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CONNECTED STABLE RANK OF THE ALGEBRAS OF CONTINUOUS FUNCTIONS FROM SPACES TO C^*-ALGEBRAS

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Abstract. We estimate the connected stable rank of the C^*-algebras of continuous functions from spaces to C^*-algebras in terms of the spaces and C^*-algebras in the images. As a corollary, we obtain a result in the K-theory of C^*-algebras.

0. Introduction

The stable rank and connected stable rank for C^*-algebras were introduced by Rieffel [Rf1] to study the dimension theory of C^*-algebras and its close connection with their K-theory. These stable ranks also played a fundamental role in the study of homotopy groups of the unitary groups of noncommutative tori [Rf2]. In particular, it has turned out that for a C^*-algebra $\mathfrak{A}$, the estimate of the connected stable rank for the C^*-algebras $C(\mathbb{T}^n, \mathfrak{A})$ of continuous $\mathfrak{A}$-valued functions on the tori $\mathbb{T}^n$ is useful for estimating the homotopy groups of the general linear groups $GL_n(\mathfrak{A})$ over $\mathfrak{A}$ ([Rf2, Theorem 3.3]). Inspired by this fact, our first motivation of this paper is to estimate the connected stable rank of the C^*-algebras $C(\mathbb{T}^n, \mathfrak{A})$ in terms of $\mathbb{T}^n$ and $\mathfrak{A}$. This estimate has not been obtained previously, while the estimate of the same type when spaces are compact and contractible was obtained by Nistor [Ns] and Elhage Hassan [Eh], and the estimate of the stable rank of that type was obtained by [Rf1, Corollary 7.2] and by Nagisa, Osaka and Phillips [NOP] and improved by the author [Sd]. Also, the estimate of the real rank of that type was obtained as one of the main results of [NOP].

Our strategy is as follows. As the main result, we estimate the connected stable rank of the C^*-algebras $C_0(\mathbb{R}, \mathfrak{A})$ of continuous $\mathfrak{A}$-valued functions vanishing at infinity on the real line $\mathbb{R}$ in terms of $\mathbb{R}$ and $\mathfrak{A}$. Moreover, we deduce several consequences from this result, one of which is the estimate mentioned above. In particular, we obtain an improved result in the homotopy theory of C^*-algebras and a somewhat interesting consequence in the (non-stable) K-theory of C^*-algebras.

Notation and facts. Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$, and $\mathrm{gsr}(\mathfrak{A})$ the stable rank, connected stable rank and general stable rank of $\mathfrak{A}$ respectively ([Rf1]). Recall that for $\mathfrak{A}$ unital, $\mathrm{sr}(\mathfrak{A})$ is the smallest $n \in \mathbb{N}$ such that $L_n(\mathfrak{A}) = \{(a_j) \in \mathfrak{A}^n \mid \sum_{j=1}^n a_j^* a_j$ is invertible in $\mathfrak{A}\}$ is dense in $\mathfrak{A}^n$, $\mathrm{csr}(\mathfrak{A})$ is the smallest $n \in \mathbb{N}$ such that $L_m(\mathfrak{A})$ is path-connected for any $m \geq n$, and for $\mathfrak{A}$ non-unital, the respective ranks are defined by those of the unitization $\mathfrak{A}^+$.

$(F1)$: $\mathrm{gsr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$ [Rf1, Corollary 4.10 and p.328]. Let $C_0(X)$ be the C^*-algebra of all continuous functions on a locally compact Hausdorff space X vanishing at infinity.

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$(F2)$: $\mathrm{sr}(C(X)) = [\dim X/2] + 1$, and $\mathrm{csr}(C(X)) \leq [(\dim X + 1)/2] + 1$, where $[x]$ means the maximum integer $\leq x$, X is compact, and $C_0(X) = C(X)$ ([Rf1, Proposition 1.7], [Ns]). For a C^*-algebra $\mathfrak{A}$, we denote by $C_0(X, \mathfrak{A})$ the C^*-algebra of continuous $\mathfrak{A}$-valued functions on X vanishing at infinity. It is known that $C_0(X, \mathfrak{A})$ is isomorphic to the C^*-tensor product $C_0(X) \otimes \mathfrak{A}$ (cf.[Mp, Theorem 6.4.17]). If X is compact, set $C_0(X, \mathfrak{A}) = C(X, \mathfrak{A})$.

