LUKASIEWICZ FINITELY LOCAL ALGEBRAS

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Abstract. In this paper finitely local Ł-algebras are introduced as a generalization of quasi-local Ł-algebras. The class of finitely local algebras includes the semilocal Ł-algebras. Some properties are studied and characterizations are given

1 Introduction A L-algebra is said to be local if it has a unique maximal ideal [5]. Local L-algebras are also characterized as the L-algebre where for each element x exists a positive integer n such that $nx = 1$ or $nx' = 1$ [1].
A generalization of these algebras are semilocal L-algebras which are defined and studied in [4]. They are characterized as the L-algebre with finitely many maximal ideals. Another generalization, that arises from the second characterization of the local L-algebras, are the quasi-local L-algebras in which for each element x exists a positive integer n such that nx or nx' is a boolean element [7].
These two generalizations, semilocal and quasi-local, are independent, i.e. there are semilocal algebras that are not quasi-local and vice versa.
In this paper we define a new class of L-algebras, called finitely local, containing both quasi-local and semilocal L-algebras.

2 Preliminaries Following [9] we recall that a L-algebra $< A, +, ' ; 0, 1 >$ (Łukasiewicz-algebra or MV-algebra [1], [2], [4]) is a sistem such that, $\forall x, y \in A$,

1) $< A, +, 0 >$ is an Abelian monoid
2) $x + 1 = 1$
3) $(x')' = x$
4) $0' = 1$
5) $x + x' = 1$
6) $(x' + y)' + y = (x + y')' + x$.

Setting as well

i) $x \cdot y = (x' + y')'$
ii) $x \vee y = (x' + y)' + y$
iii) $x \wedge y = (x' \vee y')'$
iv) $x \leq y$ if and ony if $x' + y = 1$.

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The structure $< A, \vee, \wedge, \leq; 0, 1 >$ is a bounded distributive lattice.
A L-algebra A is said to be a L-chain if the $\leq$ order is linear. Every L-algebra is a subdirect product of L-chains [2].
Given a L-algebra A, let B_A denote the set of its boolean (idempotent) elements, i.e. the set of all $x \in A$ with $2x = x$. The set B_A is a Boolean subalgebra of A [2] and, $\forall x, y \in B_A$, $x + y = x \vee y$ and $x \cdot y = x \wedge y$.
A non-empty subset $I \subseteq A$ is an ideal if it is closed under $+$ and if $x \in I$, $y \in A$ with $y \leq x$ imply $y \in I$.
For $a, b \in A$, $a \leq b$, let $A_{a,b} = \{x \in A : a \leq x \leq b\}$.
The system $< A_{a,b}, \oplus, \bar{'}; a, b >$ is a L-algebra with respect to the following operations:

$$x \oplus y = a + [(x' + a)(y' + a)]'(b' + a)'$$
$$x^{\bar{'}} = a + (b' + x)'$$

Every L-algebra is isomorphic to $A_{0,b'} \times A_{0,b}$, where $b \in B_A$ with $b \neq 0, 1$.
Recall that an element $a \in A$ has finite order n if n is the least positive integer such that $na = 1$ and we write $\text{ord}(a) = n$. If no such n exists, we say that a has infinite order.
An element $a \in A$ is said to be quasi-archimedean if $na \in B_A$ for some integer $n > 0$. If no such n exists the element a is said to be non-archimedean.
A L-algebra A is said to be
– local if, for each $x \in A$, ord$x$ or ordx' is finite, i.e. if and only if A has a unique maximal ideal;
– quasi-local if, for each $a \in A$, a or a' is quasi-archimedean;
–semilocal if it has only finitely many maximal ideals (we refer the reader to [5], [7] and [4]).

3 Finitely local L-algebras

Definition 1 *A L-algebra A is called n-local, $n \geq 2$, if the following properties hold:*

1) *For each non-archimedean $x \in A$ there exist $b \in B_A$, $b \neq 0$, and a positive integer m such that $mx \wedge b$ is non-archimedean and $mx' \wedge b$ is boolean;*

2) *For any $b_1, b_2, ..., b_n \in B_A$, for which $b_i \wedge b_j = 0$ $\forall i \neq j$, there exists k, $1 \leq k \leq n$, such that $x < b_k$ is not true for every non-archimedean element $x \in A$.*

A is called 1-local if it is a local L-algebra.
Hence, we say that A is a finitely local L-algebra if it is n-local for some $n \geq 1$.
We remark that if A is n-local, then it is m-local for any $m \geq n$.
Throughout this paper, Rad(A) denotes the radical of A, that is, the intersection of all maximal ideals of A and π denotes the canonical omomorphism of A on $\frac{A}{\text{Rad}(A)}$.

