HUYGENS' PRINCIPLE FOR THE DIRICHLET BOUNDARY VALUE PROBLEM FOR THE WAVE EQUATION

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ABSTRACT. We prove Huygens' principle for the Dirichlet boundary value problem by using the Herglotz-Petrovskii-Leray formula.

1. INTRODUCTION.

Huygens' principle for the initial value problem is well-known. However, that for the boundary value problem is not a general fact. One of approaches to prove Huygens' principle is to derive the Herglotz-Petrovskii-Leray formula. This formula was effectively used to prove the existence of lacuna for the hyperbolic initial value problem in Atiyah-Bott-Garding[1]. By the formula, the fundamental solution $E(x)$ is described as the integration on a homology class depending on x, and if the homology class is homologous to 0, the lacuna exists in a neighborhood of x. On the other hand, in Yura[2], the Herglotz-Petrovskii-Leray formula for the hyperbolic boundary value problem was derived. The formula is described as the integration on a "chain", not a "homology class". But this difference does not give any effect on studying the existence of lacuna.

In this paper, the proof of the existence of strong lacuna is a concrete calculation of the formula. Primarily, we should prove the chain of integration is homotopic to 0. But it is very complicated because the integrand has a branch locus. In the case of the initial value problem, any branch locus does not appear. Our calculation is simple, however, we believe that the lacuna's theory which was abstract is materialized by our calculation.

2. HUYGENS' PRINCIPLE.

We denote $x' = (x_1, x_2, x_3)$ for $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$ and consider the Dirichlet boundary value problem in four-dimensional space-time

$$\begin{cases} P(D)F(x) = 0, & x \in \{x \in \mathbf{R}^4\,;\, x_4 > 0\}, \\ F(x)\Big|_{x_4=0} = \delta(x'). \end{cases} \tag{2.1}$$

Here $P(D)$ is the wave operator $D_1^2 - D_2^2 - D_3^2 - D_4^2$ and D means $\frac{1}{i}\frac{\partial}{\partial x}$. Of course, P is hyperbolic with respect to $\vartheta = (1, 0, 0, 0)$. δ is Dirac's δ function. $F(x)$ is the forward fundamental solution whose support is included in $\{x \in \mathbf{R}^4\,;\, x_1 \geq 0\}$ and expresses the propagation of the wave in the case where the delta shock is given at $\{x \in \mathbf{R}^4\,;\, x_4 = 0\}$.

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$F(x)$ is written in integral form (see Yura[2])

$$F(x)=(2\pi)^{-4}i^{-1}\int_{\mathbf{R}^4-i\vartheta} e^{ix\zeta}P_+(\zeta)^{-1}\,d\zeta. \tag{2.2}$$

Here $P_+(\zeta)=\zeta_4-\sqrt{\zeta_1^2-\zeta_2^2-\zeta_3^2}$ and the branch of $\sqrt{\zeta_1^2-\zeta_2^2-\zeta_3^2}$ is selected so as to have positive imaginary part for $\zeta'\in\mathbf{R}^3-i\vartheta'$, $\vartheta'=(1,0,0)$. $\zeta_1^2-\zeta_2^2-\zeta_3^2\neq 0$ if $\zeta'\in\mathbf{R}^3-i\vartheta'$. $F(x)$ is interpreted in the distribution sense with respect to x.

Theorem 2.1. *Huygens' principle holds for the Dirichlet boundary value problem* (2.1). *In other words, $F(x)$ vanishes in the inside of the propagation cone.*

We shall prove Theorem 2.1 by evaluating $F(x)$ transformed into the Herglotz-Petrovskii-Leray formula.

For convenience, we put $B(\zeta')=\zeta_1^2-\zeta_2^2-\zeta_3^2$. $\{\zeta\in\mathbf{C}^4\,;\,B(\zeta')=0\}$ represents the branch locus of $P_+(\zeta)$. Let

$$\Gamma'(P,\vartheta)=\{\xi'\in\mathbf{R}^3\,;\,B(\xi')>0,\ \xi_1>0\}. \tag{2.3}$$

$\Gamma'(P,\vartheta)$ coincides with $\Gamma'(P,\vartheta)$ defined in [2]. Since $B(\zeta')$ does not take a value in $[0,\infty)$ if $\zeta'\in\mathbf{R}^3-i\Gamma'(P,\vartheta)$, $P_+(\zeta)$ is single-valued, holomorphic and non-zero in $\mathbf{R}^4-i\Gamma'(P,\vartheta)\times\{0\}$. To expand such a domain, let us make use of localization. The following is the definition of localization.