1. The main results

Theorem 1. *Let $\mathfrak{A}$ be a C^*-algebra. Then* $\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A})) \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\}$.

Proof. We first suppose that $\mathfrak{A}$ is unital. Since $C_0(\mathbb{R}, \mathfrak{A})$ is nonunital, we consider its unitization $C_0(\mathbb{R}, \mathfrak{A})^+$. As usual, the elements of the unitization are taken to be ordered pairs (a, λ) with $a \in C_0(\mathbb{R}, \mathfrak{A})$ and $\lambda \in \mathbb{C}$. Take any $x = ((a_j, \lambda_j))_{j=1}^n \in L_n(C_0(\mathbb{R}, \mathfrak{A})^+)$ with $\sum_{j=1}^n (b_j, \mu_j)(a_j, \lambda_j) = (0, 1) \in C_0(\mathbb{R}, \mathfrak{A})^+$ for some $((b_j, \mu_j))_{j=1}^n \in (C_0(\mathbb{R}, \mathfrak{A})^+)^n$. Then $x = ((a_j, \lambda_j))_{j=1}^n$ is regarded as a continuous function Φ_x from the one-point compactification $\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ to $L_n(\mathfrak{A}^+)$ by $\Phi_x(\infty) = ((0, \lambda_j))_{j=1}^n$, and $\Phi_x(t) = ((a_j(t), \lambda_j))_{j=1}^n \in L_n(\mathfrak{A}^+)$ for $t \in \mathbb{R}$ since $\sum_{j=1}^n (b_j(t), \mu_j)(a_j(t), \lambda_j) = (0, 1) \in \mathfrak{A}^+$. Also, we have the following inclusion: $L_n(C_0(\mathbb{R}, \mathfrak{A})^+) \subset C(\mathbb{R}^+, (\mathfrak{A}^+)^n)$.

