# A TWO-VERSUS-ONE SILENT DUEL WITH EQUAL ACCURACY FUNCTIONS UNDER ARBITRARY MOTION

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Abstract. In this paper we examine a two-person zero-sum timing game with the following structure: Player I has a gun with two bullets and player II has a gun with one bullet and they fight a duel. Both guns are silent so that neither player can determine whether his opponent has fired the bullets or not. Player I is at the place 0 at the moment when the duel begins and he can move as he likes and player II is always at the place 1. Accuracy functions, which denote the probability of hitting the opponent when each player fires his bullet, are identical for both players. If player I hits player II without being hit himself first, then the payoff is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0.

The objective of this paper is to obtain the game value and the optimal strategies for the timing game.

## 1. Introduction

A duel under arbitrary motion is a two-person zero-sum timing game with the following structure: Each of two competitors, denoted by player I and player II, has a gun and he can fire his bullets aiming at his opponent. At the moment when the duel begins these two players are one distance apart on a line and each player can move on the line as he likes. The maximum speed of player I is $v_1$, the maximum speed of player II is $v_2$ and we assume $v_1 > v_2 \geqq 0$. Without loss of generality, we can suppose $v_1 = 1$ and $v_2 = 0$, and hence, player II is motionless. Thus we assume that player II is at the place 1 all the time and player I is at the place 0 at the moment when the duel begins and he can move towards player II, he can move away from player II, and he can stay in one place. If player I or player II fires his bullet when player I is at a place $x$, he hits his opponent with probability $p(x)$ or $q(x)$, respectively. The functions $p(x)$ and $q(x)$ are called accuracy functions for players I and II, respectively, and they are continuous and strictly increasing on $[0, 1]$ with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$. The duel ends when at least one player is hit or both players fire all of their bullets; otherwise it continues indefinitely. The gun is said to be silent if the shot of the owner is not heard by his opponent and the gun is said to be noisy if the shot of the owner is heard by his opponent as soon as the owner of the gun fires the bullet. Thus if a player has a silent gun, then his opponent does not know whether the owner of the gun has fired or not. On the other hand, if a player has a noisy gun, then his opponent always knows whether the owner has fired or not. If each player has a silent gun, the duel is said to be silent and if each player has a noisy gun, the duel is said to be noisy. If player I hits player II without being hit himself first, then the payoff of the duel is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0. The objective of player I is to maximize the expected payoff and the objective of player II is to minimize it.

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Trybula [8, 9] solved a silent duel with arbitrary accuracy functions under arbitrary motion. In his model, each player has a silent gun with one bullet and accuracy functions $p(x)$ and $q(x)$ increase with a continuous second derivative each. Trybula [5-7] also solved noisy duels under arbitrary motion. Furthermore Trybula [10] solved an $m$-versus-$n$ silent duel with arbitrary accuracy functions under arbitrary motion. In the model, player I has a silent gun with $m$ bullets and he has to fire all his bullets simultaneously, whereas player II has a silent gun with $n$ bullets and he can fire each of his bullets at different moments.

The author [2] dealt with a silent-versus-noisy duel under arbitrary motion in which player I has a silent gun with one bullet and player II has a noisy gun with one bullet and the accuracy functions are $p(x)$ and $q(x)$ for players I and II, respectively. The author [3] also dealt with a noisy-versus-silent duel under arbitrary motion in which player I has a noisy gun with one bullet and player II has a silent gun with one bullet and the accuracy functions are arbitrary. Further the author [4] solved a one-noisy-versus-two-silent duel with arbitrary accuracy functions under arbitrary motion.

Further researches on duels under arbitrary motion have been done by Trybula [11, 12] and general researches on games of timing are summarized by Karlin [1].

In this paper, we examine a duel with equal accuracy functions under arbitrary motion. In the duel, player I has a silent gun with two bullets and he may fire these two bullets at different moments, and player II has a silent gun with one bullet. We assume that the accuracy functions are identical.

## 2. Problem

In this paper, we examine a two-versus-one silent duel with equal accuracy functions under arbitrary motion. Player I has a silent gun with two bullets and he is at the place 0 at the moment when the duel begins. He can move as he likes. On the other hand, player II has a silent gun with one bullet and he is always at the place 1. The accuracy functions $p(x)$ and $q(x)$ are identical for both players so that, without loss of generality, we suppose $p(x) = q(x) = x$ for all $x$ over $[0, 1]$. If player I hits player II without being hit himself first, then the payoff of the duel is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0. The objective of player I is to maximize the expected payoff and the objective of player II is to minimize it. We denote the game mentioned above by $G^*$. Note that, in the paper by Trybula [10], player I has to fire all his bullets simultanuously, whereas in our model player I may fire his bullets at different moments.