Definition 2.2. *Let Γ be an open connected cone in $\mathbf{R}^n$ and let f be a homogeneous holomorphic function in $\mathbf{R}^n-i\Gamma$. Then localization f_ξ of f at $\xi\in\mathbf{R}^n$ is defined by the first non-vanishing homogeneous term in Puiseux expansion*

$$f(\xi+t\zeta)=t^p f_\xi(\zeta)+o(t^p)\quad \text{as } t\to+0 \tag{2.4}$$

for $\zeta\in\mathbf{R}^n-i\Gamma$. When f is a polynomial, (2.4) *becomes Taylor expansion.*

We also define the local hyperbolic cone of B,P_+ at ξ by

$$\begin{aligned}\Gamma_\xi(B,\vartheta)&=\text{the connected component of}\\ &\qquad\{\eta\in\mathbf{R}^4\,;\,\eta=(\eta',\eta_4),\ \eta_4\in\mathbf{R},\ B_{\xi'}(\eta')\neq 0\}\text{ which contains }\vartheta,\\ \Gamma_\xi(P_+,\vartheta)&=\text{the connected component of}\\ &\qquad\{\eta\in\Gamma_\xi(B,\vartheta)\,;\,P_{+\xi}(-i\eta)\neq 0\}\text{ which contains }\vartheta.\end{aligned} \tag{2.5}$$

Lemma 2.3. *$\Gamma_\xi(B,\vartheta)$ coincides with $\dot\Gamma_{\xi'}\times\mathbf{R}$ in* [2].

Proof. First, let us remember the definition of $\dot\Gamma_{\xi'}\times\mathbf{R}$. Let $\xi'\in\mathbf{R}^3\setminus\{0\}$ be arbitrarily fixed and λ be a real multiple root of $P(\xi',\lambda)=0$. Then, $\dot\Gamma_{\xi'}\times\mathbf{R}$ is defined by $\Gamma_{(\xi',\lambda)}(P,\vartheta)$, which is independent of λ. If $P(\xi',\lambda)=0$ has no real multiple root, $\dot\Gamma_{\xi'}\times\mathbf{R}$ is $\mathbf{R}^4$ by definition.

When $\xi'\in\mathbf{R}^3\setminus\{0\}$, $P(\xi',\lambda)=0$ has a real multiple root if and only if $B(\xi')=0$. For $\xi'\in\{\xi'\in\mathbf{R}^3\setminus\{0\}\,;\,B(\xi')=0\}$,

$$\begin{aligned}\dot\Gamma_{\xi'}\times\mathbf{R}=\Gamma_\xi(B,\vartheta)&=\text{the connected component of}\\ &\qquad\{\eta\in\mathbf{R}^4\,;\,2(\xi_1\eta_1-\xi_2\eta_2-\xi_3\eta_3)\neq 0\}\text{ which contains }\vartheta.\end{aligned} \tag{2.6}$$

On the other hand, for $\xi'\in\{\xi'\in\mathbf{R}^3\setminus\{0\}\,;\,B(\xi')\neq 0\}$,

$$\dot\Gamma_{\xi'}\times\mathbf{R}=\Gamma_\xi(B,\vartheta)=\mathbf{R}^4. \tag{2.7}$$

□

In virtue of Lemma 2.3, $\Gamma_\xi(P_+,\vartheta)$ is also the same as $\Gamma_\xi(P_+,\vartheta)$ in [2] and $P_+(\xi-it\eta)$ is non-zero where t is small enough and $\eta\in\Gamma_\xi(P_+,\vartheta)$.