Now suppose $\max\{\mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\} \leq N$ (and in addition $N \geq 2$ later). Note that $\mathrm{csr}(\mathfrak{A}^+) \leq \max\{\mathrm{csr}(\mathfrak{A}), \mathrm{csr}(\mathbb{C})\} = \mathrm{csr}(\mathfrak{A})$ by [Sh, Theorem 3.9] and $\mathrm{sr}(\mathfrak{A}) = \mathrm{sr}(\mathfrak{A}^+)$ for $\mathfrak{A}$ unital. Thus, $L_n(\mathfrak{A}^+)$ is path-connected and dense in $(\mathfrak{A}^+)^n$ for any $n \geq N$. Hence any element of $L_n(\mathfrak{A}^+)$ is connected to $((1, 1), \cdots, (1, 1)) \in L_n(\mathfrak{A}^+)$. In the following we use the identification $C(\mathbb{R}^+, (\mathfrak{A}^+)^n) \cong C(\mathbb{R}^+) \otimes (\mathfrak{A}^+)^n$ (cf.[Mp, Theorem 6.4.17], that is, any element of the left hand side is approximated by finite sums of simple tensors in the right hand side. Since $L_n(\mathfrak{A}^+)$ is open in $(\mathfrak{A}^+)^n$, it follows from the above inclusion and the standard approximation in the above identification that for any $t \in \mathbb{R}$, the restriction $x|_U$ of Φ_x (or x) to some open neighborhood U of t is approximated closely by a sum of two simple tensors, $f_U \otimes (c_k, 0)_{k=1}^n + 1_U \otimes (0, \nu_k)_{k=1}^n$ for $1_U \in C(U)$ and $f_U \in C_0(U)$ with $1_U = 1$ and $0 \leq f_U \leq 1$ with $f_U(t) = 1$, and $(f_U(s)c_k, \nu_k)_{k=1}^n \in L_n(\mathfrak{A}^+)$ for $s \in U$. Thus, $x|_U$ is connected to $f_U \otimes (c_k, 0)_{k=1}^n + 1_U \otimes (0, \nu_k)_{k=1}^n$. Since $(c_k, \nu_k)_{k=1}^n$ is connected to $((1, 1), \cdots, (1, 1)) \in L_n(\mathfrak{A}^+)$, the restriction $x|_U$ is connected to $f_U \otimes (1, 0)_{k=1}^n + 1_U \otimes (0, 1)_{k=1}^n$, which is regarded as the restriction of an element of $L_n(C(\mathbb{R}^+))$ to U. We may assume that U is an open interval. Similarly, for $s \notin U$ (near U) we can take an open interval V such that $V \cap U$ is nonempty, and $x|_{V \cup U}$ is approximated by $f_V \otimes (d_k, 0)_{k=1}^n + f_U \otimes (c_k, 0)_{k=1}^n + 1_{V \cup U} \otimes (0, \nu_k)_{k=1}^n$ for $f_V \in C_0(V)$ with $0 \leq f_V \leq 1$ and $f_V(s) = 1$, and $(f_V(t)d_k + f_U(t)c_k, \nu_k)_{k=1}^n \in L_n(\mathfrak{A}^+)$ for $t \in V \cup U$, where in this process we may change f_V, c_k and ν_k as chosen above. Then the sum of three simple tensors is connected to $f_V \otimes (1, 0)_{k=1}^n + f_U \otimes (1, 0)_{k=1}^n + 1_{V \cap U} \otimes (0, 1)_{k=1}^n$, and so is $x|_{V \cap U}$. Note also that the space $C(V \cup U, L_n(\mathfrak{A}^+))$ is contractible since $C(V \cap U)$ and $L_n(\mathfrak{A}^+)$ are contractible (cf. [Mp, Theorem 7.5.3]). By using this argument inductively for a suitable locally finite open covering $\{U_j\}$ (possibly finite) of $\mathbb{R}$ associated with Φ_x, and using the density of $L_n(\mathfrak{A}^+)$ for replacing the restrictions with such sums of simple tensors as above, we obtain that Φ_x is connected to an element $\Psi_x = f_x \otimes (1, 0)_{k=1}^n + 1_{\mathbb{R}^+} \otimes (0, 1)_{k=1}^n$ of $L_n(C(\mathbb{R}^+))$, where f_x is regarded as an element of $C_0(\mathbb{R})$ and $1_{\mathbb{R}^+}$ is the identity of $C(\mathbb{R}^+)$. Note that it is possible to make f_x by putting together f_{U_j} corresponding to the restrictions U_j by adjusting values of f_{U_j} at points near the boundary of U_j (if necessary by using partition of unity of $\mathbb{R}$ associated with the covering $\{U_j\}$ or its (inductive) refinements), where it seems that in the final step we are using of $\mathbb{R}$ contractible, and note that the space of elements of $C_0(\mathbb{R}, \mathfrak{A})$ with compact supports are dense in $C_0(\mathbb{R}, \mathfrak{A})$, and since $\dim \mathbb{R} = 1$ (the covering dimension one) there exists an open covering of $\mathbb{R}$ such that intersections of more than two members of the covering must be empty (cf. [Ng]). Since $\mathrm{csr}(C_0(\mathbb{R})) = 2$

[Sh, p.381], $L_n(C_0(\mathbb{R})^+) = L_n(C(\mathbb{R}^+))$ is connected for any $n \geq 2$. Thus, Ψ_x is connected to $(1, \cdots, 1) \in L_n(C_0(\mathbb{R})^+)$, and $L_n(C_0(\mathbb{R})^+) \subset L_n((C_0(\mathbb{R}) \otimes \mathfrak{A})^+)$ since $\mathfrak{A}$ is unital.

