_1$,

(ii) $\overline{\pi^{-1}}(w_1 \circ w_2) = \overline{\pi^{-1}}(w_1)(\overline{\pi^{-1}}(w_2)(x_1))$.

Then we have

Lemma 2.6 $\overline{\pi} = \varphi$ *and* $\overline{\pi^{-1}} = \varphi^{-1}$

Proof. We give a proof by induction on the term complexity. It is clear that for the variable x_1 we have $\overline{\pi}(x_1) = \lambda = \varphi(x_1)$. Assume inductively that $t = f_i(t')$ and that $\overline{\pi}(t') = \varphi(t')$. Then $\overline{\pi}(f_i(t')) = \pi(f_i) \circ \overline{\pi}(t') = F_i \circ \varphi(t') = \varphi(f_i(t'))$. For $\overline{\pi^{-1}}$ we have $\overline{\pi^{-1}}(\lambda) = x_1 = \varphi^{-1}(\lambda)$ and $\overline{\pi^{-1}}(w_1 \circ w_2) = \overline{\pi^{-1}}(w_1)(\overline{\pi^{-1}}(w_2)(x_1)) = \varphi^{-1}(w_1)(\varphi^{-1}(w_2)(x)) = \varphi^{-1}(w_1 \circ w_2)$. ∎

The diagram (D) below shows that for every hypersubstitution σ of unary type, the mapping $\eta := \varphi \circ \sigma \circ \pi^{-1}$ is a substitution.

Proposition 2.7 *If η is a substitution of the form $\varphi \circ \sigma \circ \pi^{-1}$, then the following are satisfied:*

(i) $\overline{\eta} = \varphi \circ \hat{\sigma} \circ \varphi^{-1}$,

(ii) $\varphi \circ (\sigma_1 \circ_h \sigma_2) \circ \pi^{-1} = (\varphi \circ \hat{\sigma}_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1})$.

Proof. (i) For the base case we have $\overline{\eta}(F_i) = \eta(F_i) = (\varphi \circ \sigma \circ \pi^{-1})(F_i) = \varphi(\sigma(f_i)) = \varphi(\hat{\sigma}[f_i(x_1)]) = \varphi(\hat{\sigma}[\varphi^{-1}(F_i)]) = (\varphi \circ \hat{\sigma} \circ \varphi^{-1})(F_i)$. Since the extension of η is uniquely determined, we get (i). (ii) This follows from $\varphi \circ (\sigma_1 \circ_h \sigma_2) \circ \pi^{-1} = (\varphi \circ \hat{\sigma}_1 \circ \sigma_2 \circ \pi^{-1}) = (\varphi \circ \hat{\sigma}_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_2 \circ \pi^{-1})$. ■

$$
\begin{tikzcd}
\{f_i \mid i \in I\} \arrow[r, "\sigma"] \arrow[d, shift right, "\pi"'] & W_{\tau_0}(\{x_1\}) \arrow[r, "\varphi"] & \langle \{F_i \mid i \in I\} \cup \{\lambda\} \rangle \\
\{F_i \mid i \in I\} \arrow[u, shift right, "\pi^{-1}"'] \arrow[urr, "\eta"'] & &
\end{tikzcd}
\qquad (D)
$$

Now we can establish our isomorphism.

Theorem 2.8 *The monoid $(Hyp(\tau_0); \circ_h, \sigma_{id})$ is isomorphic to $(Subst_M(2); \odot, id)$.*

Proof. Consider the mapping $\psi : Hyp(\tau_0) \longrightarrow Subst_M(2)$ defined by $\psi(\sigma) := \varphi \circ \sigma \circ \pi^{-1}$. Clearly ψ is well-defined. To see that ψ is one-to-one, suppose that $\psi(\sigma_1) = \psi(\sigma_2)$. Then $\varphi \circ \sigma_1 \circ \pi^{-1} = \varphi \circ \sigma_2 \circ \pi^{-1}$ and since φ and π^{-1} are both bijections, we get $\sigma_1 = \sigma_2$. For surjectivity of ψ, let $\eta \in Subst_M(2)$. Then the mapping $\varphi^{-1} \circ \eta \circ \pi$ from $\{f_i \mid i \in I\}$ to $W_{\tau_0}(\{x_1\})$ is a hypersubstitution of unary type, and applying ψ gives $\psi(\varphi^{-1} \circ \eta \circ \pi) = \varphi \circ (\varphi^{-1} \circ \sigma \circ \pi) \circ \pi^{-1} = \sigma$. Finally, ψ is a homomorphism, since $\psi(\sigma_1 \circ_h \sigma_2) = \varphi \circ (\sigma_1 \circ_h \sigma_2) \circ \pi^{-1} = (\varphi \circ \hat{\sigma}_1 \circ \varphi^{-1}) \circ (\varphi \circ \sigma_1 \circ \pi^{-1}) = (\varphi \circ \sigma_1 \circ \pi^{-1}) \circ (\varphi \circ \sigma_1 \circ \pi^{-1}) = (\varphi \circ \sigma_1 \circ \pi^{-1}) \odot (\varphi \circ \sigma_1 \circ \pi^{-1})$, by Proposition 2.7 and the definition of the operation $\odot$ on $Subst_M(2)$. ■

We noted above that the free monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ is isomorphic to the unary clone $Clone^{(1)}(\tau_0)$ of the type τ_0. If in diagram (D) above we replace the monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ by $Clone^{(1)}(\tau_0)$ and the set $\{F_i \mid i \in I\}$ by $\{f_i \mid i \in I\}$, then instead of $Subst_M(2)$ we can consider the set of all clone substitutions. Then Theorem 2.8 can be interpreted as asserting the existence of a one-to-one mapping between sets of hypersubstitutions and clone substitutions. A more general version of this interconnection, using the full clone of terms for arbitrary types, can be found in [Den-G;96] (Corollary 3.5); see also [Den-W;00]. The results of this section can be modified to the semigroup case, excluding the variable x_1 and the empty word λ, by using pre-hypersubstitutions. These are unary hypersubstitutions σ which map all the operation symbols f_i to terms other than the variable x_1. In this case we consider the restriction φ_S of the mapping φ which maps $W_{\tau_0}(\{x_1\})\backslash\{x_1\}$ to the free semigroup $\mathcal{F}_S(\{F_i \mid i \in I\})$ generated by the variables $\{F_i \mid i \in I\}$. Then instead of monoid substitutions we consider semigroup substitutions $\eta_S : \{F_i \mid i \in I\} \longrightarrow \langle\{F_i \mid i \in I\}\rangle$ and the monoid $(Subst_S; \odot, id)$. The mappings of $\overline{\pi_S}$ and $\overline{\pi_S^{-1}}$ are defined by the equations $\overline{\pi_S}(f_i(t)) = \pi_S(f_i) \circ \overline{\pi_S}(t)$ and $\overline{\pi_S^{-1}}(w_1 \circ w_2) = \overline{\pi_S^{-1}}(w_1)(\overline{\pi_S^{-1}}(w_2)(x_1))$. Then by a proof similar to that of Theorem 2.8, we have the following analogous result.