Let $\Gamma^\circ_\xi(P_+,\vartheta)$ be the dual cone of $\Gamma_\xi(P_+,\vartheta)$, that is,

$$\Gamma^\circ_\xi(P_+,\vartheta)=\{x\in\mathbf{R}^4\,;\,x\eta\geq 0 \text{ for all } \eta\in\Gamma_\xi(P_+,\vartheta)\}. \tag{2.8}$$

Then, by Theorem 4.1 in [2], $F(x)$ can be described as

$$F(x)=(2\pi i)^{-3}\int_{\gamma(\xi)=1}\delta^{(2)}(x\xi)P_+(\zeta)^{-1}\,\omega_4(\zeta),\quad \zeta=\xi-iv(\xi) \tag{2.9}$$

when $x\notin\cup_{\xi\in\mathbf{R}^4\setminus\{0\}}\Gamma^\circ_\xi(P_+,\vartheta)\cup\{x\in\mathbf{R}^4\,;\,x_1<0\}$. Here $v(\xi)$ is a special kind of C^∞ vector field satisfying the following conditions:

(i) If $\lambda\in\mathbf{R}\setminus\{0\}$,

$$v(\lambda\xi)=|\lambda|v(\xi). \tag{2.10}$$

(ii) For any $\xi\in\mathbf{R}^4\setminus\{0\}$,

$$v(\xi)\in\Gamma_\xi(P_+,\vartheta)\cap\{\xi\in\mathbf{R}^4\,;\,x\xi=0\}. \tag{2.11}$$

(iii) If $\xi\in\mathbf{R}^4\setminus\{0\},\ 0<t\leq 1$,

$$P_+(\xi-itv(\xi))\neq 0. \tag{2.12}$$

$\gamma(\xi)$ is a special kind of C^∞ function which is positive homogeneous of degree 1 outside a neighborhood of $\xi=0$ and satisfies $d\gamma|_{\gamma=1}\neq 0$. $\omega_n(\zeta)$ is the Kronecker form,

$$\omega_n(\zeta)=\sum_{j=1}^{n}(-1)^{j-1}\zeta_j d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n. \tag{2.13}$$

In (2.9), $\delta^{(2)}(x\xi)$ is interpreted in the distribution on $\gamma(\xi)=1$.

Now, $\operatorname{supp}F(x)\subset\Gamma^\circ_0(P_+,\vartheta)$ from (2.2),(2.5) and

$$\Gamma^\circ_0(P_+,\vartheta)=\{x\in\mathbf{R}^4\,;\,x_1^2-x_2^2-x_3^2-x_4^2\geq 0,\ x_1\geq 0,\ x_4\geq 0\}. \tag{2.14}$$

In order to prove Theorem 2.1, we prove that $F(x)$ vanishes at any point of the interior of the propagation cone $\Gamma^\circ_0(P_+,\vartheta)$. Because of homogeneity of the integrand and rotational invariance in ζ_2,ζ_3, all we have to do is to prove that $F(a)$ vanishes for $a=(1,0,\alpha,\beta)$ where α,β are real numbers satisfying $1-\alpha^2-\beta^2>0,\ \beta>0$.

3. Preliminaries.

Let us evaluate $F(a)$ by using (2.9). We change the variables from ζ to σ by rotation, which makes calculation easy.

$$\begin{gathered}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}=\begin{pmatrix}\cos\theta_1 & 0 & 0 & -\sin\theta_1\\ 0&1&0&0\\ 0&0&1&0\\ \sin\theta_1 & 0 & 0 & \cos\theta_1\end{pmatrix}\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ 0&0&\cos\theta_2 & -\sin\theta_2\\ 0&0&\sin\theta_2&\cos\theta_2\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\\ \zeta_4\end{pmatrix},\\ \cos\theta_1=\frac{\sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2+1}},\quad \sin\theta_1=\frac{1}{\sqrt{\alpha^2+\beta^2+1}},\\ \cos\theta_2=\frac{\beta}{\sqrt{\alpha^2+\beta^2}},\quad \sin\theta_2=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}.\end{gathered} \tag{3.1}$$

By this change of variables, we have $a=(0,0,0,\sqrt{\alpha^2+\beta^2+1})$ in the new coordinates $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$. Hence $\delta^{(2)}(a\xi)=\delta^{(2)}(\sqrt{\alpha^2+\beta^2+1}\sigma_4)$. However, B becomes complicated by this rotation. Hence, we change the varibles again by linear transformation. Let us consider

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix} = \begin{pmatrix}\cos\theta_1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta_1\sin\theta_2 & 0 & \cos\theta_2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}. \tag{3.2}$$

By this linear transformation, B goes back to the original form $\eta_1^2 - \eta_2^2 - \eta_3^2$ when $\eta_4 = 0$ (see Lemma 3.1). For a function $Q(\zeta)$ in ζ, we denote $\bar{Q}(\eta)$ the function $Q(\zeta(\eta))$ in η given by the transformations (3.1) and (3.2).

