IDEAL THEORY OF SUBTRACTION ALGEBRAS

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ABSTRACT. The notion of ideals in subtraction algebras is considered. Characterizations of ideals are given.

1. Introduction.

B. M. Schein [3] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [2]).

In this paper, we define the notion of ideals of subtraction algebras, and then we give some of its characterizations. We show that any ideal A of a subtraction algebra X can be represented as the union of ideals of the form $X(a, b)$ for all $a, b \in A$.

2. Preliminaries

A *subtraction algebra* is defined as an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, b]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $(x - y) - x = 0$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \leq x - z$.

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3. Ideal Theory of Subtraction Algebras

We begin with the following propositions.

Proposition 3.1. *In a subtraction algebra X, we have*

(i) $x \leq y$ *if and only if* $x = y - w$ *for some* $w \in X$.

(ii) $x \leq y$ *implies* $x - z \leq y - z$ *and* $z - y \leq z - x$ *for all* $z \in X$.

(iii) $x, y \leq z$ *implies* $x - y = x \wedge (z - y)$.

Proof. (i) If $x \leq y$, then by taking $w = y - x$ we have

$$x = x - 0 = x - (x - y) = y - (y - x) = y - w.$$

Conversely, if $x = y - w$ for some $w \in X$, then

$$x - y = (y - w) - y = (y - y) - w = 0 - w = 0,$$

that is, $x \leq y$.

(ii) If $x \leq y$, then $x = y - w$ for some $w \in X$. Hence $x - z = (y - w) - z = (y - z) - w$, and so $x - z \leq y - z$ by (i). Next if $x \leq y$, then $x - y = 0$. Thus

$$\begin{aligned}(z - y) - (z - x) &= (z - (z - x)) - y = (x - (x - z)) - y \\ &= (x - y) - (x - z) = 0 - (x - z) = 0,\end{aligned}$$

that is, $z - y \leq z - x$.

(iii) If $x \leq z$, then $x - y \leq z - y$ by (ii). But $x - y \leq x$, and thus $x - y \leq x \wedge (z - y)$. Let $w = x \wedge (z - y)$. Then $w \leq x$, and so $w = x \wedge w = x - (x - w)$. Also

$$y \wedge (z - y) = (z - y) - ((z - y) - y) = (z - y) - (z - y) = 0,$$

hence $w - (w - y) = y \wedge w = x \wedge y \wedge (z - y) = 0$. Therefore

$$\begin{aligned}w - (x - y) &= (w - 0) - (x - y) \\ &= (w - (w - (w - y))) - (x - y) \\ &= (w - y) - (x - y) \\ &= ((x - (x - w)) - y) - (x - y) \\ &= ((x - y) - (x - w)) - (x - y) \\ &= ((x - y) - (x - y)) - (x - w) \\ &= 0 - (x - w) = 0,\end{aligned}$$

and thus $x \wedge (z - y) = w \leq x - y$. Consequently, $x - y = x \wedge (z - y)$. □

Proposition 3.2. *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

Proof. Let $w \in X$ be an upper bound for x and y. Since $w - y \leq w$, it follows from Proposition 3.1(iii) that

$$(w - y) - x = (w - y) \wedge (w - x) \leq w - x$$

which implies from Proposition 3.1(ii) that

$$x = x \wedge w = w - (w - x) \leq w - ((w - y) - x).$$

Similarly, $y \leq w - ((w - x) - y) = w - ((w - y) - x)$. Hence $w - ((w - y) - x)$ is an upper bound for x and y. Let a be any other upper bound for x and y. Using Proposition 3.1(ii), we have $w - a \leq w - y$ and $w - a \leq w - x$. It follows that $w - a \leq (w - y) \wedge (w - x)$. Applying Proposition 3.1(ii) again, we get

$$w - ((w - y) - x) = w - ((w - y) \wedge (w - x)) \leq w - (w - a) \leq a.$$

Therefore $x \vee y := w - ((w - y) - x)$ is a least upper bound for x and y. $\square$

Definition 3.3. A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies:

(I1) $a - x \in A$ for all $a \in A$ and $x \in X$.
(I2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Let A be a nonempty subset of a subtraction algebra X satisfying (I1). If $a \in A$ and $x \leq a$, then $x = a - w$ for some $w \in X$, and hence $x \in A$ by (I1). Thus every ideal of a subtraction algebra contains the zero element 0.

Theorem 3.4. *Let w be a nonzero element of a subtraction algebra X. Then the set*

$$(w] := \{x \in X \mid x \leq w\}$$

is the least nonzero ideal of X containing w.

