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PSEUDO-MTL ALGEBRAS AND PSEUDO-R_0 ALGEBRAS

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Abstract. The relations between pseudo-MTL algebras and pseudo-R_0 algebras are discussed. The main results are as follows: pseudo-$IMTL$ algebras are equivalent to weak pseudo-R_0 algebras; pseudo-NM algebras are equivalent to pseudo-R_0 algebras.

1. Introduction The notion of MTL algebras was introduced by Esteva and Godo in [3] as a generalization of BL algebras [6]. In [1,2,4], Georgescu et.al proposed the notion of pseudo-BL algebras as a noncommutative extension of BL algebras. Afterwards, Georgescu and Popescu [5] proposed the notion of pseudo-MTL algebras (called weak pseudo-BL algebras) as a noncommutative extension of MTL algebras. In [7], we generalized Georgescu's ideas to R_0 algebras and proposed the concept of pseudo-R_0 algebras. In this paper, we discuss the relations between pseudo-MTL algebras and pseudo-R_0 algebras. We prove that pseudo-$IMTL$ algebras are equivalent to weak pseudo-R_0 algebras, and that pseudo-NM algebras are equivalent to pseudo-R_0 algebras.

Now let us recall the definition of MTL algebras (see [3]).

An MTL algebra is a structure $L = (L, \vee, \wedge, \odot, \to, 0, 1)$ such that

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(L, \odot, 1)$ is an abelian monoid, i.e. $\odot$ is commutative and associative and $x \odot 1 = 1 \odot x = x$,

(iii) the following conditions hold for all $x, y, z \in L$:

(1) $x \odot y \leq z$ if and only if $x \leq y \to z$ (residuation),

(2) $(x \to y) \vee (y \to x) = 1$ (prelinearity).

An MTL algebra L is called an $IMTL$ algebra, if the following condition holds:

(3) $(x \to 0) \to 0 = x$.

An $IMTL$ algebra L is called a NM algebra, if the following condition holds:

(4) $(x \odot y \to 0) \vee (x \wedge y \to x \odot y) = 1$.

2. Pseudo-MTL algebras **Definition 2.1.** (see [5]) A pseudo-MTL algebra is a structure $L = (L, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ of type (2,2,2,2,2,0,0), which satisfies the following axioms:

(C1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(C2) $(L, \odot, 1)$ is a monoid, i.e. $\odot$ is associative and $x \odot 1 = 1 \odot x = x$,

(C3) $x \odot y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \rightsquigarrow z$,

(C4) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

The following example shows that pseudo-MTL algebras exist.

Example 2.2. Let $L = \{0, a, b, c, 1\}$ and satisfy $0 \leq a \leq b \leq c \leq 1$. We define $x \wedge y = \min\{x, y\}, x \vee y = \max\{x, y\}$, and define $\odot, \to, \rightsquigarrow$ as follows:

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$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

It is easily checked that $(L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL algebra.

Lemma 2.3. (see [5]) Let L be a pseudo-MTL algebra. The following properties hold:

(1) $x \to x = x \rightsquigarrow x = 1$,
(2) $1 \to x = 1 \rightsquigarrow x = x$,
(3) $x \odot y \le x \wedge y, y \odot x \le x \wedge y$,
(4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(5) $x \odot (x \rightsquigarrow y) \le x \wedge y, (x \to y) \odot x \le x \wedge y$,
(6) If $x \le y$, then $x \odot z \le y \odot z, z \odot x \le z \odot y$,
(7) $x \le y$ if and only if $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$,
(8) $y \to z \le (x \to y) \to (x \to z), y \rightsquigarrow z \le (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$,
(9) $y \to z \le (z \to x) \rightsquigarrow (y \to x), y \rightsquigarrow z \le (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,
(10) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z), (x \odot y) \to z = x \to (y \to z)$,
(11) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z), (y \vee z) \odot x = (y \odot x) \vee (z \odot x)$,
(12) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z), (y \wedge z) \odot x = (y \odot x) \wedge (z \odot x)$.