Lemma 3.1. *$F(a)$ can be written as*

$$\begin{aligned} F(a) &= (2\pi i)^{-3}\beta^{-1}(\alpha^2+\beta^2+1)^{-1} \\ &\quad \times \int_{\bar{\gamma}(\xi',0)=1} \Big(-\frac{\partial}{\partial\xi_4}\Big)^2 \bar{P}_+(\eta',\xi_4)^{-1}\Big|_{\xi_4=0}\omega_3(\eta'), \quad \eta' = \xi' - i\bar{v}(\xi',0)' \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \bar{P}_+(\eta',\xi_4) &= -\frac{1}{\beta}\eta_1 - \frac{\alpha}{\beta}\eta_3 + \frac{\beta}{\sqrt{\alpha^2+\beta^2+1}}\xi_4 \\ &\quad - \sqrt{\Big(\eta_1 + \xi_4/\sqrt{\alpha^2+\beta^2+1}\Big)^2 - \eta_2^2 - \Big(\eta_3 + \alpha\xi_4/\sqrt{\alpha^2+\beta^2+1}\Big)^2}. \end{aligned} \tag{3.4}$$

Proof. ω_4 is invariant by (3.1). By (3.2), ω_4 is multiplied by $\beta^{-1}\sqrt{\alpha^2+\beta^2+1}$. Therefore, from (2.9),

$$\begin{aligned} F(a) &= (2\pi i)^{-3}\beta^{-1}\sqrt{\alpha^2+\beta^2+1} \\ &\quad \times \int_{\bar{\gamma}(\xi)=1} \delta^{(2)}(\sqrt{\alpha^2+\beta^2+1}\,\xi_4)\bar{P}_+(\eta)^{-1}\,\omega_4(\eta), \quad \eta = \xi - i\bar{v}(\xi). \end{aligned} \tag{3.5}$$

We remark $\bar{v}_4 = 0$ by the condition (2.11). Thus we have

$$\omega_4(\eta) = \omega_3(\eta') \wedge d\xi_4 - \xi_4 d\eta_1 \wedge d\eta_2 \wedge d\eta_3, \quad \eta' = \xi' - i\bar{v}(\xi)'. \tag{3.6}$$

Since the support of integrand is $\{\xi \in \mathbf{R}^4\,;\, \xi_4 = 0\}$, we can make $\bar{\gamma}(\xi)$ independent of ξ_4. Also by the lower semi-continuity of $\Gamma_\xi(P_+,\vartheta)$ with respect to ξ, we can make $\bar{v}(\xi)$ independent of ξ_4 near $\{\xi \in \mathbf{R}^4\,;\, \xi_4 = 0\}$. Thus we have

$$\omega_4(\eta) = \omega_3(\eta') \wedge d\xi_4, \quad \eta' = \xi' - i\bar{v}(\xi',0)' \tag{3.7}$$

near $\{\xi \in \mathbf{R}^4\,;\, \bar{\gamma}(\xi) = 1,\ \xi_4 = 0\}$. In terms of integration by parts with respect to ξ_4, we can obtain (3.3). (3.4) follows from (3.1),(3.2) and $P_+(\zeta) = \zeta_4 - \sqrt{\zeta_1^2 - \zeta_2^2 - \zeta_3^2}$. □

4. Proof of Theorem 2.1.

In Lemma 3.1, $F(a)$ was transformed into a convenient form to calculate. By selecting $\bar{\gamma}(\xi',0) = |\xi'|$, let us show $F(a) = 0$.