Theorem 2.9 *The monoid $(Pre(\tau_0); \circ_h, \sigma_{id})$ is isomorphic to $(Subst_S(2); \odot, id)$.*

We can use the isomorphism of Theorem 2.8 in the following way. Sets of substitutions with certain properties which are preserved by composition of substitutions will correspond to submonoids of hypersubstitutions. For instance, in Section 4 we will consider the following two monoids of hypersubstitutions:

$$\mathcal{B} := \{\sigma \mid \sigma \in Hyp(\tau_0) \quad \text{and} \quad \varphi \circ \hat{\sigma} \circ \varphi^{-1} \text{is bijective}\}$$

and

$$\mathcal{O} := \{\sigma \mid \sigma \in Hyp(\tau_0) \quad \text{and} \quad \varphi \circ \hat{\sigma} \circ \varphi^{-1} \text{is surjective}\}.$$

3 Identities and Hyperidentities Let s and t be unary terms of type τ_0, and let $\mathcal{A}$ be a unary algebra of type τ_0. The identity $s \approx t$ is satisfied in $\mathcal{A}$ if the term operations induced by s and t are equal, that is, if $s^{\mathcal{A}} = t^{\mathcal{A}}$. An equivalent condition that we shall frequently use is that an identity $s \approx t$ is satisfied in $\mathcal{A}$ iff for every valuation mapping $v : X \to \mathcal{A}$ we have $\overline{v}(s) = \overline{v}(t)$ where $\overline{v}$ is the uniquely determined extension of v. We shall denote by IdA the set of all identities satisfied in the algebra $\mathcal{A}$. If V is a variety of unary type τ_0, then IdV denotes the set of all identities which are satisfied in all algebras of V. An identity $s \approx t$ is called a hyperidentity of a variety V if for every $\sigma \in Hyp(\tau_0)$ the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is in IdV. The identity is called a pre-hyperidentity of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is in IdV for every pre-hypersubstitution σ. In the first case we write $V \underset{hyp}{\models} s \approx t$ and in the second case $V \underset{pre\text{-}hyp}{\models} s \approx t$. If w_1 and w_2 are monoid words from $\mathcal{F}_M(\{F_i \mid i \in I\})$, then the pair $w_1 \approx w_2$ is an identity in the monoid $T^{(1)}(\mathcal{A})$ if $w_1^{T^{(1)}(\mathcal{A})} = w_2^{T^{(1)}(\mathcal{A})}$. Analogously, in the semigroup case where w_1 and w_2 are semigroup words from $\mathcal{F}_S(\{F_i \mid i \in I\})$, then $w_1 \approx w_2$ is an identity in the semigroup $S(\mathcal{A})$ when $w_1^{S(\mathcal{A})} = w_2^{S(\mathcal{A})}$. We need one more mapping h which assigns to each variable F_i the fundamental operation $f_i^{\mathcal{A}}$ of our algebra $\mathcal{A}$. The mapping h can be uniquely extended to a homomorphism $\overline{h} : \mathcal{F}_M(\{F_i \mid i \in I\}) \to (T^{(1)}(\mathcal{A}), \circ, id_A)$ since $\mathcal{F}_M(\{F_i \mid i \in I\})$ is free. For $S(\mathcal{A})$ and $\mathcal{F}_S(\{F_i \mid i \in I\})$ we denote the corresponding mapping by h_S. We use our isomorphism φ from the previous section and the mapping h to relate hyperidentities on an algebra $\mathcal{A}$ to clone identities on the unary clones of the algebra $\mathcal{A}$.

Lemma 3.1 *For every term $s \in W_{\tau_0}(\{x_1\})$ and for every algebra $\mathcal{A}$ of type τ_0, we have $s^{\mathcal{A}} = \overline{h}(\varphi(s))$; and for every term $t \in W_{\tau_0}(\{x_1\}) \setminus \{x_1\}$ we have $t^{\mathcal{A}} = \overline{h}_S(\varphi_S(t))$.*

Proof. We will give a proof by induction on the complexity of the term s. If $s = x_1$, then $x_1^{\mathcal{A}} = id_A = \overline{h}(\lambda) = \overline{h}(\varphi(x_1))$. Now let $s = f_i(s_1)$ be a compound term, with $s_1^{\mathcal{A}} =$

$\overline{h}(\varphi(s_1))$. Then $s^{\mathcal{A}} = (f_i(s_1))^{\mathcal{A}} = f_i^{\mathcal{A}} \circ s_1^{\mathcal{A}} = \overline{h}(F_i) \circ \overline{h}(\varphi(s_1)) = \overline{h}(F_i \circ \varphi(s_1)) = \overline{h}(\varphi(f_i(s_1))$ $= \overline{h}(\varphi(s))$. The proof for non-variable terms using h_S is similar, but with a base case of $s = f_i(x_1)$. ■

Lemma 3.1 will allow us to compare identities $s \approx t$ (of the right form) on an algebra $\mathcal{A}$ with their images $\varphi(s) \approx \varphi(t)$ on the unary clone of $\mathcal{A}$. In order to apply φ to terms s and t, and hence to identities, the terms must be terms on the alphabet X_1 only. For unary type, any term contains at most one variable, and hence any identity contains at most two variables. We cannot apply φ to identities such as $f_1(f_2(x_1)) \approx f_2(f_1(x_2))$ which contain two different variables, and hence we restrict ourselves to identities which contain only the single variable x_1. Such identities are regular, since they have the same variable on each side. Let $s \approx t \in IdV$ be a regular identity of a variety V, so that for every algebra $\mathcal{A} \in V$ we have $s^{\mathcal{A}} = t^{\mathcal{A}}$. Then by Lemma 3.1 the elements $\overline{h}(\varphi(s))$ and $\overline{h}(\varphi(t))$ agree on $T^{(1)}(\mathcal{A})$ for every $\mathcal{A}$. This means that a set Σ of identities satisfied in an algebra $\mathcal{A}$ of type τ_0 corresponds to a set of equations between the generating elements $\{f_i^{\mathcal{A}} \mid i \in I\}$ of the unary clone $T^{(1)}(\mathcal{A})$. Such sets of equations are also called relations of the algebra. For more information on presentations of an algebra by generators and relations the reader is referred to [Bur-S;81] in the arbitrary case and to [Cli-P;67] for semigroups.
Now let $\mathcal{A}$ be an algebra of type τ_0. On the free monoid $\mathcal{F}_M(\{F_i \mid i \in I\})$ we define the so-called Myhill congruence μ_A of $\mathcal{A}$ ([Pet-C-B;02]) by

$$(w_1, w_2) \in \mu_A :\Longleftrightarrow \varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \in Id\mathcal{A}.$$

For V a variety, we define the congruence μ_V to be the intersection of all the congruences $\mu_{\mathcal{A}}$ for all $\mathcal{A} \in V$. Analogously we define $\mu_A^{\star}$ on $\mathcal{F}_S(\{F_i \mid i \in I\})$ by

$$(w_1, w_2) \in \mu_A^{\star} :\Longleftrightarrow \varphi_S^{-1}(w_1) \approx \varphi_S^{-1}(w_2) \in Id\mathcal{A}.$$

It is easy to see that μ_A is a congruence and therefore we may form the quotient monoid $\mathcal{F}_M(\{F_i \mid i \in I\})/\mu_A$ or for a variety $\mathcal{F}_M(\{F_i \mid i \in I\})/\mu_V$. In a corresponding way we may consider $\mathcal{F}_S(\{F_i \mid i \in I\})/\mu_A^{\star}$ and $\mathcal{F}_S(\{F_i \mid i \in I\})/\mu_V^{\star}$, respectively. Then we have