Remark 2.4. The identity $x \wedge y = x \odot (x \rightsquigarrow y) = (x \to y) \odot x$ does not hold in pseudo-MTL algebras. In fact, in Example 2.2, take $x = b, y = a$, we have $x \wedge y = b \wedge a = a$, but $(x \to y) \odot x = (b \to a) \odot b = a \odot b = 0$.

Let L be a pseudo-MTL algebra. Define

$$\neg x = x \to 0, \sim x = x \rightsquigarrow 0.$$

Obviously, $\neg$ and $\sim$ are unary operations on L.

Definition 2.5. A pseudo-MTL algebra is called a pseudo-$IMTL$ algebra, if it satisfies (C5) $x = \neg \sim x = \sim \neg x$.

Definition 2.6. A pseudo-$IMTL$ algebra is called a pseudo-NM algebra, if it satisfies (C6) $(x \odot y \to 0) \vee (x \wedge y \to x \odot y) = 1, (x \odot y \rightsquigarrow 0) \vee (x \wedge y \rightsquigarrow x \odot y) = 1$.

Lemma 2.7. Let L be a pseudo-$IMTL$ algebra. The following properties hold:

(1) $x \rightsquigarrow y = \sim y \to \sim x, x \to y = \neg y \rightsquigarrow \neg x$,
(2) $x \le y$ if and only if $\neg y \le \neg x$ if and only if $\sim y \le \sim x$,
(3) $\sim (x \vee y) = \sim x \wedge \sim y, \neg(x \vee y) = \neg x \wedge \neg y$,
(4) $\sim (x \wedge y) = \sim x \vee \sim y, \neg(x \wedge y) = \neg x \vee \neg y$,
(5) $x \odot y = \neg(y \rightsquigarrow \sim x) = \sim (x \to \neg y)$,
(6) $x \rightsquigarrow y = \sim (\neg y \odot x), x \to y = \neg(x \odot \sim y)$,
(7) $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z), x \to (y \vee z) = (x \to y) \vee (x \to z)$.

Proof. (1) By Lemma 2.3 and (C5), we have $x \rightsquigarrow y \le (y \rightsquigarrow 0) \to (x \rightsquigarrow 0) = \sim y \to \sim x \le (\sim x \to 0) \rightsquigarrow (\sim y \to 0) = \neg \sim x \rightsquigarrow \neg \sim y = x \rightsquigarrow y$, and so $x \rightsquigarrow y = \sim y \to \sim x$. Similarly, $x \to y = \neg y \rightsquigarrow \neg x$.

(2) From Lemma 2.3 and (1), it follows that $x \le y$ if and only if $x \rightsquigarrow y = 1$ if and only if $\sim y \to \sim x = 1$ if and only if $\sim y \le \sim x$. Similarly, $x \le y$ if and only if $\neg y \le \neg x$.

(3) Since $x, y \le x \vee y$, we have $\sim (x \vee y) \le \sim x, \sim (x \vee y) \le \sim y$. Let $t \le \sim x, t \le \sim y$, by (2) and (C5), we have $x = \neg \sim x \le \neg t, y = \neg \sim y \le \neg t$, hence $x \vee y \le \neg t$. Using (2) and (C5) again, we have $t = \sim \neg t \le \sim (x \vee y)$, thus $\sim (x \vee y) = \sim x \wedge \sim y$. Similarly, $\neg(x \vee y) = \neg x \wedge \neg y$.

(4) The proof is similar to (3).

(5) By (C3), (C5), (1) and (2), we have $x \odot y \le t$ if and only if $x \le y \to t = \neg t \rightsquigarrow \neg y$ if and only if $\neg t \le x \to \neg y$ if and only if $\sim (x \to \neg y) \le \sim \neg t = t$. Therefore $x \odot y = \sim (x \to \neg y)$. On the other hand, $x \odot y \le t$ if and only if $y \le x \rightsquigarrow t = \sim t \to \sim x$ if and only if $\sim t \le y \rightsquigarrow \sim x$ if and only if $\neg(y \rightsquigarrow \sim x) \le \neg \sim t = t$. Thus $x \odot y = \neg(y \rightsquigarrow \sim x)$.