We introduce spherical coordinates on $\{\xi' \in \mathbf{R}^3\,;\, |\xi'| = 1\}$, that is,

$$\xi_1 = \cos\theta,\ \xi_2 = \sin\theta\cos\phi,\ \xi_3 = \sin\theta\sin\phi \quad (0 \le \theta \le \pi,\ 0 \le \phi < 2\pi). \tag{4.1}$$

Let us define $\bar{v}(\xi',0)'$ as

$$\begin{aligned} \bar{v}_1(\xi',0) &= -\varepsilon(\xi_1^2 - \xi_2^2 - \xi_3^2)/|\xi'| = -\varepsilon\cos 2\theta, \\ \bar{v}_2(\xi',0) &= -2\varepsilon\xi_1\xi_2/|\xi'| = -\varepsilon\sin 2\theta\cos\phi, \\ \bar{v}_3(\xi',0) &= -2\varepsilon\xi_1\xi_3/|\xi'| = -\varepsilon\sin 2\theta\sin\phi \end{aligned} \tag{4.2}$$

near $\{\xi' \in \mathbf{R}^3 \,;\, |\xi'| = 1,\ \xi_4 = 0\}$. Here $\varepsilon(> 0)$ is sufficiently small. Let us show that $\bar{v}(\xi', 0)'$ satisfies the conditions (2.10), (2.11) and (2.12).

Lemma 4.1. *$\bar{v}(\xi', 0)'$ in (4.2) satisfies the following conditions:*

(i) When $\lambda \in \mathbf{R} \setminus \{0\}$,

$$\bar{v}(\lambda\xi', 0)' = |\lambda|\bar{v}(\xi', 0)'. \tag{4.3}$$

(ii) For any $\xi' \in \{\xi' \in \mathbf{R}^3 \,;\, |\xi'| = 1\}$,

$$(\bar{v}(\xi', 0)', 0) \in \Gamma_{(\xi',0)}(\bar{P}_+, \bar{\vartheta}). \tag{4.4}$$

(iii) When $\xi' \in \{\xi' \in \mathbf{R}^3 \,;\, |\xi'| = 1\},\ 0 < t \le 1$,

$$\bar{P}_+(\xi' - it\bar{v}(\xi', 0)', 0) \neq 0. \tag{4.5}$$

Here $\bar{\vartheta}$ is ϑ after the transformations (3.1) *and* (3.2)*, that is,*

$$\bar{\vartheta} = (\cos^2\theta_1, 0, -\sin\theta_1\cos\theta_1\sin\theta_2, \sin\theta_1). \tag{4.6}$$

Proof. (4.3) is trivial. (4.5) follows from the definition of localization. Therefore, let us show (4.4).

Remember that $a = (1, 0, \alpha, \beta)$ is in the propagation cone. Hence localization of P at $\{\xi \in \mathbf{R}^4 \setminus \{0\} \,;\, a\xi = 0\}$ is non-zero constant and so is P_+. Therefore, localization of $\bar{P}_+$ is also non-zero constant at $\{\xi \in \mathbf{R}^4 \setminus \{0\} \,;\, \xi_4 = 0\}$ and $\Gamma_{(\xi',0)}(\bar{P}_+, \bar{\vartheta})$ coincides with $\Gamma_{(\xi',0)}(\bar{B}, \bar{\vartheta})$ when $\xi' \in \{\xi' \in \mathbf{R}^3 \,;\, |\xi'| = 1\}$. $\bar{B}(\xi', 0) = 0$ if and only if $\theta = \pi/4, 3\pi/4$ in (4.1). When $\xi' \in \{\xi' \in \mathbf{R}^3 \,;\, |\xi'| = 1,\ \bar{B}(\xi', 0) = 0\}$,

$$\begin{aligned}\bar{B}_{(\xi',0)}(\zeta) = 2\{&\xi_1(\zeta_1 + \zeta_4/\sqrt{\alpha^2 + \beta^2 + 1}) \\ &- \xi_2\zeta_2 - \xi_3(\zeta_3 + \alpha\zeta_4/\sqrt{\alpha^2 + \beta^2 + 1})\}\end{aligned} \tag{4.7}$$

and $\bar{B}_{(\xi',0)}(\bar{v}(\xi', 0)', 0) > 0,\ \bar{B}_{(\xi',0)}(\bar{\vartheta}) > 0$. Hence (4.4) holds. □