Next suppose that $\mathfrak{A}$ is nonunital. Then we can follow almost the same argument as in the unital case. Note that the element $((1,1), \cdots, (1,1)) \in L_n(\mathfrak{A}^+)$ taken above should be replaced with a certain fixed $((d_1, 1), \cdots, (d_n, 1))$ with each $d_k \in \mathfrak{A}$ nonzero and small enough so that $((d_k, 1))_{k=1}^n \in L_n(\mathfrak{A}^+)$ since it is near $((0,1), \cdots, (0,1)) \in L_n(\mathfrak{A}^+)$, and the continuous functions on $\mathbb{R}$ of the form $f_x \otimes ((d_k, 0))_{k=1}^n + 1_{\mathbb{R}^+} \otimes ((0,1))_{k=1}^n$ can be identified with elements of $L_n(C(\mathbb{R}^+)) = L_n(C_0(\mathbb{R})^+)$. Also, under this identification the space $L_n(C_0(\mathbb{R})^+)$ is regarded as a subspace of $L_n((C_0(\mathbb{R}) \otimes \mathfrak{A})^+)$. $\square$

Corollary 2. *Let $\mathfrak{A}$ be a C^*-algebra. Then* $\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A})) \leq \mathrm{sr}(\mathfrak{A}) + 1$.

Proof. Use the fact that $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$ (F1). $\square$

Remark. The estimate: $\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A})) \leq \mathrm{sr}(C_0(\mathbb{R}, \mathfrak{A})) + 1 \leq \mathrm{sr}(\mathfrak{A}) + 2$ can be obtained by (F1) and [NOP, Theorem 1.13]. When $\mathfrak{A} = \mathbb{C}$ in Theorem 1 and Corollary 2 we obtain the equality of each estimate. In this sense those estimates are the best possible. On the other hand, if the K_1-group of $C_0(\mathbb{R}, \mathfrak{A})$ is nontrivial, then $\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A})) \geq 2$ by [Eh, Corollary 1.6]. Note that the K_1-group $K_1(C_0(\mathbb{R}, \mathfrak{A})) \cong K_0(\mathfrak{A})$ the K_0-group of $\mathfrak{A}$.

Theorem 3. *Let $\mathfrak{A}$ be a C^*-algebra. Then for any $k \geq 1$,*

$$\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A})) \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\}.$$

Furthermore, it follows that $\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A})) \leq \max\{\mathrm{sr}(\mathfrak{A}) + 1, \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\}$. *In particular,* $\mathrm{gsr}(C(\mathbb{T}, \mathfrak{A})) \leq \mathrm{csr}(C(\mathbb{T}, \mathfrak{A})) \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\} \leq \mathrm{sr}(\mathfrak{A}) + 1$. *Moreover, when $\mathfrak{A}$ is unital, if $p \geq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\}$ for all k, then for $n \geq p - 1$ and $k \geq 0$,*

$$\pi_k(GL_n(\mathfrak{A})) \cong \begin{cases} K_1(\mathfrak{A}) & \text{if } k \text{ even}, \\ K_0(\mathfrak{A}) & \text{if } k \text{ odd}, \end{cases}$$

where $\pi_k(GL_n(\mathfrak{A}))$ mean the homotopy groups of $GL_n(\mathfrak{A})$.

Proof. We consider the exact sequence: $0 \to C_0(\mathbb{R}, \mathfrak{A}) \to C(\mathbb{T}, \mathfrak{A}) \to \mathfrak{A} \to 0$ when $k = 1$. Using [Sh, Theorem 3.9] and Theorem 1, we obtain

$$\mathrm{csr}(C(\mathbb{T}, \mathfrak{A})) \leq \max\{\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A})), \mathrm{csr}(\mathfrak{A})\} \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\}.$$

Our first idea for calculating the rank $\mathrm{csr}(C(\mathbb{T}, \mathfrak{A}))$ is the reduction to the estimate of $\mathrm{csr}(C_0(\mathbb{R}, \mathfrak{A}))$. For $k \geq 2$, we use this estimate inductively as follows. Since $C(\mathbb{T}^k, \mathfrak{A}) \cong C(\mathbb{T}) \otimes C(\mathbb{T}^{k-1}, \mathfrak{A})$, we have

$$\begin{aligned} \mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A})) &\leq \max\{2, \mathrm{csr}(C(\mathbb{T}^{k-1}, \mathfrak{A})), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\} \\ &\leq \max\{2, \mathrm{csr}(C(\mathbb{T}^{k-2}, \mathfrak{A})), \mathrm{sr}(C(\mathbb{T}^{k-2}, \mathfrak{A})), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\} \\ &\leq \cdots\cdots\cdots\cdots\cdots\cdots \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\}. \end{aligned}$$