Lemma 1 *Let A be a L-algebra. Then B_A is isomorphic to $B_{\frac{A}{Rad(A)}}$.*

Proof. It is suffices to check that if $[b]$ is a boolean element of $\frac{A}{\text{Rad}(A)}$, then there exists a unique $y \in B_A$ such that $\pi(y) = [b]$.
Let x be an element of A such that $\pi(x) = [b]$ and put $y = (2(2x)')'$. Thus

$$\pi(y) = \pi((2(2x)')') = (2(2\pi(x))')' = \pi(x) = [b]. \tag{1}$$

Now we note that

$$
\begin{aligned}
((2y)'+y)' &= ((2(2(2x)')')'+(2(2x)')')' = ((2(2x)'+2(2x)')'+2(2x)')' = \\
&= (((3(2x)')'+2x)'+(3(2x)')'+(2x)')' = \\
&= (((3(2x)')'+2x)'+((2(2x)')'+2x)'+2x)'. \qquad (2)
\end{aligned}
$$

Since

$$\pi(3(2x)')'+2x) = \pi(x) \text{ and } \pi((2(2x)')'+2x) = \pi(x) \qquad (3)$$

yield

$$((3(2x)')'+2x)'+x)' \in \mathrm{Rad}(A) \text{ and } (((2(2x)')'+2x)'+x)' \in \mathrm{Rad}(A); \qquad (4)$$

we obtain $((2y)'+y)' = 0$ i.e. $2y = y$.
The uniqueness of y follows from the fact that the $\mathrm{Rad}(A)$ contains only one boolean element, that is 0. □

Proposition 2 *A is a semilocal L-algebra if and only if A is direct product of finitely many local L-algebras.*

Proof. Let A be direct product of finitely many local L-algebras. Then, by Theorem 2.6 in [4], A is semilocal.
Conversely, suppose that A is a semilocal L-algebra. Thus the L-algebra $\frac{A}{\mathrm{Rad}(A)}$ is a direct product of finitely many simple L-chains (see [4]).
Let $[b_1], [b_2], ..., [b_n]$ be the atoms of $B_{\frac{A}{\mathrm{Rad}(A)}}$. By Lemma 7, we can suppose that $b_1, b_2, ..., b_n$ are atoms of B_A. Then A is isomorphic to the direct product $A_{0,b_1} \times A_{0,b_2} \times A_{0,b_n}$, where each $A_{0,b_i} \simeq \frac{A}{<b_i>}$, $i = 1, 2, ..., n$, is semilocal. Since $B_{A_{0,b_i}} = \{0, b_i\}$, the L-algebras A_{0,b_i} are local. □

Proposition 3 *Let A be a semilocal L-algebra. Then A is a finitely local L-algebra.*

Proof. Take $b_1, b_2, ..., b_n \in B_A$ as in Proposition 1. Let x be a non-archimedean element of A and let $B_x = \{b_i : x \wedge b_i \text{ is non-archimedean}\}$.
We remark that B_x is not empty: otherwise, by Proposition 7 *ii*) in [7], the element $\bigvee(x \wedge b_i) = x$ would be quasi-archimedean.
Thus, put $b = \bigvee\{b_i : b_i \in B_x\}$, the element $x \wedge b$ is non-archimedean. But, since for each b_i, $i = 1, 2, ..., n$, $x \wedge b_i$ or $x' \wedge b_i$ is quasi-archimedean, the element $x' \wedge b$ is quasi-archimedean.
Hence we conclude that A is a m-local L-algebra with $m \leq n$. □

Proposition 4 *A is a quasi-local L-algebra if and only if it is a 2-local L-algebra.*