¿From (3.4), we have

$$\begin{aligned}\Big(-\frac{\partial}{\partial\xi_4}\Big)^2 \bar{P}_+(\eta', \xi_4)^{-1}\Big|_{\xi_4=0} = &\frac{\beta^2\{(1-\alpha^2)B(\eta') - (\eta_1 - \alpha\eta_3)^2\}}{(\alpha^2 + \beta^2 + 1)(\eta_1 + \alpha\eta_3 + \beta\sqrt{B(\eta')})^2\sqrt{B(\eta')}^3} \\ &- \frac{2\beta^3(\beta\sqrt{B(\eta')} - \eta_1 + \alpha\eta_3)^2}{(\alpha^2 + \beta^2 + 1)(\eta_1 + \alpha\eta_3 + \beta\sqrt{B(\eta')})^3 B(\eta')}\end{aligned} \tag{4.8}$$

where $\eta' = \xi' - i\bar{v}(\xi', 0)'$. By using (4.8) we prove Theorem 2.1.

Proof of Theorem 2.1. We have only to show $F(a) = 0$. Let us evaluate the integral part in (3.3). Setting

$$\begin{aligned}\eta^{\pm}_{\varepsilon,1}(\theta) &= \cos\theta \pm i\varepsilon\cos 2\theta, \\ \eta^{\pm}_{\varepsilon,2}(\theta, \phi) &= \sin\theta\cos\phi \pm i\varepsilon\sin 2\theta\cos\phi, \\ \eta^{\pm}_{\varepsilon,3}(\theta, \phi) &= \sin\theta\sin\phi \pm i\varepsilon\sin 2\theta\sin\phi, \\ {\eta^{\pm}_{\varepsilon}}'(\theta, \phi) &= (\eta^{\pm}_{\varepsilon,1}(\theta), \eta^{\pm}_{\varepsilon,2}(\theta, \phi), \eta^{\pm}_{\varepsilon,3}(\theta, \phi))\end{aligned} \tag{4.9}$$

by spherical coordinates, we can obtain

(4.10)
$$\begin{aligned}&\int_{|\xi'|=1}\Big(-\frac{\partial}{\partial\xi_4}\Big)^2\bar{P}_+(\eta_\varepsilon^{+'}(\theta,\phi),\xi_4)^{-1}\Big|_{\xi_4=0}\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))\\
&\underset{(4.8)}{=}\int_0^{2\pi}\int_0^{\pi}\Big[\frac{\beta^2\{(1-\alpha^2)B(\eta_\varepsilon^{+'}(\theta,\phi))-(\eta_{\varepsilon,1}^{+}(\theta)-\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2\}}{(\alpha^2+\beta^2+1)\{\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi)+\beta\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\}^2\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}^3}\\
&-\frac{2\beta^3(\beta\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}-\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2}{(\alpha^2+\beta^2+1)\{\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi)+\beta\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\}^3B(\eta_\varepsilon^{+'}(\theta,\phi))}\Big]\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))\end{aligned}$$

where

$$\begin{aligned}B(\eta_\varepsilon^{+'}(\theta,\phi))&=\eta_{\varepsilon,1}^{+}(\theta)^2-\eta_{\varepsilon,2}^{+}(\theta,\phi)^2-\eta_{\varepsilon,3}^{+}(\theta,\phi)^2\\
&=\cos2\theta-\varepsilon^2\cos4\theta+2i\varepsilon\cos3\theta,\\
\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))&=\eta_{\varepsilon,1}^{+}(\theta)d\eta_{\varepsilon,2}^{+}(\theta,\phi)\wedge d\eta_{\varepsilon,3}^{+}(\theta,\phi)\\
&\quad-\eta_{\varepsilon,2}^{+}(\theta,\phi)d\eta_{\varepsilon,1}^{+}(\theta)\wedge d\eta_{\varepsilon,3}^{+}(\theta,\phi)\\
&\quad+\eta_{\varepsilon,3}^{+}(\theta,\phi)d\eta_{\varepsilon,1}^{+}(\theta)\wedge d\eta_{\varepsilon,2}^{+}(\theta,\phi)\\
&=\sin\theta(1+2i\varepsilon\cos\theta)(1+3i\varepsilon\cos\theta-2\varepsilon^2)\,d\theta\wedge d\phi.\end{aligned}\tag{4.11}$$

Let us divide the range of θ integration into three intervals $0\le\theta\le\frac{\pi}{4}-\delta$, $\frac{\pi}{4}-\delta\le\theta\le\frac{3}{4}\pi+\delta$ and $\frac{3}{4}\pi+\delta\le\theta\le\pi$, $(0<\delta\ll1)$ because $B(\eta_0^{+'}(\theta,\phi))$ has zeros at $\theta=\pi/4,3\pi/4$.