Proposition 3.2 *The monoid $(T^{(1)}(\mathcal{A}), \circ, id_A)$ is isomorphic to the quotient monoid $\mathcal{F}_M(\{F_i \mid i \in I\})/\mu_A$, and in the semigroup case $S(\mathcal{A})$ is isomorphic to $\mathcal{F}_S(\{F_i \mid i \in I\})/\mu_A^{\star}$.*

Proof. We show first that μ_A is the kernel of $\overline{h}$. We have

$$\begin{array}{lll}
(w_1, w_2) \in \mu_A & \Longleftrightarrow & \varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \in Id\mathcal{A} \\
& \Longleftrightarrow & \varphi^{-1}(w_1)^{\mathcal{A}} = \varphi^{-1}(w_2)^{\mathcal{A}} \\
& \Longleftrightarrow & \overline{h}(\varphi(\varphi^{-1}(w_1))) = \overline{h}(\varphi(\varphi^{-1}(w_2))) \quad \text{by Lemma 3.1 and the bijectiv-} \\
& \Longleftrightarrow & \overline{h}(w_1) = \overline{h}(w_2) \\
& \Longleftrightarrow & (w_1, w_2) \in \ker \ \overline{h},
\end{array}$$

ity of φ. Since $\overline{h}$ is a surjective homomorphism from $\mathcal{F}_M(\{F_i \mid i \in I\})$ onto $T^{(1)}(\mathcal{A})$, by the homomorphism theorem we have $T^{(1)}(\mathcal{A}) \cong \mathcal{F}_M(\{F_i \mid i \in I\})/ker\overline{h} \cong \mathcal{F}_M(\{F_i \mid i \in I\})/\mu_A$. The proof for $S(\mathcal{A})$ is similar. ■

If $\eta : \{F_i \mid i \in I\} \to \mathcal{F}_M(\{F_i \mid i \in I\})$ is a substitution then the composition $\overline{h} \circ \eta : \{F_i \mid i \in I\} \cup \to T^{(1)}(\mathcal{A})$ is a valuation of the variables F_i by elements of the clone $T^{(1)}(\mathcal{A})$. The next Theorem connects hyperidentities of the algebra or variety with identities of the unary clone.

Theorem 3.3 *A regular identity $s \approx t$ is a hyperidentity in an algebra $\mathcal{A}$ of type τ_0 iff the equation $\varphi(s) \approx \varphi(t)$ is an identity in the monoid $T^{(1)}(\mathcal{A})$. That is,*

$$\mathcal{A} \underset{hyp}{\models} s \approx t \quad \Leftrightarrow \quad T^{(1)}(\mathcal{A}) \models \varphi(s) \approx \varphi(t) \ .$$

Proof. We first assume that $s \approx t$ is a hyperidentity of $\mathcal{A}$. Then for every $\sigma \in Hyp(\tau_0)$ we have $\hat{\sigma}[s]^{\mathcal{A}} = \hat{\sigma}[t]^{\mathcal{A}}$. Let $v : \{F_i \mid i \in I\} \to T^{(1)}(\mathcal{A})$ be an arbitrary valuation mapping. Since $\overline{h}$ is surjective, there exists a substitution η_v with $v = \overline{h} \circ \eta_v$, using the axiom of choice. Then we have $\overline{v}(\varphi(s)) = (\overline{h} \circ \overline{\eta}_v)(\varphi(s)) = \overline{h} \circ (\varphi \circ \hat{\sigma}_{\eta_v} \circ \varphi^{-1})(\varphi(s)) = (\overline{h} \circ \varphi \circ \hat{\sigma}_{\eta_v})(s) = (\hat{\sigma}_{\eta_v}[s])^{\mathcal{A}} = (\hat{\sigma}_{\eta_v}[t])^{\mathcal{A}} = \overline{v}(\varphi(t))$, and this means that $\varphi(s) \approx \varphi(t) \in IdT^{(1)}(\mathcal{A})$. Conversely, let $\varphi(s) \approx \varphi(t) \in IdT^{(1)}(\mathcal{A})$. Then $s, t \in W_{\tau_0}(\{x_1\})$ and for every valuation v we have $\overline{v}(\varphi(s)) = \overline{v}(\varphi(t))$. Let σ be a hypersubstitution. Then, using Lemma 3.1 and the fact that $\overline{h} \circ \varphi \circ \hat{\sigma} \circ \varphi^{-1}$ is the extension of a valuation mapping, we obtain $\hat{\sigma}[s]^{\mathcal{A}} = \overline{h}(\varphi(\hat{\sigma}[s])) = \overline{h}(\varphi(\hat{\sigma}(\varphi^{-1}(\varphi(s))))) = \overline{h}(\varphi(\hat{\sigma}(\varphi^{-1}(\varphi(t))))) = \hat{\sigma}[t]^{\mathcal{A}}$. This shows that $\hat{\sigma}[s]^{\mathcal{A}} = \hat{\sigma}[t]^{\mathcal{A}}$, and $s \approx t$ is a hyperidentity in $\mathcal{A}$. ∎

A normal regular identity, of type τ_0, is a regular identity $s \approx t$ in which either both s and t are equal to the variable x_1 or neither are. In the following analogue of Theorem 3.3, normal pre-hyperidentities of an algebra correspond to identities in the transition semigroups of the algebra.

Theorem 3.4 *Let $s \approx t$ be a non-trivial regular normal identity of a unary algebra $\mathcal{A}$. Then $s \approx t$ is a pre-hyperidentity of $\mathcal{A}$ iff the equation $\varphi_S(s) \approx \varphi_S(t)$ is an identity in the transition semigroup $S(\mathcal{A})$.*

Proof. The proof is similar to that of Theorem 3.3. ∎

A variety V of type τ_0 is called solid if every identity in V is satisfied as a hyperidentity, and is called presolid if every identity in V is satisfied as a pre-hyperidentity. The presolid varieties of type τ_0 form a complete sublattice of the lattice of all varieties of type τ_0, and the solid varieties form a complete sublattice of the lattice of all presolid varieties. Solidity and presolidity are special cases of the general concept of M-solidity, for any submonoid M of hypersubstitutions; see [Den-W;00] for more information on M-solid varieties. Let V be a variety defined by a set Σ of regular equations of type τ_0, so that $V = Mod\ \Sigma$ is the class of all algebras of type τ_0 which satisfy every equation from Σ as an identity. It is well-known that V is solid or presolid iff every equation from Σ is satisfied as a hyperidentity or pre-hyperidentity, respectively, of V.