(6) From (5) and (C5), it follows that $\sim (\neg y \odot x) = \sim \neg(x \rightsquigarrow \sim (\neg y)) = x \rightsquigarrow y, \neg(x \odot \sim y) = \neg \sim (x \to \neg(\sim y)) = x \to y$.

(7) By (6), (3), (4) and Lemma 2.3, we have $x \rightsquigarrow (y \vee z) = \sim (\neg(y \vee z) \odot x) = \sim ((\neg y \wedge \neg z) \odot x) = \sim ((\neg y \odot x) \wedge (\neg z \odot x)) = \sim (\neg y \odot x) \vee \sim (\neg z \odot x) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$. Similarly, $x \to (y \vee z) = (x \to y) \vee (x \to z)$. □

Remark 2.8. Lemma 2.7(5) shows that operator $\odot$ can be defined by operators $\neg, \sim, \to$(or $\rightsquigarrow$) in pseudo-$IMTL$ algebras.

3. Main results Pseudo-R_0 algebras were introduced by the authors in [7] as a noncommutative extension of R_0 algebras [8]. In this section, we discuss the relations between pseudo-MTL algebras and the pseudo-R_0 algebras.

Definition 3.1. (see [7]) A pseudo-R_0 algebra is a structure

$$L = (L, \vee, \wedge, \to, \rightsquigarrow, \sim, \neg, 0, 1)$$

of type (2,2,2,2,1,1,0,0), which satisfies the following axioms, for all $x, y, z \in L$:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) The following conditions hold:

(PR1) $\sim x \to \sim y = y \rightsquigarrow x$, $\neg x \rightsquigarrow \neg y = y \to x$,

(PR2) $1 \to x = 1 \rightsquigarrow x = x$,

(PR3) $(y \rightsquigarrow z) \vee ((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$,
$(y \to z) \vee ((x \to y) \to (x \to z)) = (x \to y) \to (x \to z)$,

(PR4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(PR5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
$x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$,

(PR6) $(x \to y) \vee ((x \to y) \rightsquigarrow (\neg x \vee y)) = (x \rightsquigarrow y) \vee ((x \rightsquigarrow y) \to (\sim x \vee y)) = 1$,

(PR7) $\neg x = x \to 0, \sim x = x \rightsquigarrow 0$.

If a bounded lattice L satisfies (PR1)-(PR5) and (PR7), then L is called a weak pseudo-R_0 algebra.

Lemma 3.2 (see [7]). Let L be a pseudo-R_0 algebra. The following hold:

(1) $x = \sim \neg x = \neg \sim x$,

(2) $x \to x = x \rightsquigarrow x = 1$,

(3) $x \le y$ if and only if $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$,

(4) $x \le y \to z$ if and only if $y \le x \rightsquigarrow z$,

(5) $\neg(x \vee y) = \neg x \wedge \neg y, \sim (x \vee y) = \sim x \wedge \sim y$,

(6) $\neg(x \wedge y) = \neg x \vee \neg y, \sim (x \wedge y) = \sim x \vee \sim y$.

The following theorem is a characterization of a pseudo-R_0 algebra.

Theorem 3.3. Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. $\neg$ and $\sim$ are two unary operations on L, and $\neg 0 = \sim 0 = 1$, $\to$ and $\rightsquigarrow$ are two binary operations on L. Then $(L, \vee, \wedge, \to, \rightsquigarrow, \neg, \sim, 0, 1)$ is a pseudo-R_0 algebra if and only if L satisfies (PR1), (PR4), (PR5), (PR6) and (PR8), where

(PR8) $x \le y$ if and only if $x \rightsquigarrow y = 1$ if and only if $x \to y = 1$.