We first evaluate θ integration from $\frac{3}{4}\pi+\delta$ to π. Let $\theta=-\tau+\pi$, $\phi=-\psi+2\pi$ and rewrite τ,ϕ with θ,ψ, then we have

(4.12)
$$\begin{aligned}&\int_0^{2\pi}\int_{\frac{3}{4}\pi+\delta}^{\pi}\Big(-\frac{\partial}{\partial\xi_4}\Big)^2\bar{P}_+(\eta_\varepsilon^{+'}(\theta,\phi),\xi_4)^{-1}\Big|_{\xi_4=0}\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))\\
&=\int_0^{2\pi}\int_0^{\frac{\pi}{4}-\delta}\Big[\frac{\beta^2\{(1-\alpha^2)B(\eta_\varepsilon^{-'}(\theta,\phi))-(\eta_{\varepsilon,1}^{-}(\theta)-\alpha\eta_{\varepsilon,3}^{-}(\theta,\phi))^2\}}{(\alpha^2+\beta^2+1)\{\eta_{\varepsilon,1}^{-}(\theta)+\alpha\eta_{\varepsilon,3}^{-}(\theta,\phi)-\beta\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}\}^2\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}^3}\\
&+\frac{2\beta^3(\beta\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}+\eta_{\varepsilon,1}^{-}(\theta)-\alpha\eta_{\varepsilon,3}^{-}(\theta,\phi))^2}{(\alpha^2+\beta^2+1)\{\eta_{\varepsilon,1}^{-}(\theta)+\alpha\eta_{\varepsilon,3}^{-}(\theta,\phi)-\beta\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}\}^3B(\eta_\varepsilon^{-'}(\theta,\phi))}\Big]\omega_3(\eta_\varepsilon^{-'}(\theta,\phi)).\end{aligned}$$

Here we remark

$$\begin{aligned}\mathrm{Re}\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}&=-\mathrm{Re}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))},\\
\mathrm{Im}\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}&=\mathrm{Im}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\end{aligned}\tag{4.13}$$

by selection of the branch $\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}=\sqrt{\eta_{\varepsilon,1}^{+}(\theta)^2-\eta_{\varepsilon,2}^{+}(\theta,\phi)^2-\eta_{\varepsilon,3}^{+}(\theta,\phi)^2}$.

By Stokes' formula, (4.10) does not depend on ε. Therefore, we can let $\varepsilon\to+0$, then $\mathrm{Im}\sqrt{B(\eta_\varepsilon^{-'}(\theta,\phi))}$ and $\mathrm{Im}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}$ tend to 0. Thus, it follows that

$$\int_0^{2\pi}\int_I\Big(-\frac{\partial}{\partial\xi_4}\Big)^2\bar{P}_+(\eta_\varepsilon^{+'}(\theta,\phi),\xi_4)^{-1}\Big|_{\xi_4=0}\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))\to0\quad\text{as }\varepsilon\to+0\tag{4.14}$$

where $I=[0,\frac{\pi}{4}-\delta]\cup[\frac{3}{4}\pi+\delta,\pi]$.