Corollary 3.5 *Let $\mathcal{A}$ be an algebra of unary type τ_0 with the property that the variety $V(\mathcal{A})$ which is generated by $\mathcal{A}$ is regular. Then $V(\mathcal{A})$ is solid iff the monoid $T^{(1)}(\mathcal{A})$ is free with respect to itself (Hall-free), meaning that every mapping from $\{f_i^{\mathcal{A}} \mid i \in I\}$ to $T^{(1)}(\mathcal{A})$ can be extended to an endomorphism of $T^{(1)}(\mathcal{A})$.*

Proof. Using the equivalence from Theorem 3.3, we will show that $T^{(1)}(\mathcal{A})$ is free iff for every identity $s \approx t \in Id\mathcal{A}$ the equation $\varphi(s) \approx \varphi(t)$ is an identity in $T^{(1)}(\mathcal{A})$. Suppose first that $T^{(1)}(\mathcal{A})$ is free with respect to itself, freely generated by the independent set $\{f_i^{\mathcal{A}} \mid i \in I\}$, and let $s \approx t \in Id\mathcal{A}$. Then by Lemma 3.1, the elements $\overline{h}(\varphi(s))$ and $\overline{h}(\varphi(t))$ are equal in $T^{(1)}(\mathcal{A})$. Let $v : \{F_i \mid i \in I\} \to T^{(1)}(\mathcal{A})$ be an arbitrary valuation mapping. We define a mapping $\alpha_v : \{f_i^{\mathcal{A}} \mid i \in I\} \to T^{(1)}(\mathcal{A})$ by $\alpha_v(f_i^{\mathcal{A}}) := v(F_i)$ for every $i \in I$. The mapping α_v is well-defined since $\{f_i^{\mathcal{A}} \mid i \in I\}$ is an independent set and so, for $i, j \in I$ we have

$$f_i^{\mathcal{A}} = f_j^{\mathcal{A}} \Rightarrow i = j \Rightarrow F_i = F_j \Rightarrow v(F_i) = v(F_j) \Rightarrow \alpha_v(f_i^{\mathcal{A}}) = \alpha_v(f_j^{\mathcal{A}}).$$

Since $\mathcal{F}_M(\{F_i \mid i \in I\})$ is free in the class of all monoids, the mapping $v = \alpha_v \circ h$ can be uniquely extended to $\overline{v} = \overline{\alpha_v} \circ \overline{h}$. Then

$$\overline{h}(\varphi(s)) = \overline{h}(\varphi(t)) \Rightarrow \overline{\alpha_v}(\overline{h}(\varphi(s))) = \overline{\alpha_v}(\overline{h}(\varphi(t))) \Rightarrow \overline{v}(\varphi(s)) = \overline{v}(\varphi(t)) \Rightarrow \varphi(s) \approx \varphi(t) \in IdT^{(1)}(\mathcal{A}).$$

Therefore $s \approx t$ is a hyperidentity in $\mathcal{A}$, and we have shown that $V(\mathcal{A})$ is solid.
For the converse direction, we assume that $V(\mathcal{A})$ satisfies the solidity condition. To show that $T^{(1)}$ is free, we let $\alpha : \{f_i^{\mathcal{A}} \mid i \in I\} \to T^{(1)}(\mathcal{A})$ be any mapping, and show that it can be extended to an endomorphism. Our extension will be the mapping $\overline{\alpha} : T^{(1)}(\mathcal{A}) \to T^{(1)}(\mathcal{A})$ defined by $\overline{\alpha}(u^{\mathcal{A}}) := \overline{\alpha \circ h}(\varphi(u))$, for each $u^{\mathcal{A}} \in T^{(1)}(\mathcal{A})$. Note that $\overline{\alpha \circ h}$ is the uniquely defined extension of the valuation $\alpha \circ h : \{F_i \mid i \in I\} \to T^{(1)}(\mathcal{A})$. To check that $\overline{\alpha}$ is well-defined, we let $u^{\mathcal{A}} = t^{\mathcal{A}}$. Then $u \approx t \in Id\mathcal{A}$ and by the solidity assumption we have $\varphi(u) \approx \varphi(t) \in IdT^{(1)}(\mathcal{A})$. Then $\overline{\alpha \circ h}(\varphi(u)) = \overline{\alpha \circ h}(\varphi(t))$ since $\alpha \circ h$ is a valuation, and by definition we get $\overline{\alpha}(u^{\mathcal{A}}) = \overline{\alpha}(t^{\mathcal{A}})$. Next we show that $\overline{\alpha}$ is a homomorphism: for any $u^{\mathcal{A}}, t^{\mathcal{A}} \in T^{(1)}(\mathcal{A})$, we have $\overline{\alpha}(u^{\mathcal{A}} \circ t^{\mathcal{A}}) = \overline{\alpha}(u(t)^{\mathcal{A}}) = \overline{\alpha \circ h}(\varphi(\underline{u(t)})) = \overline{\alpha \circ h}(\varphi(u)) \circ \overline{\alpha \circ h}(\varphi(t)) = \overline{\alpha}(u^{\mathcal{A}}) \circ \overline{\alpha}(t^{\mathcal{A}})$. Moreover $\overline{\alpha}$ extends α, since $\overline{\alpha}(f_i^{\mathcal{A}}) = \overline{\alpha \circ h}(\varphi(f_i(x_1))) = \overline{\alpha \circ h}(F_i) = (\alpha \circ h)(F_i) = \alpha(h(F_i)) = \alpha(f_i^{\mathcal{A}})$. ■

If V is a regular variety of algebras of type τ_0 and if $\mathcal{F}_V(X)$ is the free algebra generated by the countably infinite alphabet X, then $T^{(1)}(\mathcal{F}_V(X))$ is called the transition monoid of the variety V. Corollary 3.5 says that V is solid iff this transition monoid $T^{(1)}(\mathcal{F}_V(X))$ is free with respect to itself. If $V = Mod\Sigma$ then V is solid iff $T^{(1)}(\mathcal{F}_V(X))$ is free with respect to itself and has the free presentation $(\{\overline{f}_i \mid i \in I\}; \varphi(\Sigma))$. ([Mar;66]). We remark that Corollary 3.5 is a special case of the equivalence of the solidity of a variety of arbitrary type τ and the freeness of the clone of that variety, if we regard the clone as a multibased (heterogeneous) algebra. In a similar way, Theorem 3.4 can be used to show that for every set Σ of normal and regular equations of type τ_0, the variety $V = Mod\Sigma$ is presolid iff the semigroup presented by $(\{f_i \mid i \in I\}, \overline{h}(\varphi_S(\Sigma)))$ is free with respect to itself. Theorem 3.3 can be generalized in the following way. The isomorphism $\psi : Hyp(\tau_0) \longrightarrow Subst_M(2)$ used in the proof of Theorem 2.8 maps submonoids of $Hyp(\tau_0)$ to submonoids of $Subst_M(2)$. Let $G \subseteq Hyp(\tau_0)$ be a submonoid and let $\psi(G) \subseteq Subst_M(2)$ be the corresponding monoid of substitutions.

Definition 3.6 *An equation $w_1 \approx w_2$ between monoid words from $\mathcal{F}_M(\{F_i \mid i \in I\})$ is called a $\psi(G)$-identity in a semigroup S if for all $\eta \in \psi(G)$ the equations $\overline{h}(\overline{\eta}(w_1)) = \overline{h}(\overline{\eta}(w_2))$ are satisfied.*

Then analogously to Theorem 3.3 we get the following Theorem.