Proof. Let L be a bounded lattice and satisfy (PR1), (PR4)-(PR6) and (PR8). Since $x \le x$, by (PR8) we have $x \rightsquigarrow x = x \to x = 1$. By (PR4), $x \to (1 \rightsquigarrow x) = 1 \rightsquigarrow (x \to x) = 1 \rightsquigarrow 1 = 1$, so by (PR8) $x \le 1 \rightsquigarrow x$. Conversely, from (PR4) it follows that $1 \rightsquigarrow ((1 \rightsquigarrow x) \to x) = (1 \rightsquigarrow x) \to (1 \rightsquigarrow x) = 1$. By (PR8) we have $(1 \rightsquigarrow x) \to x = 1$ and $1 \rightsquigarrow x \le x$. Therefore $1 \rightsquigarrow x = x$. Similarly, $1 \to x = x$. Thus (PR2) holds. From (PR1) and (PR2) , it follows that $x \rightsquigarrow 0 = \sim 0 \rightarrow \sim x = 1 \rightarrow \sim x = \sim x, x \rightarrow 0 = \neg 0 \rightsquigarrow \neg x = 1 \rightsquigarrow \neg x = \neg x$. Thus (PR7) holds. Finally, from (PR5) we have if $x \leq y$, then $z \rightsquigarrow x \leq z \rightsquigarrow y, z \rightarrow x \leq z \rightarrow y$. On the other hand, from (PR4) and (PR8), it follows that $x \leq y \rightsquigarrow z$ if and only if $y \leq x \rightarrow z$. Now let $t \leq y \rightarrow z$, then $y \leq t \rightsquigarrow z$ and $x \rightarrow y \leq x \rightarrow (t \rightsquigarrow z) = t \rightsquigarrow (x \rightarrow z)$. Hence $t \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$. This implies $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$. Therefore $(y \rightarrow z) \vee ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (x \rightarrow z)$. Similarly, $(y \rightsquigarrow z) \vee ((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$. Thus (PR3) holds. Consequently, L is a pseudo-R_0 algebra. The converse is obvious. □

Corollary 3.4. Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. $\neg$ and $\sim$ are two unary operations on L, and $\neg 0 = \sim 0 = 1$, $\rightarrow$ and $\rightsquigarrow$ are two binary operations on L. Then $(L, \vee, \wedge, \rightarrow, \rightsquigarrow, \neg, \sim, 0, 1)$ is a weak pseudo-R_0 algebra if and only if L satisfies (PR1), (PR4), (PR5) and (PR8).

From Lemmas 2.3, 2.7 and Corollary 3.4, we have the following theorem:

Theorem 3.5. Each pseudo-$IMTL$ algebra is a weak pseudo-R_0 algebra.

Corollary 3.6. Each pseudo-NM algebra is a pseudo-R_0 algebra.

Proof. Let L be a pseudo-NM algebra. By Theorem 3.5, L is a weak pseudo-R_0 algebra. Now we only prove that L satisfies (PR6). By Lemma 2.7(6) we have $x \rightsquigarrow y = \sim(\neg y \odot x), x \rightarrow y = \neg(x \odot \sim y)$. Hence $(x \rightarrow y) \vee ((x \rightarrow y) \rightsquigarrow (\neg x \vee y)) = (\neg(x \odot \sim y)) \vee ((\neg(x \odot \sim y)) \rightsquigarrow (\neg x \vee y)) = (\neg(x \odot \sim y)) \vee (\sim(\neg x \vee y) \rightarrow \sim(\neg(x \odot \sim y))) = (\neg(x \odot \sim y)) \vee ((x \wedge \sim y) \rightarrow (x \odot \sim y)) = 1$. Similarly, $(x \rightsquigarrow y) \vee ((x \rightsquigarrow y) \rightarrow (\sim x \vee y)) = (\sim(\neg y \odot x)) \vee ((\sim(\neg y \odot x)) \rightarrow (\sim x \vee y)) = (\sim(\neg y \odot x)) \vee (\neg(\sim x \vee y) \rightsquigarrow (\neg y \odot x)) = (\sim(\neg y \odot x)) \vee ((\neg y \wedge x) \rightsquigarrow (\neg y \odot x)) = 1$. Thus (PR6) holds. By Theorem 3.3, L is a pseudo-R_0 algebra. □

Corollary 3.7. Each $IMTL$ algebra is a weak R_0 algebra. Further, each NM algebra is a R_0 algebra.