Proof. If A is a quasi-local L-algebra, then the claim follows from Proposition 8 in [7].
Suppose that A is a 2-local L-algebra. Let x be a non-archimedean element of A. By hypothesis, there exist $b \in B_A$, $b \neq 0$, and a positive integer m such that $mx \wedge b$ is non-archimedean and $mx' \wedge b$ is boolean.
Since, by (2) of Definition 1, $mx \wedge b$ non-archimedean gives $mx' \wedge b'$ quasi-archimedean, then the element $mx' = (mx' \wedge b) \vee (mx' \wedge b')$ is quasi-archimedean. □

Let $I_A = \{x \in A : x \wedge z \text{ is quasi-archimedean}, \forall z \in A\}$.
We show that

Proposition 5 *I_A is an ideal of A.*

Proof. Take $x \in I_A$ and $y \in A$ with $y \leq x$. Since $y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$ is quasi-archimedean, the element y lies in I_A.
Let $x, y \in I_A$, then we can find an integer $n \geq 1$ such that $nx, ny \in B_A$. Since $x \wedge z$ and $y \wedge z$ are quasi-archimedean, from $n((x+y) \wedge z) = (nx + ny) \wedge nz = (nx \vee ny) \wedge nz = (nx \wedge nz) \vee (ny \wedge nz) = n((x \wedge z) \vee (y \wedge z))$ follows that the element $(x + y) \wedge z$ is also quasi-archimedean. Hence $x + y \in I_A$. □

Proposition 6 *Let A be a n-local L-algebra which is not $(n-1)$-local. Then there exist $b_1, b_2, ..., b_{n-1} \in B_A$, with $b_i \wedge b_j = 0$ for $i \neq j$, with the following properties*

1) $(\bigvee b_i)' \in I_A$

2) *For each x non-archimedean, there is $0 < j < n$ such that $x \wedge b_j$ is non-archimedean.*

Proof. Since A is not $(n-1)$-local, there exist $b_1, b_2, ..., b_{n-1} \in B_A$, $b_i \wedge b_j = 0$ for $i \neq j$, such that for each b_i there is a non-archimedean element $x \in A$ with $x < b_i$. Since A is n-local, for each non-archimedean element $x \in A$, $x \not< (\bigvee b_i)'$. Then $(\bigvee b_i)' \in I_A$. This implies that the element $(x \wedge (\bigvee b_i)')$ is quasi-archimedean, for each x non-archimedean.
Then, since $x = x \wedge (b_1 \vee b_2 \vee ... \vee b_{n-1} \vee (\bigvee b_i)') = (x \wedge b_1) \vee .. \vee (x \wedge b_{n-1}) \vee (x \wedge (\bigvee b_i)')$, at least one element $x \wedge b_j$ must be non-archimedean. □

Theorem 7 *A is a finitely local L-algebra if and only if $\frac{A}{I_A}$ is a semilocal L-algebra.*

Proof. Suppose that $\overline{A} = \frac{A}{I_A}$ is semilocal. First we show that if $[x] \in B_{\frac{A}{I_A}}$, then there exists $b \in B_A$ such that $[x] = [b]$. From $[x] = 2[x]$ we have $z = ((2x)' + x)' \in I_A$. Let n be the positive integer such that $nz \in B_A$. Then $x + nz = x + (n+1)z = 2x + nz = 2x + 2nz = 2(x + nz)$, that is $x + nz \in B_A$ and $[x + nz] = [x]$.
Now let x be a non-archimedean element of A.
We remark that $[x]$ is also non-archimedean: otherwise would exist a non-archimedean $a \in A$ and $b \in B_A$ such that $[a] = [b]$. This implies that there is a boolean $c \in I_A$ such that $a + c = a \vee c = b$ which gives $(c' \wedge (a \vee c)) \vee (a \wedge c) = a$. Hence $a \wedge c$ would be non-archimedean which is a contradiction being $c \in I_A$.
Since $\frac{A}{I_A}$ is semilocal, we can take the atoms $[b_1], [b_2], ..., [b_n]$ of $B_{\frac{A}{I_A}}$ (see proposition 1). Then, from $[x] = ([b_1] \wedge [x]) \vee ([b_2] \wedge [x]) \vee ... \vee ([b_n] \wedge [x])$, follows that there is at least a $[b_i] \wedge [x]$ which is non-archimedean. Hence $b_i \wedge x$ is non-archimedean.
On the other hand, since $\overline{A}_{0,b_i}$ is local, we have $[b_i] \wedge [x'] = [b_i]$ which implies $b_i \wedge x'$ quasi-archimedean. Now it is easy to conclude that A is $n+1$-locale.
Conversely, suppose A finitely local. Let $b_1, b_2, ..., b_{n-1} \in B_A$ as in the above proposition. Let J_i be the ideal generated by

$$H_i = \{b \in B_A : b \wedge (x \wedge b_i) \text{ quasi-archimedean } \forall x\} \cup \{x \wedge b_i \text{ non-archimedean}\}.$$