Theorem 3.7 *Let $\mathcal{G}$ be a monoid of hypersubstitutions of type τ_0 and let $\mathcal{A}$ be an algebra of type τ_0. Then the regular identity $s \approx t$ is a G-hyperidentity in $\mathcal{A}$ iff $\varphi(s) \approx \varphi(t)$ is a $\psi(G)$-identity in $T^{(1)}(\mathcal{A})$. In this case we write* $T^{(1)}(\mathcal{A}) \underset{\psi(G)-id}{\models} \varphi(s) \approx \varphi(t)$. ■

For the semigroup case, for terms $s, t \neq x_1$ we get a similar result:

Theorem 3.8 *Let $\mathcal{G}$ be a monoid of hypersubstitutions of type τ_0 and let $\mathcal{A}$ be an algebra of type τ_0. Then the regular and normal identity $s \approx t$ is a G-hyperidentity in V iff $\varphi_S(s) \approx \varphi_S(t)$ is a $\psi(G)$-identity in $S(\mathcal{A})$. In this case we write* $S(\mathcal{A}) \underset{\psi(G)-id}{\models} \varphi_S(s) \approx \varphi_S(t)$. ■

4 σ-Closed and υ-Closed Varieties of Unary Type There are some interesting interconnections between varieties V of unary type τ_0 and the classes $\{T^{(1)}(\mathcal{A}) \mid \mathcal{A} \in V\}$ of transition monoids or $\{S(\mathcal{A}) \mid \mathcal{A} \in V\}$ of transition semigroups formed from algebras in V. One such connection uses the concept of σ-closed varieties of type τ_0 introduced in [Pet-C-B;02].

Definition 4.1 *Let V be a variety of unary type τ_0. Then V is called σ-closed if whenever $\mathcal{A} \in V$ and $T^{(1)}(\mathcal{B}) \cong T^{(1)}(\mathcal{A})$, then also $\mathcal{B} \in V$. The variety V is called $\sigma^\star$-closed if whenever $\mathcal{A} \in V$ and $S(\mathcal{B}) \cong S(\mathcal{A})$ then also $\mathcal{B} \in V$.*

We recall from the end of Section 2 the definition of the submonoid $\mathcal{O}$ of hypersubstitutions:

$$\mathcal{O} := \{\sigma \mid \sigma \in Hyp(\tau_0) \quad \text{and} \qquad \varphi \circ \hat{\sigma} \circ \varphi^{-1} \;\; \text{is surjective}\}.$$

A unary variety V is called $\mathcal{O}$-solid if for all identities $s \approx t$ in IdV and all $\sigma \in \mathcal{O}$ we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\ V$. We shall show that σ-closed varieties and $\mathcal{O}$-solid varieties are closely related to each other. To formulate and prove this interrelation we introduce some new notation and concepts. For any algebra $\mathcal{A}$ (of arbitrary type) we denote by $Con\mathcal{A}$ the lattice of congruences on $\mathcal{A}$. By the second isomorphism theorem, if two congruences θ and ρ on $\mathcal{A}$ satisfy $\theta \subseteq \rho$, it follows that there is a homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\rho$; but the converse implication is not always true.

Definition 4.2 *Let $\mathcal{A}$ be an algebra of arbitrary type. A congruence $\theta \in Con\mathcal{A}$ is said to be weakly invariant if for every $\rho \in Con\mathcal{A}$ the following condition is satisfied: if there exists a homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\rho$ then $\theta \subseteq \rho$.*

Definition 4.3 *A set C of congruences of an algebra $\mathcal{A}$ of arbitrary type τ is said to be isomorphically closed if whenever $\theta \in C$ and $\mathcal{A}/\theta \cong \mathcal{A}/\rho$ it follows that $\rho \in C$.*

The following results were proved for semigroups in [Pet-C-B;02], and can easily be generalized to algebras of arbitrary type τ.

Theorem 4.4 *Let $\mathcal{A}$ be an algebra of arbitrary type τ, let V be a variety of type τ and let $\mathcal{F}_V(X)$ be the free algebra with respect to V, freely generated by X. Then (i) A congruence θ on $\mathcal{A}$ is weakly invariant iff the principal filter generated by θ in $Con\mathcal{A}$ is isomorphically closed.*
(ii) Every weakly invariant congruence on $\mathcal{A}$ is invariant under all surjective endomorphisms of $\mathcal{A}$.
(iii) The set $Con_{wi}\mathcal{A}$ of all weakly invariant congruences of $\mathcal{A}$ forms a complete lattice.
(iv) Every fully invariant congruence on $\mathcal{F}_V(X)$ is weakly invariant. ■

Lemma 4.5 *([Pet-C-B;02]) Let V be a unary variety of type τ_0. If V is σ-closed, then $\mu_V \in Con\mathcal{F}_M(\{F_i \mid i \in I\})$ is weakly invariant.* ■

Now we are able to prove:

Theorem 4.6 *A regular variety V of type τ_0 is σ-closed iff it is $\mathcal{O}$-solid.*

Proof. Assume that V is $\mathcal{O}$-solid. It follows from the theory of M-hyperidentities and M-solid varieties that there is a set Σ of regular equations of type τ_0 such that V is the $\mathcal{O}$-hypermodel class $V = H_{\mathcal{O}}Mod\Sigma$ defined by Σ, meaning that V is the class of all algebras of type τ_0 satisfying all the equations from Σ as $\mathcal{O}$-hyperidentities. Thus $\mathcal{A} \in V$

iff $\mathcal{A}$ $\mathcal{O}$-hypersatisfies Σ. We will use the notation $\varphi(\Sigma)$ to denote the set of identities obtained by applying φ to each identity in Σ; this makes sense because the identities in Σ are all regular. Now let $\mathcal{A}$ be any algebra in V. Then ¿from $\mathcal{A} \underset{O\text{-}hyp}{\models} \Sigma$ it follows that $T^{(1)}(\mathcal{A}) \underset{\psi(O)\text{-}id}{\models} \varphi(\Sigma)$. If $T^{(1)}(\mathcal{A}) \cong T^{(1)}(\mathcal{B})$, then $T^{(1)}(\mathcal{B}) \underset{\psi(O)\text{-}id}{\models} \varphi(\Sigma)$; this gives $\mathcal{B} \underset{O\text{-}hyp}{\models} \Sigma$ and thus $\mathcal{B} \in V = H_{\mathcal{O}}Mod\Sigma$. This shows that V is σ-closed. Conversely, we suppose that V is a regular variety of type τ_0 which is σ-closed. Then by Lemma 4.5 the Myhill congruence $\mu_V \in Con\ \mathcal{F}_M(\{F_i | i \in I\})$ is weakly invariant. By Theorem 4.4 μ_V is invariant under all surjective endomorphisms of $\mathcal{F}_M(\{F_i | i \in I\})$. Therefore for every pair $(w_1, w_2) \in \mu_V$ and every surjective endomorphism $\overline{\eta} : \mathcal{F}_M(\{F_i | i \in I\}) \longrightarrow \mathcal{F}_M(\{F_i | i \in I\})$ we have $(\overline{\eta}(w_1), \overline{\eta}(w_2)) \in \mu_V$. Then by Theorem 2.8 there is a hypersubstitution σ such that $((\varphi \circ \hat{\sigma} \circ \varphi^{-1})(w_1), (\varphi \circ \hat{\sigma} \circ \varphi^{-1})(w_2)) \in \mu_V$. By definition of μ_V for every algebra $\mathcal{A} \in V$ and hypersubstitution σ such that $\varphi \circ \hat{\sigma} \circ \varphi^{-1}$ is surjective we get that from $\varphi^{-1}(w_1) \approx \varphi^{-1}(w_2) \in Id\ \mathcal{A}$ there follows $\varphi^{-1} \circ (\varphi \circ \hat{\sigma} \circ \varphi^{-1})(w_1) \approx \varphi^{-1} \circ (\varphi \circ \hat{\sigma} \circ \varphi^{-1})(w_2) \in Id\ \mathcal{A}$ and thus $\hat{\sigma}[\varphi^{-1}(w_1)] \approx \hat{\sigma}[\varphi^{-1}(w_2)] \in Id\ \mathcal{A}$. This means that $\varphi^{-1}(w_1) \approx \varphi^{-1}(w_2)$ is an $\mathcal{O}$-hyperidentity in V. Since every regular identity $s \approx t \in Id\ V$ can be written in the form $\varphi^{-1}(w_1) \approx \varphi^{-1}(w_2)$, this completes the proof. ∎