Theorem 3.8. Let $L = (L, \vee, \wedge, \rightarrow, \rightsquigarrow, \neg, \sim, 0, 1)$ be a weak pseudo-R_0 algebra. For all $x, y \in L$, we define

$$x \odot y = \neg(y \rightsquigarrow \sim x).$$

Then $L = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, \neg, \sim, 0, 1)$ is a pseudo-$IMTL$ algebra.

Proof. It suffices to check the conditions (C1)-(C5) hold. From Definition 3.1 and Lemma 3.2, it follows that (C1) and (C5) hold.

(C2) By the definition of $\odot$, we have $x \odot 1 = \neg(1 \rightsquigarrow \sim x) = \neg \sim x = x, 1 \odot x = \neg(x \rightsquigarrow \sim 1) = \neg(x \rightsquigarrow 0) = \neg \sim x = x$. And $(x \odot y) \odot z = \neg(z \rightsquigarrow \sim (x \odot y)) = \neg(z \rightsquigarrow (y \rightsquigarrow \sim x)) = \neg(z \rightsquigarrow (\sim(\sim x) \rightarrow \sim y)) = \neg(\sim(\sim x) \rightarrow (z \rightsquigarrow \sim y)) = \neg(\neg(z \rightsquigarrow \sim y) \rightsquigarrow \neg \sim (\sim x)) = \neg((y \odot z) \rightsquigarrow \sim x) = x \odot (y \odot z)$. This shows that $\odot$ is associative. Hence (C2) holds.

(C3) If $x \odot y \leq z$, by Lemma 3.2(3) we have $(x \odot y) \rightsquigarrow z = 1$, i.e., $\neg(y \rightsquigarrow \sim x) \rightsquigarrow z = 1$. Using (PR1) we have $\neg(y \rightsquigarrow \sim x) \rightsquigarrow z = \sim z \rightarrow \sim \neg(y \rightsquigarrow \sim x) = \sim z \rightarrow (y \rightsquigarrow \sim x)$, and so $\sim z \rightarrow (y \rightsquigarrow \sim x) = 1$. From (PR4), it follows that $y \rightsquigarrow (x \rightsquigarrow z) = y \rightsquigarrow (\sim z \rightarrow \sim x) = \sim z \rightarrow (y \rightsquigarrow \sim x) = 1$. This implies that $y \leq x \rightsquigarrow z$. Conversely, if $y \leq x \rightsquigarrow z$, then $y \leq \sim z \rightarrow \sim x$, thus $y \rightsquigarrow (\sim z \rightarrow \sim x) = 1$. By (PR1) and (PR4) we have, $x \odot y \rightsquigarrow z = \neg(y \rightsquigarrow \sim x) \rightsquigarrow z = \sim z \rightarrow (y \rightsquigarrow \sim x) = y \rightsquigarrow (\sim z \rightarrow \sim x) = 1$. Hence $x \odot y \leq z$. On the other hand, $y \leq x \rightsquigarrow z$ if and only if $y \rightarrow (x \rightsquigarrow z) = 1$ if and only if $x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z) = 1$ if and only if $x \leq y \rightarrow z$. Thus (C3) holds.