Note that $\mathrm{sr}(C(\mathbb{T}^l, \mathfrak{A})) \leq \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))$ for $0 \leq l \leq k-2$ by [Rf1, Theorem 4.3] since $C(\mathbb{T}^l, \mathfrak{A})$ is a quotient of $C(\mathbb{T}^{k-1}, \mathfrak{A})$. By [Rf2, Theorem 3.3], when $\mathfrak{A}$ is unital, if $p \geq \mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A}))$ for all k, then for $n \geq p - 1$ and $k \geq 0$, we obtain

$$\pi_k(GL_n(\mathfrak{A})) \cong \begin{cases} K_1(\mathfrak{A}) & \text{if } k \text{ even}, \\ K_0(\mathfrak{A}) & \text{if } k \text{ odd}. \end{cases}$$

Combining this result with the above inequality for $\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A}))$, we obtain the desired conclusion. □

Remark 3.1. The first and second estimates of the connected stable rank of Theorem 3 suggest that it is impossible to estimate the rank $\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A}))$ in terms of $\mathfrak{A}$ only in general. Also, $\max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A}))\}$ is not bounded as k varies in general (cf. [Rf2, Theorem 3.3] the result for calculating homotopy groups). However, for example, if we take $\mathfrak{A}$ to be an inductive limit of homogeneous C^*-algebras of slow dimension growth, then $\mathrm{sr}(C(\mathbb{T}^{k-1}, \mathfrak{A})) \leq 2$ and $\mathrm{csr}(\mathfrak{A}) \leq 2$ by using [Rf1, Theorems 5.1 and 6.1], [Rf2, Theorem 4.7] and (F2) and by induction, but $\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A}))$ and $\mathrm{gsr}(C(\mathbb{T}, \mathfrak{A}))$ can be strictly smaller than $\mathrm{csr}(\mathfrak{A})$. Also, it is harder to calculate the ranks csr, gsr than the stable rank in general.

Remark 3.2. Note that $C(\mathbb{T}^k, \mathfrak{A})$ is regarded as a crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}^k$ with α the trivial action since $\mathfrak{A} \rtimes_\alpha \mathbb{Z}^k \cong \mathfrak{A} \otimes C^*(\mathbb{Z}^k)$ and $C^*(\mathbb{Z}^k) \cong C(\mathbb{T}^k)$. When $\mathfrak{A}$ is unital, we obtain $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}^k) \leq \mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}^{k-1}) + 1 \leq \mathrm{sr}(\mathfrak{A}) + k$ by using [Rf1, Corollary 8.6 and Theorem 7.1]. This estimate is valid for α any nontrivial action. Also note that for any $k \geq 1$, $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}^k) \geq 2$ for $\mathfrak{A}$ a unital C^*-algebra by [Eh, Theorem 2.2]. In particular, for any $k \geq 1$, $\mathrm{csr}(C(\mathbb{T}^k, \mathfrak{A})) \geq 2$ for $\mathfrak{A}$ unital.

Remark 3.3. For X any locally compact Hausdorff space and $\mathfrak{A}$ a C^*-algebra, the estimate: $\mathrm{csr}(C_0(X, \mathfrak{A})) \leq \max\{\mathrm{csr}(C_0(X)), \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\}$ is wrong in general. For example, let $C_0(X) = C_0(\mathbb{R}^m) = \mathfrak{A}$ for $m \geq 2$. Note that $\mathrm{csr}(C_0(\mathbb{R}^2)) = 1$, and $\mathrm{csr}(C_0(\mathbb{R}^n)) = [(n+1)/2] + 1$ for $n \geq 3$ (cf. [Sh, p.381]).