In a similar way it can be shown that a regular and normal variety V of type τ_0 is $\sigma^\star$-closed iff it is $\mathcal{O}$-solid. We generalize Definition 4.1 in the following way:

Definition 4.7 *A variety V of type τ_0 is said to be s-closed if for every algebra $\mathcal{B}$ of type τ_0, whenever $\mathcal{A} \in V$ and $T^{(1)}(\mathcal{B})$ is isomorphic to a submonoid of $T^{(1)}(\mathcal{A})$, it follows that $\mathcal{B} \in V$. The variety V is said to be v-closed if for every algebra $\mathcal{B}$ of type τ_0, whenever $\mathcal{A} \in V$ and $Id\ T^{(1)}(\mathcal{B}) \supseteq Id\ T^{(1)}(\mathcal{A})$ (that is, if $T^{(1)}(\mathcal{B}) \in V(T^{(1)}(\mathcal{A}))$), then it follows that $\mathcal{B} \in V$. (Analogous definitions can be made for $v^\star$-closed and $s^\star$-closed varieties, using the transition semigroups $S(\mathcal{A})$ and $S(\mathcal{B})$.)*

Proposition 4.8 *If a variety V of type τ_0 is v-closed, then it is σ-closed and s-closed.*

Proof. Let V be v-closed, and let $\mathcal{A} \in V$ and $T^{(1)}(\mathcal{B}) \cong T^{(1)}(\mathcal{A})$. Then $Id\ T^{(1)}(\mathcal{B}) = Id\ T^{(1)}(\mathcal{A})$, and so $\mathcal{B} \in V$ since V is v-closed. Therefore V is σ-closed. If $T^{(1)}(\mathcal{B})$ is isomorphic to a submonoid of $T^{(1)}(\mathcal{A})$, then $Id\ T^{(1)}(\mathcal{B}) \supseteq Id\ T^{(1)}(\mathcal{A})$ and so $\mathcal{B} \in V$ since V is v-closed. This shows that V is s-closed. ∎

In a corresponding way it can be shown that every $v^\star$-closed variety of type τ_0 is $\sigma^\star$-closed and $s^\star$-closed.

Theorem 4.9 *A regular variety V of type τ_0 is v-closed iff it is solid.*

Proof. Let V be a solid regular variety. Then there exists a set Σ of regular equations with $V = HMod\Sigma$, such that for every algebra $\mathcal{A} \in V$ we have $\mathcal{A} \underset{hyp}{\models} \Sigma$. Then by Theorem 3.3 for every $\mathcal{A} \in V$ we have $T^{(1)}(\mathcal{A}) \models \varphi(\Sigma)$. If $\mathcal{B}$ is an algebra with $Id\ T^{(1)}(\mathcal{B}) \supseteq Id\ T^{(1)}(\mathcal{A})$ then $T^{(1)}(\mathcal{B}) \models \varphi(\Sigma)$ and by Theorem 3.3 we have $\mathcal{B} \underset{hyp}{\models} \Sigma$; thus $\mathcal{B} \in V = HMod\ \Sigma$, and V is v-closed. Now let V be v-closed, and let $\mathcal{A} \in V$ with $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$. We consider the derived algebra $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^{\mathcal{A}})_{i \in I})$ for a hypersubstitution σ. Since the operations $\sigma(f_i)^{\mathcal{A}}$ are term operations of $\mathcal{A}$ we have $\langle \{\sigma(f_i)^{\mathcal{A}} | i \in I\} \cup \{id_A\}\rangle = T^{(1)}(\sigma(\mathcal{A})) \subseteq T^{(1)}(\mathcal{A}) = \langle \{(f_i)^{\mathcal{A}} | i \in I\} \cup \{id_A\}\rangle$. Therefore $Id\ (T^{(1)}(\sigma(\mathcal{A})) \supseteq Id\ T^{(1)}(\mathcal{A})$. Since V is v-closed, we have $\sigma(\mathcal{A}) \in V$. This shows that any derived algebra formed from an algebra in V is also in V, and this is known to be equivalent to solidity of V (see [Den-W;00]). ∎

In a corresponding way it can be shown that a regular and normal variety of type τ_0 is $v^\star$-closed iff it is pre-solid. Now we prove that v-closure and s-closure are equivalent concepts for varieties of type τ_0.

Theorem 4.10 *Let V be a regular variety of type τ_0. Then V is v-closed iff it is s-closed.*

Proof. One direction is given by Proposition 4.8. Conversely, suppose that V is s-closed, and let $\mathcal{A} \in V$. Then as in the proof of Theorem 4.9 we have $T^{(1)}(\sigma(\mathcal{A})) \subseteq T^{(1)}(\mathcal{A})$ and then $\sigma(\mathcal{A}) \in V$. This shows that V is solid, and hence by Theorem 4.9 it is also v-closed. ■

This means that for a regular variety of type τ_0, the concepts of v-closure, s-closure and solidity are equivalent. An analogous result holds for $v^\star$-closure, $s^\star$-closure and presolidity of normal regular varieties. In Section 2 we introduced the concept of a $\mathcal{B}$-solid variety. Since $\mathcal{B}$ is a submonoid of $\mathcal{O}$, every $\mathcal{O}$-solid variety of type τ_0 is also $\mathcal{B}$-solid. Let V be a $\mathcal{B}$-solid regular variety of type τ_0. Then there exists a set Σ of regular equations of type τ_0 such that $V = H_{\mathcal{B}}Mod\Sigma$ and if $\mathcal{C} \in V$, then $\mathcal{C} \underset{B\text{-}hyp}{\models} \Sigma$. Therefore $T^{(1)}(\mathcal{C}) \underset{\psi(B)\text{-}id}{\models} \varphi(\Sigma)$. If $T^{(1)}(\mathcal{A}) \cong T^{(1)}(\mathcal{C})$ then $T^{(1)}(\mathcal{A}) \underset{\psi(B)\text{-}id}{\models} \varphi(\Sigma)$ and $\mathcal{A} \underset{B\text{-}hyp}{\models} \Sigma$ by Theorem 3.7, and it follows that $\mathcal{A} \in V$ and V is σ-closed. By Theorem 4.6 V is $\mathcal{O}$-solid. This proves the following result.