(C4) Since $x \leq x \vee y$, then $x \rightarrow (x \vee y) = \neg(x \vee y) \rightsquigarrow \neg x = 1$. By (PR3) we have $(\neg y \rightsquigarrow \neg(x \vee y)) \rightsquigarrow (\neg y \rightsquigarrow \neg x) = ((\neg(x \vee y)) \rightsquigarrow \neg x) \vee ((\neg y \rightsquigarrow \neg(x \vee y)) \rightsquigarrow (\neg y \rightsquigarrow \neg x)) = 1$, that is, $((x \vee y) \rightarrow y) \rightsquigarrow (x \rightarrow y) = 1$. Therefore $x \vee y \rightarrow y \leq x \rightarrow y$. Similarly, $x \vee y \rightarrow x \leq y \rightarrow x$. Thus $((x \vee y) \rightarrow y) \vee ((x \vee y) \rightarrow x) \leq (x \rightarrow y) \vee (y \rightarrow x)$. Since $1 = (x \vee y) \rightarrow (x \vee y) = ((x \vee y) \rightarrow x) \vee ((x \vee y) \rightarrow y)$, we have $(x \rightarrow y) \vee (y \rightarrow x) = 1$. Similarly, $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$. By Definition 2.5, L is a pseudo-$IMTL$ algebra. □

Corollary 3.9. Let $L = (L, \vee, \wedge, \rightarrow, \rightsquigarrow, \neg, \sim, 0, 1)$ be a pseudo-R_0 algebra. For all $x, y \in L$, we define

$$x \odot y = \neg(y \rightsquigarrow \sim x).$$

Then $L = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, \neg, \sim, 0, 1)$ is a pseudo-NM algebra.

Proof. From Theorem 3.8, it follows that L is a pseudo-$IMTL$ algebra. Now we only prove that L satisfies (C6). Firstly, we prove

$$x \odot y = \sim (x \rightarrow \neg y).$$

Indeed, from Theorem 3.8, it follows that $x \odot y \leq t$ if and only if $x \leq y \rightarrow t = \neg t \rightsquigarrow \neg y$ if and only if $\neg t \leq x \rightarrow \neg y$ if and only if $\sim (x \rightarrow \neg y) \leq \sim \neg t = t$. Consequently, $x \odot y = \sim (x \rightarrow \neg y)$.

(C6) Since $x \odot y = \sim (x \rightarrow \neg y)$, we have $(x \odot y) \rightarrow 0 = x \rightarrow \neg y$. Then $((x \odot y) \rightarrow 0) \vee ((x \wedge y) \rightarrow (x \odot y)) = (x \rightarrow \neg y) \vee (\neg(x \odot y) \rightsquigarrow \neg(x \wedge y)) = (x \rightarrow \neg y) \vee ((x \rightarrow \neg y) \rightsquigarrow (\neg x \vee \neg y))$. By (PR6) we have $(x \rightarrow \neg y) \vee ((x \rightarrow \neg y) \rightsquigarrow (\neg x \vee \neg y)) = 1$. Thus $((x \odot y) \rightarrow 0) \vee ((x \wedge y) \rightarrow (x \odot y)) = 1$. On the other hand, since $x \odot y = \neg(y \rightsquigarrow \sim x)$, then $(x \odot y) \rightsquigarrow 0 = y \rightsquigarrow \sim x$. Hence $((x \odot y) \rightsquigarrow 0) \vee ((x \wedge y) \rightsquigarrow (x \odot y)) = (y \rightsquigarrow \sim x) \vee (\sim (x \odot y) \rightarrow \sim (x \wedge y)) = (y \rightsquigarrow \sim x) \vee ((y \rightsquigarrow \sim x) \rightarrow (\sim x \vee \sim y)) = 1$. By Definition 2.6, L is a pseudo-NM algebra. □

Corollary 3.10. Each weak R_0 algebra is an $IMTL$ algebra. Further, each R_0 algebra is a NM algebra.

Remark 3.11. Theorems 3.5 and 3.8 show that pseudo-$IMTL$ algebras are equivalent to weak pseudo-R_0 algebras. Corollaries 3.6 and 3.9 show that pseudo-NM algebras are equivalent to pseudo-R_0 algebras.

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