Proof. Obviously $w \in (w]$. Let $a \in (w]$ and $x \in X$. Then $a - x \leq a \leq w$, and so $a - x \in (w]$. Now let $a, b \in (w]$ and $x \in X$. Then $a, b \leq w$, and hence $a \vee b$ exists by Proposition 3.2. It follows from (I2) that $a \vee b \in (w]$. Therefore $(w]$ is an ideal of X containing w. Let B be any ideal of X containing w and let $y \in (w]$. Then $y \leq w$, and thus $y \in B$, that is, $(w] \subseteq B$. This completes the proof. $\square$

We provide characterizations of ideals.

Theorem 3.5. *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies* (I1) *and*

(I3) $x - ((x - a) - b) \in A$ *for all* $a, b \in A$ *and* $x \in X$.

Proof. Let A be a nonempty subset of X satisfying (I1) and (I3). Suppose that $a \vee b$ exists for $a, b \in A$. Putting $w = a \vee b$, we get

$$a \vee b = w - ((w - a) - b) \in A$$

by Proposition 3.2 and (I3). Hence A is an ideal of X.

Conversely, let A be an ideal of X, and let θ_A be a relation on X defined by

$$(x, y) \in \theta_A \Leftrightarrow x - y,\ y - x \in A,\ \forall x, y \in X.$$

It is routine to check that θ_A is a congruence relation on X. Let $a, b \in A$ and $x \in X$. Then $(x, x) \in \theta_A$, $(a, 0) \in \theta_A$, and $(b, 0) \in \theta_A$. Hence

$$(x - ((x - a) - b),\ 0) = (x - ((x - a) - b),\ x - ((x - 0) - 0)) \in \theta_A,$$

and so $x - ((x - a) - b) \in A$. This completes the proof. $\square$

Lemma 3.6. *Let A be a nonempty subset of a subtraction algebra X such that*

(I4) $0 \in A$.
(I5) $(x - y) - z \in A$ *and* $y \in A$ *imply* $x - z \in A$ *for all* $x, y, z \in X$.

If $a \in A$ and $x \leq a$, then $x \in A$.

Proof. Let $x \in X$ and $a \in A$ be such that $x \leq a$. Then $(x - a) - 0 = 0 \in A$, and so $x = x - 0 \in A$. $\square$

Theorem 3.7. *Let X be a subtraction algebra. A nonempty subset A of X is an ideal of X if and only if it satisfies the conditions* (I4) *and* (I5).

Proof. Assume that A is an ideal of X. Obviously $0 \in A$. Let $x, y, z \in X$ be such that $(x - y) - z \in A$ and $y \in A$. Taking $b = 0$ and $a = y$ in (I3) and using (p2), we get $x - (x - y) \in A$. It follows from (p2), (p7), and (I3) that

$$\begin{aligned} x - z &= (x - z) - 0 \\ &= (x - z) - (((x - z) - ((x - y) - z)) - (x - (x - y))) \in A, \end{aligned}$$

proving (I5). Conversely suppose that A satisfies (I4) and (I5). Then $(a - a) - x = 0 - x = 0 \in A$ for all $a \in A$ and $x \in X$, and so $a - x \in A$ by (I5). Since $(x - a) - (x - a) = 0 \in A$ for all $a \in A$ and $x \in X$, it follows from (I5) that $x - (x - a) \in A$. Note that $((x - b) - ((x - a) - b)) - (x - (x - a)) = 0$, that is, $(x - b) - ((x - a) - b) \leq x - (x - a)$ for all $b \in A$. Using Lemma 3.6, we have $(x - b) - ((x - a) - b) \in A$. Since $b \in A$, it follows from (I5) that $x - ((x - a) - b) \in A$ which shows (I3). Hence A is an ideal of X. □

Theorem 3.8. *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies* (I4) *and*

(I6) *$y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.*

Proof. Let A be an ideal of X. (I6) is by taking $z = 0$ in (I5) and using (p2). Conversely assume that A satisfies (I4) and (I6). Let $x \in X$ and $y \in A$. Since $(y - x) - y = (y - y) - x = 0 - x = 0 \in A$, it follows from (I6) that $y - x \in A$ which proves (I1). Note that

$$(x - ((x - a) - b)) - b = (x - b) - ((x - a) - b) \leq x - (x - a) \leq a,$$

that is, $((x - ((x - a) - b)) - b) - a = 0 \in A$ for all $a, b \in A$ and $x \in X$. Using (I6), we get $x - ((x - a) - b) \in A$, that is, (I3) is valid. Hence, by Theorem 3.5, A is an ideal of X. □

Theorem 3.9. *Let A be a nonempty subset of a subtraction algebra X. Then A is an ideal of X if and only if it satisfies:*