We show that J_i is a maximal ideal of A containing I_A. Let $\overline{J}$ be an ideal of A with $J_i \subset \overline{J}$. Take $a \in \overline{J} - J_i$. This element a is quasi-archimedean: otherwise, by proposition 5, we would have $a < b_j$, for $j \neq i$, which implies $a \in J_i$, since $b_j \in H_i$. Then there exists n such that $na = \overline{b} \in B_A$. Since $\overline{b} \notin H_i$, there is an element z of A such that $\overline{z} = \overline{b} \wedge z \wedge b_i$ is non-archimedean. If $\overline{b}' \wedge x \wedge b_i$ is quasi-archimedean for each x, then $\overline{b}' \in H_i$ wich gives $1 \in \overline{J}$, that is $\overline{J} = A$. Conversely, if there is $y \in A$ with $\overline{b}' \wedge y \wedge b_i = \overline{y}$ non-archimedean, then $\overline{b}' \wedge b_i > \overline{y}$, $\overline{b} \wedge b_i > \overline{z}$ and $(\overline{b}' \wedge b_i) \wedge (\overline{b} \wedge b_i) = 0$. Since A is n-local, we obtain a contradiction.
Now we take a maximal ideal J of A different from $J_1, J_2, ..., J_{n-1}$ and show that $I_A \not\subseteq J$.

If, for each $i = 1, 2, ...n-1$, $b_i \in J$, then $(\bigvee b_i)' \notin J$ which implies $I_A \nsubseteq J$.
Hence we can suppose that there exists $1 \leq k \leq n-1$ such that $b_k \notin J$. Since $H_k \nsubseteq J$, we distinguish two cases:
1) There is $b \in B_A$ such that $b \wedge (x \wedge b_k)$ is quasi-archimedean, for each $x \in A$, and $b \notin J$;
2) There is $x \wedge b_k$ non-archimedean such that $x \wedge b_k \notin J$.
In the first case, being J a prime ideal of A, the element $b \wedge b_k$ lies in I_A but not in J.
In the second case, by Theorem 4.7 in [2], there is a positive integer n such that $n(x \wedge b_k)' \in J$. Since $(x \wedge b_k)' = x' \vee b'_k = (x' \wedge b_k) \vee b'_k$, the element $(x \wedge b_k)'$ is quasi-archimedean. Thus, for some integer m, $b = m(x \wedge b_k)' \in J \cap B_A$ that is $b' \notin J$. We remark that $b + b_k = m(x \wedge b_k)' + b_k = mx' \vee b'_k + b_k = 1$ that is $b' \leq b_k$. Thus we have $b_k \wedge x = [(b \wedge b_k) \vee b'] \wedge x = [(b \wedge b_k) \wedge x] \vee (b' \wedge x)$. Since $b' \wedge x$ is quasi-archimedean, the above relation gives $(b \wedge b_k) \wedge x$ non-archimedean. It follows that, for each $y \in A$, the element $b' \wedge y$ is quasi-archimedean, hence $b' \in I_A$ and $b' \notin J$.
We can conclude that $\frac{A}{I_A}$ has n maximal ideals that is, by [4], it is semilocale. □

Corollary 8 *A L-algebra A is quasi-local if and only if* $\frac{A}{I_A}$ *is a local L-algebra.*

Theorem 9 *A L-algebra A is finitely local if and only if it is isomorphic to a direct product of finitely many quasi local L-algebras.*

Proof. We remark that if $b \in B_A$, then $I_{A_{0,b}} = \{x \wedge b : x \in I_A\}$. Hence the claim follows by the propositions 1 and 6 and the corollary 1. □

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