Theorem 4.11 *A regular variety V of type τ_0 is $\mathcal{B}$-solid iff it is $\mathcal{O}$-solid.* ■

5 Example

In this section we illustrate the application of Theorems 4.9 and 4.10 with an example. We consider the unary type $\tau_0 = (1, 1)$, so that we have two unary operation symbols f and g. Let V be the model class of the following equations: $f(f(x)) \approx x$, $g(g(x)) \approx x$, $f(g(f(x))) \approx g(x)$, $g(f(g(x))) \approx f(x)$.

We want to show that V is solid. Since the four identities defining V are regular, the variety V is regular, and we can use the fact that such varieties are solid iff they are s-closed. Let $\mathcal{A} \in V$, so $\mathcal{A} = (A; f^A, g^A)$ is an algebra of type $(1, 1)$ satisfying the four identities above. Then the transition monoid $T^{(1)}(\mathcal{A})$ has the presentation $T^{(1)}(\mathcal{A}) := (\{f^A, g^A\}, \{(f^A)^2 = id_A, (g^A)^2 = id_A, f^A \circ g^A \circ f^A = g^A, g^A \circ f^A \circ g^A = f^A\})$. $T^{(1)}(A)$ consists of the four elements id_A, f^A, g^A, and f^Ag^A. The Cayley-table of $T^{(1)}(\mathcal{A})$ has the form

$\circ$	id_A	f^A	g^A	f^Ag^A
id_A	id_A	f^A	g^A	f^Ag^A
f^A	f^A	id_A	f^Ag^A	g^A
g^A	g^A	f^Ag^A	id_A	f^A
f^Ag^A	f^Ag^A	g^A	f^A	id_A

$T^{(1)}(\mathcal{A})$ has exactly 4 proper submonoids, which have the following sets as universes:

$$\{id_A\}, \{id_A, f^A\}, \{id_A, g^A\}, \{id_A, f^Ag^A\}.$$

Since these submonoids are isomorphic to trivial monoids or two-element groups, by commutativity the identities $x^2 \approx x, xyx \approx y, yxy \approx x$ are satisfied. Let C be an arbitrary submonoid of $T^{(1)}(\mathcal{A})$. If $\mathcal{B}$ is an algebra of type $(1, 1)$ such that $T^{(1)}(\mathcal{B}) \cong C$, then $T^{(1)}(\mathcal{B})$ satisfies the identities $x^2 \approx x, xyx \approx y, yxy \approx x$, and this means $\mathcal{B} \in V$. Therefore, V is $s-$closed and using Theorem 4.9 and Theorem 4.10 V is solid.

6 Extension to n-Clones and Type (n,n, ...) In this section we show that some results of the previous sections also hold for n-clones of algebras whose operations are all n-ary, for some $n \geq 1$. We will call a type of algebras n-ary if all its operations are n-ary, and an algebra of this type will be called an n-ary algebra. Throughout this section we assume that τ is such an n-ary type, for some fixed $n \geq 1$, and that the operation symbols of type τ are $(f_i)_{i \in I}$, indexed by some set I. The set $W_\tau(X_n)$ of all n-ary terms of type τ, on the alphabet $X_n = \{x_1, x_2, \ldots, x_n\}$, forms the universe of an algebra of type $(n+1, 0, 0 \ldots, 0)$, which has an $n+1$-ary operation $S_n^{n, W_\tau(X_n)}$ called superposition and n nullary operations or special elements $e_1, \ldots, e_n$, and which satisfies the three clone axioms (see [Den-W;00]): (C1) $S_n^n(F_j, S_n^n(F_{i_1}, F_2, \ldots, F_{n+1}), \ldots, S_n^n(F_{i_n}, F_2, \ldots, F_{n+1}))$
$= S_n^n(S_n^n(F_j, F_{i_1}, \ldots, F_{i_n}), F_2, \ldots, F_{n+1})$,
(C2) $S_n^n(e_i, F_1, \ldots, F_n) = F_i$, and
(C3) $S_n^n(F_i, e_1, \ldots, e_n) = F_i$
for $1 \leq i \leq n$. The terms occurring in these axioms are terms of an language of type $\tau = (n+1, 0, \ldots, 0)$ built up by operation symbols $S_n^n, e_1, \ldots, e_n$ and an alphabet $\{F_i \mid i \in I\}$ with a bijection π between $\{F_i \mid i \in I\}$ and $\{f_i \mid i \in I\}$. Then an *n-clone* is defined as any algebra of type $\tau = (n+1, 0, \ldots, 0)$ satisfying (C1), (C2), (C3). The special n-clone whose universe is the set $W_\tau(X_n)$ of all n-ary terms is called the n-clone of type τ, and is denoted by $Clone_n(\tau)$. (For convenience, the $n+1$-ary operation of this concrete n-clone will also be denoted by S_n^n.) Notice that in the case $n = 1$ the clone axioms just say that the superposition operation S_1^1 is associative and that e_1 is a two-sided identity for S_1^1; that is, that the 1-clone is a monoid. Then we can define a map $\varphi : W_\tau(X_n) \to \langle \{F_i \mid i \in I\} \cup \{e_1, \ldots, e_n\} \rangle$, inductively by $\varphi(x_j) = e_j$, for $1 \leq j \leq n$,
$\varphi(f_i(t_1, \ldots, t_n)) = S_n^n(F_i, \varphi(t_1), \ldots, \varphi(t_n))$. Using this mapping φ, all the results of Section 2 can be generalized to the n-clone case. Most of the proofs are very similar; we give here only the proof that φ is a homomorphism, the analogue of Proposition 2.2, in order to show the flavour of the proofs.

Proposition 6.1 *The map φ is a homomorphism from $Clone_n(\tau)$ to the free n-clone $\mathcal{F}_n$ generated by $\{F_i \mid i \in I\} \cup \{e_1, \ldots, e_n\}$.*

Proof. We prove by induction on the complexity of term t that for any n-ary terms $t, s_1, \ldots, s_n$ we have $\varphi(S_n^n(t, s_1, \ldots, s_n)) = S_n^n(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$. First, if t is a variable x_j, for $1 \leq j \leq n$, then $\varphi(S_n^n(t, s_1, \ldots, s_n))$
$= \varphi(S_n^n(x_j, s_1, \ldots, s_n))$
$= \varphi(s_j)$, by the clone axioms,
$= S_n^n(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$. Inductively, let $t = f_i(t_1, \ldots, t_n)$ for some $i \in I$ and some n-ary terms $t_1, \ldots, t_n$. Then
$\varphi(S_n^n(t, s_1, \ldots, s_n))$
$= \varphi(f_i(t_1(s_1, \ldots, s_n), \ldots, t_n(s_1, \ldots, s_n)))$, by definition of S_n^n
$= S_n^n(F_i, \varphi(S_n^n(t_1, s_1, \ldots, s_n)), \ldots, \varphi(S_n^n(t_n, s_1, \ldots, s_n)))$, by definition of φ
$= S_n^n(S_n^n(F_i, \varphi(t_1), \ldots, \varphi(t_n)), \varphi(s_1), \ldots, \varphi(s_n))$, by the clone axioms
$= S_n^n(\varphi(f_i(t_1, \ldots, t_n)), \varphi(s_1), \ldots, \varphi(s_n))$
$= S_n^n(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$. ■