(I7) *for any $a, b \in A$, $x - a \leq b$ implies $x \in A$.*

Proof. Assume that A is an ideal of X and let $a, b \in A$. If $x - a \leq b$, then $(x - a) - b = 0 \in A$, and so $x \in A$ by (I6). Conversely suppose that A satisfies (I7). Since A is nonempty, we can take $a \in A$, and then $0 - a = 0 \leq a$. Using (I7), we get $0 \in A$. Let $x, y \in X$ be such that $y \in A$ and $x - y \in A$. Since $x - (x - y) \leq y$, we obtain $x \in A$ by (I7). It follows from Theorem 3.8 that A is an ideal of X. □

Lemma 3.10. *Every subtraction algebra satisfies the right self-distributive law, that is, the equality $(x - y) - z = (x - z) - (y - z)$ is valid.*

Proof. Let X be a subtraction algebra and $x, y, z \in X$. Then

$$\begin{aligned} &((x - y) - z) - ((x - z) - (y - z)) \\ = \;& ((x - z) - y) - ((x - z) - (y - z)) \\ \leq \;& (y - z) - y = 0, \end{aligned}$$

and so $((x - y) - z) - ((x - z) - (y - z)) = 0$. Since $(x - y) - (x - z) \leq z - y$, it follows from (S3) and Proposition 3.1(ii) that

$$((x - y) - w) - (x - z) = ((x - y) - (x - z)) - w \leq (z - y) - w$$

for all $w \in X$. Substituting $x - z$ for x, $y - z$ for y, $(x - z) - z$ for z, and $(x - y) - z$ for w in the above inequality, we have

$$\begin{aligned} &((x - z) - (y - z)) - ((x - y) - z) \\ \leq \;& (((x - z) - z) - (y - z)) - ((x - y) - z) \\ \leq \;& ((x - z) - y) - ((x - y) - z) = 0, \end{aligned}$$

and thus $((x-z)-(y-z))-((x-y)-z)=0$. Therefore the right self-distributive law is valid. $\square$

Theorem 3.11. *If A is an ideal of a subtraction algebra X, then the set*

$$A_w := \{x \in X \mid x - w \in A\},\ w \in X$$

is the least ideal of X containing A and w.

Proof. Let $w \in X$. Since $0 - w = 0 \in A$, we have $0 \in A_w$. Let $x, y \in X$ be such that $y \in A_w$ and $x - y \in A_w$. Then $y - w \in A$ and $(x-y)-w \in A$. It follows from Lemma 3.10 that

$$(x-w)-(y-w) = (x-y)-w \in A$$

so from (I6) that $x - w \in A$, that is, $x \in A_w$. Hence A_w is an ideal of X. Obviously A_w contains A and w. Let B be an ideal of X containing A and w. If $x \in A_w$, then $x - w \in A \subseteq B$, and hence $x \in B$ by (I6). Thus $A_w \subseteq B$, and consequently A_w is the least ideal of X containing A and w. $\square$

Theorem 3.12. *Let X be a subtraction algebra. For $u, v \in X$, the set*

$$X(u,v) := \{x \in X \mid (x-u)-v = 0\}$$

is an ideal of X, and $u, v \in X(u,v)$.

Proof. Obviously $0, u, v \in X(u,v)$. Let $x, y \in X$ be such that $y \in X(u,v)$ and $x - y \in X(u,v)$. Then $(y-u)-v = 0$ and $((x-y)-u)-v = 0$. It follows from (p2) and Lemma 3.10 that

$$\begin{aligned}
0 &= ((x-y)-u)-v \\
&= ((x-u)-(y-u))-v \\
&= ((x-u)-v)-((y-u)-v) \\
&= ((x-u)-v)-0 \\
&= (x-u)-v
\end{aligned}$$

so that $x \in X(u,v)$. Hence $X(u,v)$ is an ideal of X. $\square$

Lemma 3.13. *Every ideal A of a subtraction algebra X contains the ideal $X(a,b)$ for all $a, b \in A$.*

Proof. Let $x \in X(a,b)$. Then $(x-a)-b = 0 \in A$, and hence $x \in A$. This shows that $X(a,b) \subseteq A$ for all $a, b \in A$. $\square$

Theorem 3.14. *Every ideal A of a subtraction algebra X can be represented as the union of ideals of the form $X(a,b)$ for all $a, b \in A$, that is, $A = \bigcup_{a,b \in A} X(a,b)$.*

Proof. Let A be an ideal of X and $x \in A$. Since $x \in X(x,0)$, we have

$$A \subseteq \bigcup_{x \in A} X(x,0) \subseteq \bigcup_{a,b \in A} X(a,b).$$

Now let $x \in \bigcup_{a,b \in A} X(a,b)$. Then there exist $u, v \in A$ such that $x \in X(u,v)$. It follows from Lemma 3.13 that $x \in A$ so that $\bigcup_{a,b \in A} X(a,b) \subseteq A$. This completes the proof. $\square$

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