Let us evaluate θ integration from $\frac{\pi}{4}-\delta$ to $\frac{3}{4}\pi+\delta$. We remark (4.10) does not also depend on δ. Therefore, by rationalization of denominator and integration by parts, we can obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_{\frac{\pi}{4}-\delta}^{\frac{3}{4}\pi+\delta}\Big(-\frac{\partial}{\partial\xi_4}\Big)^2\bar{P}_+(\eta_\varepsilon^{+'}(\theta,\phi),\xi_4)^{-1}\Big|_{\xi_4=0}\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))\\
&=\int_0^{2\pi}\int_{\frac{\pi}{4}-\delta}^{\frac{3}{4}\pi+\delta}\frac{-\beta^2(\eta_{\varepsilon,1}^{+}(\theta)^2-\alpha^2\eta_{\varepsilon,3}^{+}(\theta,\phi)^2)^2(\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2}{(\alpha^2+\beta^2+1)\{(\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2-\beta^2B(\eta_\varepsilon^{+'}(\theta,\phi))\}^3}\\
&\times\frac{1}{(\sin 2\theta-2\varepsilon^2\sin 4\theta+3i\varepsilon\sin 3\theta)}\times\frac{\partial}{\partial\theta}\Big(\frac{1}{\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}}\Big)\omega_3(\eta_\varepsilon^{+'}(\theta,\phi))+o(\delta)\\
&=\int_0^{2\pi}\Big[\frac{-\beta^2(\eta_{\varepsilon,1}^{+}(\theta)^2-\alpha^2\eta_{\varepsilon,3}^{+}(\theta,\phi)^2)^2(\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2}{(\alpha^2+\beta^2+1)\{(\eta_{\varepsilon,1}^{+}(\theta)+\alpha\eta_{\varepsilon,3}^{+}(\theta,\phi))^2-\beta^2B(\eta_\varepsilon^{+'}(\theta,\phi))\}^3}\\
&\times\frac{\sin\theta(1+2i\varepsilon\cos\theta)(1+3i\varepsilon\cos\theta-2\varepsilon^2)}{(\sin 2\theta-2\varepsilon^2\sin 4\theta+3i\varepsilon\sin 3\theta)\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}}\Big]_{\frac{\pi}{4}-\delta}^{\frac{3}{4}\pi+\delta}d\phi+o(\delta)\quad\text{as }\delta\to+0.
\end{aligned}\tag{4.15}$$

By using the change of variable $\phi=-\psi+2\pi$ used in (4.12) and

$$\begin{aligned}
\operatorname{Re}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\Big|_{\theta=\frac{3}{4}\pi+\delta}&=-\operatorname{Re}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\Big|_{\theta=\frac{\pi}{4}-\delta},\\
\operatorname{Im}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\Big|_{\theta=\frac{3}{4}\pi+\delta}&=\operatorname{Im}\sqrt{B(\eta_\varepsilon^{+'}(\theta,\phi))}\Big|_{\theta=\frac{\pi}{4}-\delta},
\end{aligned}\tag{4.16}$$

the first term of the last expression in (4.15) tends to 0 as ε tends to $+0$.

This just means that Huygens' principle holds for the Dirichlet boundary value problem (2.1). □

There is another proof, according to M. Uchida*. It is very simple. Here let us introduce his proof. We set

$$G(x)=(2\pi)^{-4}i^{-1}\int_{\mathbf{R}^4-i\vartheta}e^{ix\zeta}P_-(\zeta)^{-1}\,d\zeta.\tag{4.17}$$

Here $P_-(\zeta)=\zeta_4+\sqrt{\zeta_1^2-\zeta_2^2-\zeta_3^2}$ and the branch of $\sqrt{\zeta_1^2-\zeta_2^2-\zeta_3^2}$ is the same as before. Then we have

$$\begin{aligned}
F(x)\Big|_{x_4>0}&=F(x)+G(x)\\
&=-2(2\pi)^{-4}i^{-1}\int_{\mathbf{R}^4-i\vartheta}e^{ix\zeta}\zeta_4P(\zeta)^{-1}\,d\zeta\\
&=-2(2\pi)^{-4}i^{-1}D_4\int_{\mathbf{R}^4-i\vartheta}e^{ix\zeta}P(\zeta)^{-1}\,d\zeta
\end{aligned}\tag{4.18}$$

because of $G(x)=0$ in $\{x\in\mathbf{R}^4\,;\,x_4>0\}$. Consequently our problem is reduced to Huygens' principle for the initial value problem which is well-known. However, this is only applied to the case where $P(D)$ is the wave operator.

On the other hand, our proof can be applied to some other cases. For instance, we can evaluate $F(x)$ concretely when the integrand is transformed into rational function of one variable. There exists such a case because we can deform the chain of integration by the Stokes' formula. In the case of $n=4$, the integrand becomes one variable if we take a cylinder as $\{\xi'\in\mathbf{R}^3\,;\,\bar{\gamma}(\xi',0)=1\}$, introduce cylindrical coordinates and take $\bar{v}(\xi',0)'=0$

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out of a neighborhood of zero and the branch locus of P_+. In a neighborhood of them, our calculation becomes complex integration of one variable by taking $\bar{v}(\xi', 0)'$ so that the integral path does not meet them. This is possible because the integrand is closed form.

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