Corollary 4. *We have* $\mathrm{csr}(C(S^n, \mathfrak{A})) \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C_0(\mathbb{R}^{n-1}, \mathfrak{A}))\}$ *for a* C^**-algebra* $\mathfrak{A}$ *and* $n \geq 1$*, where* S^n *means the* n*-dimensional sphere.*

Proof. We have the exact sequence: $0 \to C_0(\mathbb{R}^n, \mathfrak{A}) \to C(S^n, \mathfrak{A}) \to \mathfrak{A} \to 0$. Using [Sh, Theorem 3.9] and Theorem 1 repeatedly, we obtain

$$\begin{aligned}\mathrm{csr}(C(S^n, \mathfrak{A})) &\leq \max\{\mathrm{csr}(C_0(\mathbb{R}^n, \mathfrak{A})), \mathrm{csr}(\mathfrak{A})\} \\ &\leq \max\{2, \mathrm{csr}(C_0(\mathbb{R}^{n-1}, \mathfrak{A})), \mathrm{sr}(C_0(\mathbb{R}^{n-1}, \mathfrak{A})), \mathrm{csr}(\mathfrak{A})\} \\ &\leq \cdots\cdots\cdots\cdots\cdots\cdots \leq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(C_0(\mathbb{R}^{n-1}, \mathfrak{A}))\}. \qquad \square\end{aligned}$$

Remark 4.1. By the Künneth formula (cf. [Wo, 9.3.3]), we obtain

$$\begin{aligned}K_1(C(S^n, \mathfrak{A})) &\cong K_0(C(S^n)) \otimes K_1(\mathfrak{A}) + K_1(C(S^n)) \otimes K_0(\mathfrak{A}) \\ &\cong K_0(C_0(\mathbb{R}^n)^+) \otimes K_1(\mathfrak{A}) + K_1(C_0(\mathbb{R}^n)^+) \otimes K_0(\mathfrak{A}) \\ &\cong \begin{cases} \mathbb{Z}^2 \otimes K_1(\mathfrak{A}) & n \text{ even}, \\ \mathbb{Z} \otimes K_1(\mathfrak{A}) + \mathbb{Z} \otimes K_0(\mathfrak{A}) & n \text{ odd} \end{cases}\end{aligned}$$

if $\mathfrak{A}$ is in the class X ([Wo, 11.2.3]) or the bootstrap category ([Bl, 22.3.4]) and its K-groups are torsion free.

Remark 4.2. Note that $\mathrm{csr}(C(X, \mathfrak{A})) = \mathrm{csr}(\mathfrak{A})$ for X a contractible compact space and $\mathfrak{A}$ a C^*-algebra by [Eh, Corollary 2.12] (cf. [Ns, Corollary 2.9]). For example, we may let $X = [0, 1]^n$. Also, $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(C(K, \mathfrak{A}))$ for K a compact space [Eh, 2.13] since the quotient from $C(K, \mathfrak{A})$ to $\mathfrak{A}$ by the point evaluation splits.

As an important application to the K-theory of C^*-algebras,

Theorem 5. *Let $\mathfrak{A}$ be a C^*-algebra. Then for any $n \geq \max\{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\}$, the map from the quotient $GL_{n-1}(\mathfrak{A})/GL_{n-1}(\mathfrak{A})_0$ to $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ is an isomorphism so that $GL_{n-1}(\mathfrak{A})/GL_{n-1}(\mathfrak{A})_0$ is isomorphic to the K_1-group $K_1(\mathfrak{A})$, where $GL_{n-1}(\mathfrak{A})_0$ is the connected component of the identity in $GL_{n-1}(\mathfrak{A})$.*

Proof. If $\mathfrak{A}$ is nonunital, we consider its unitization $\mathfrak{A}^+$. Let $n \geq \{2, \mathrm{csr}(\mathfrak{A}), \mathrm{sr}(\mathfrak{A})\}$. By the third estimate in Theorem 3, $n \geq \max\{\mathrm{csr}(\mathfrak{A}), \mathrm{gsr}(C(\mathbb{T}, \mathfrak{A}))\}$. The conclusion of the theorem now follows by [Rf2, Theorem 2.9]. □

Remark. This partially improves Rieffel's result [Rf2, Theorem 2.10].