Since φ is easily shown to be a bijection (the analogue of Proposition 2.3), we have an isomorphism between $Clone_n(\tau)$ and the free n-clone $\mathcal{F}_n$ on our new language. Next we define $Subst_n$ to be the set of all substitutions η from the generating set $\{F_i \mid i \in I\}$ to the free n-clone $\mathcal{F}_n$. We define a binary composition operation $\odot$ on $Subst_n$ by $(\eta_1 \odot \eta_2)(F_i)$ $= \overline{\eta_1}(\eta_2(F_i))$, where $\overline{\eta_1}$ is the obvious extension of η_1. Then $Subst_n$ forms a monoid under

this operation, with the identity substitution as identity element. Then we can prove that the monoid $Hyp(\tau)$ of all hypersubstitutions of type τ is isomorphic to the monoid $Subst_n$, using the isomorphism $\psi : \sigma \mapsto \varphi \circ \sigma \circ \pi^{-1}$, as in the proof of Theorem 2.8. Now let $\mathcal{A}$ be any algebra of our n-ary type τ. It is well known that every n-ary term t of type τ induces an n-ary term operation $t^{\mathcal{A}}$ on $\mathcal{A}$, by $t^{\mathcal{A}}$ = the j-th projection operation e_j on $\mathcal{A}$, for $t = x_j$, for $1 \leq j \leq n$,
$t^{\mathcal{A}} = f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$, for $t = f_i(t_1, \ldots, t_n)$. The collection $\{t^{\mathcal{A}} \mid t \in W_\tau(X_n)\}$ forms the universe of an n-clone $T^{(n)}(\mathcal{A})$, called the term n-clone of $\mathcal{A}$. Then we can prove inductively, as in Lemma 3.1, that for any n-ary term t, we have $t^{\mathcal{A}} = \overline{h}(\varphi(t))$. Now we want to apply this result to identities of the appropriate form, to get an analogue of Theorem 3.3. For unary type, the property we needed was that only the variable x_1 was used on either side of the identity, which is equivalent to regularity for unary. For $n \geq 2$, we need that the identity is n-ary, meaning that it uses only the variables $x_1, \ldots, x_n$ so that φ can be applied; note that this is not equivalent to regular for $n \geq 2$. We will call a variety V n-ary if $V = Mod\Sigma$ for some set Σ of n-ary identities. Note that the property on n-ary identities is not an equational theory, since a variety $V = Mod\Sigma$ for a set Σ of n-ary identities still satisfies identities which use more than just the variables $x_1, \ldots, x_n$.

Theorem 6.2 *Let V be an n-ary algebra of type τ. Then an n-ary identity $s \approx t$ is a hyperidentity of $\mathcal{A}$ iff $\varphi(s) \approx \varphi(t)$ is an identity of the term n-clone $T^{(n)}(\mathcal{A})$.*

In the unary case most of the results from Section 2 had two versions, one for the transition monoid which is the unary clone, and one for the corresponding transition semigroup. For $n \geq 2$, it is also possible to consider this additional structure. We define $S^{(n)}(\mathcal{A})$ to be the set of term operations on $\mathcal{A}$ induced by non-variable terms from $W_\tau(X_n)$. This set is closed under the $n+1$-ary operation S_n^n, and we shall call the algebra $(S^{(n)}(\mathcal{A}); S_n^n)$ the n-transition algebra or the pre-n-clone of $\mathcal{A}$. It is straightforward then to define the restriction φ_S of the isomorphism φ to the set $S^{(n)}(\mathcal{A})$, and to prove analogous results in this case. We conclude with an example. We take $n = 2$, and type $\tau = (2)$ with a single binary operation f. We let Σ be the set consisting of the following identities of type $(2) : f(x, x) \approx x,\ f(x, f(y, x)) \approx f(f(x, y), x) \approx x,\ f(f(x, y), y) \approx f(x, f(y, y)),\ f(x, f(x, y)) \approx f(f(x, x), y)$. Now let $V = Mod\ \Sigma$, the variety with the set Σ as a basis. Notice that the identities in Σ are all 2-ary, and hence V is a 2-ary variety; and that V is not a variety of semigroups, since associativity is not satisfied here, and indeed is not a 2-ary identity. It is easy to calculate that for any algebra $\mathcal{A} \in V$, we have $T^{(2)}(\mathcal{A}) = \{x_1^{\mathcal{A}}, x_2^{\mathcal{A}}, f^{\mathcal{A}}(x_1, x_2), f^{\mathcal{A}}(x_2, x_1)\}$. This 2-clone has only two 2-subclones, the trivial (projection) clone $\{x_1^{\mathcal{A}}, x_2^{\mathcal{A}}\}$ and the 2-clone itself. It is not difficult to see that any algebra $\mathcal{B}$ of type (2) with a 2-clone isomorphic to a subclone of $T^{(2)}(\mathcal{A})$ belongs to the variety V. Therefore V is s-closed. It is also straightforward to check that V is a solid variety of type (2); any hypersubstitution of this type is $\sim_V$ equivalent to one of σ_{x_1}, σ_{x_2}, $\sigma_{f(x_1,x_2)}$ or $\sigma_{f(x_2,x_1)}$, and application of any of these to the basis identities in Σ yields an identity of V.

References

[Bur-S;81] Burris, S, Sankappanavar, H. P., *A Course in Universal Algebra*, Springer-Verlag New York, 1981.

[Cli-P;67] Clifford, A. H., Preston, G. B., *The Algebraic Theory of Semigroups Vol II*, American Mathematical Society, 1967.

[Den-G;96] Denecke, K., Głazek, K., *M-solid varieties and Q-free clones*, Math. Slovaca 46 (1996), No. 5, 515 - 524.

[Den-W;00] Denecke, K., Wismath, S.L., *Hyperidentities and Clones*, Gordon and Breach Science Publishers 2000.

[Eil;7] Eilenberg, S., *Automata, Languages and Machines*, Academic Press, Vol A 1974, Vol B 1976.

[Kle;50] Kleene, S., *Introduction to Metamathematics*, Van Nostrand Co.,m Inc., 1950.

[Kro-R;65] Krohn, K., Rhodes, J., *Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines*, Trans. Amer. Math. Soc. 116 (1965), 450 - 464.

[Mar;66] Marczewski, E., *Independence in Abstract Algebras, Results and Problems*, Colloquium Mathematicum XIV (1966), 169 - 188.

[Pet-C-B;02] Petković, T., Ćirić, M., Bogdanović, St., *Unary Algebras, Semigroups and Congruences on Free Semigroups*, preprint 2002.

[Sch;65] Schützenberger, M. P., *On finite monoids having only trivial subgroups*, Inf. Control 8 (1965), 190 - 194.

[Tay;81] Taylor, W., *Hyperidentities and Hypervarieties*, Aequationes Mathematicae 23 (1981), 111 - 127.

Klaus Denecke,
University of Potsdam,
Institute of Mathematics,
Am Neuen Palais,
14415 Potsdam, Germany
kdenecke@rz.uni-potsdam.de

Prakit Jampachon,
KhonKaen University,
Department of Mathematics
KhonKaen,
40002 Thailand
prajam@.kku.ac.th

Shelly Wismath
University of Lethbridge
Dept. of Math. and C.S.
Lethbridge, Ab.,
Canada T1K-3M4
wismaths@cs.uleth.ca