Corollary 6. *Let $\mathfrak{A}$ be a C^*-algebra with $\mathrm{sr}(\mathfrak{A}) \leq 2$ and $\mathrm{csr}(\mathfrak{A}) \leq 2$. Then the K_1-group $K_1(\mathfrak{A})$ of $\mathfrak{A}$ is isomorphic to $GL_1(\mathfrak{A})/GL_1(\mathfrak{A})_0 = \mathfrak{A}^{-1}/\mathfrak{A}_0^{-1}$, where $\mathfrak{A}^{-1}$ is the group of all invertible elements of $\mathfrak{A}$.*

Remark 6.1. Let $\mathcal{T}$ be the Toeplitz algebra. It is well known that $\mathcal{T}$ is an extension of $C(\mathbb{T})$ by the C^*-algebra of compact operators (cf. [Mp, Section 3.5]). By [Rf1, Examples 4.13], [Sh, Theorem 3.9 and p.381] and [Eh, Proposition 1.15], we have $\mathrm{sr}(\mathcal{T}) = 2$ and $\mathrm{csr}(\mathcal{T}) = 2$. Note that the K_1-group of $\mathcal{T}$ is trivial (cf. [Wo, Exercises 9.L]), but $\mathcal{T}$ is not (stably) finite, from which we also have $\mathrm{csr}(\mathcal{T}) \neq 1$.

Remark 6.2. It is shown in [Bl, 8.1] (cf. [Wo]) that $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1} \cong K_1(\mathfrak{A})$ if $\mathfrak{A} = C(S^1)$, and $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1} \not\cong K_1(\mathfrak{A})$ if $\mathfrak{A} = C(S^3)$. Note that $\mathrm{sr}(C(S^1)) = 1$ and $\mathrm{csr}(C(S^1)) = 2$ while $\mathrm{sr}(C(S^3)) = 2$ and $\mathrm{csr}(C(S^3)) = 3$. For X a contractible compact space, we have $C(X)^{-1}/C(X)_0^{-1} = 0 = K_1(C(X))$ (cf. [RLL, Chapter 8]). Moreover, if $\mathfrak{A}$ is an AF-algebra, then $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1} \cong K_1(\mathfrak{A}) = 0$ since $\mathrm{sr}(\mathfrak{A}) = 1$ by [Rf2, Theorem 2.10]. Also, for any C^*-algebra $\mathfrak{A}$, we have $(\mathfrak{A} \otimes \mathbb{K})^{-1}/(\mathfrak{A} \otimes \mathbb{K})_0^{-1} = K_1(\mathfrak{A}) \cong K_1(\mathfrak{A} \otimes \mathbb{K})$, and $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \min\{2, \mathrm{sr}(\mathfrak{A})\}$, $\mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq \min\{2, \mathrm{csr}(\mathfrak{A})\}$, where $\mathbb{K}$ is the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space ([Rf1, Theorems 3.6 and 6.4], [Sh, Theorem 3.10] and [Ns, Corollaries 2.5 and 2.12]). Moreover, note that $\mathrm{csr}(C_0(\mathbb{R}^{2n}) \otimes \mathbb{K}) = 1$ for $n \geq 0$ and $\mathrm{csr}(C_0(\mathbb{R}^{2n+1}) \otimes \mathbb{K}) = 2$ for $n \geq 0$, and $K_1(C_0(\mathbb{R}^{2n})) \cong K_1(\mathbb{C}) = 0$ and $K_1(C_0(\mathbb{R}^{2n+1})) \cong K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$ (cf. [Sh, p.386] and [Wo, Sections 7 and 9]).

Remark 6.3. If $\mathfrak{A}$ is a purely infinite C^*-algebra, then $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1} \cong K_1(\mathfrak{A})$ while it is known that $\mathrm{sr}(\mathfrak{A}) = \infty$ by [Rf1, Proposition 6.5], and if $\mathfrak{A}$ is unital, simple and purely infinite, then $\mathrm{csr}(\mathfrak{A}) = \infty$ when the unit of $K_0(\mathfrak{A})$ has torsion and $\mathrm{csr}(\mathfrak{A}) = 2$ when the unit of $K_0(\mathfrak{A})$ has no torsion, and if $\mathfrak{A}$ is nonunital, simple and purely infinite, then $\mathrm{csr}(\mathfrak{A}) = 2$ (see [X]).

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