Before solving the game $G^*$, we consider the following auxiliary game $G$. In $G$, player I has a silent gun with two bullets and player I has a silent gun with one bullet, and both players' accuracy functions are identical and thus we assume, without loss of generality, $p(x) = q(x) = x$ for all $x$ in [0, 1]. Player I is at the place 0 at the moment when the duel begins and player II is at the place 1 all the time. In game $G^*$, player I can move as he likes, however, we suppose that, in game $G$, player I can move towards player II but he can not move away from player II. Further we assume that the payoff of $G$ is as follows:
(i) if player I hits player II before player II hits player I, then the payoff is +1,
(ii) if player I misses his two bullets before player II fires, then the payoff is 0,
(iii) if player II hits player I before player I fires both of his bullets, then the payoff is -1,
(iv) if both players hit each other at the same time or they miss all their bullets, then the payoff is 0.
Suppose that player I fires both his bullets and misses them before player II fires. In this case, the payoff is always 0 in the game $G$, whereas in the game $G^*$ the payoff is 0 or -1 according to player II misses (or he does not fire) his bullet or player II hits his opponent.

Let $M(x, y, z)$ be the expected payoff of the game $G$ when player I fires his first bullet and second when he is at the places $x$ and $y$, respectively ($0 \leqq x \leqq y \leqq 1$), and player II fires his bullet at the moment when player I is at the place $z$ ($0 \leqq z \leqq 1$). The function $M(x,y,z)$, called the payoff kernel of the game $G$, is of the form

$$M(x,y,z) = \begin{cases} x + (1-x)y, & \text{if } 0 \leqq x \leqq y < z, \\ x - (1-x)z + (1-x)(1-z)y, & \text{if } 0 \leqq x < z < y \leqq 1, \\ -z + (1-z)x + (1-z)(1-x)y, & \text{if } 0 \leqq z < x \leqq y \leqq 1, \\ x, & \text{if } 0 \leqq x < y = z, \\ (1-x)^2 y, & \text{if } 0 \leqq x = z < y \leqq 1, \\ x(1-x), & \text{if } 0 \leqq x = y = z. \end{cases}$$

For the game $G$, we shall search for an optimal strategy for player I with the following structure:
(i) player I fires both his bullets simultaneously with probability $\alpha$ and he fires his bullets at different moments with probability $1-\alpha$,
(ii) if player I fires both his bullets simultaneously, he fires his bullets at a place in $[a,b]$ according to the conditional distribution with a density function $f_1(x)$ under the condition that he fires both his bullets simultaneously,
(iii) if player I fires his bullets at different moments, then he fires his first bullet when he is at a place in $[b,c]$ according to the conditional distribution with a density part $f_2(x)$ under the condition that player I fires his bullets at different moments and he fires his second bullet at a place $y$ in $[c,1]$, independently of the place where he has fired his first bullet, according to the distribution with a density part $g(y)$ and mass part $\beta$ on 1, where

$$\int_a^b f_1(x)\,dx = \int_b^c f_2(x)\,dx = \int_c^1 g(y)\,dy + \beta = 1$$

and

$$0 < a < b < c < 1.$$

We denote such a strategy by $\{\alpha, f_1(x), f_2(x), g(y), \beta\}$. Further we shall search for an optimal strategy for player II which is denoted by $\{h(z)\}$. By the strategy $\{h(z)\}$, player II fires his bullet when player I is at $z$ in $[a,1]$ according to the distribution with the probability density function $h(z)$, where

$$\int_a^1 h(z)\,dz = 1.$$

## 3. Preliminary lemmas

In this section we prove two lemmas which will be used in the following sections.

It is seen that the equation

$$\log \frac{1+x}{2x} = \frac{1 - 4x^2 + 4x^3 - 5x^4}{4x^2(1-x)^2} \tag{1}$$

has a unique root in the interval $(0,1)$. We denote by $c$ the unique root in $(0,1)$ of the equation (1). We set

$$b = \frac{1-c}{1+c}\,.$$

The values of $b$ and $c$ are approximately 0.3106 and 0.5261, respectively. We note that $b$ and $c$ satisfy

$$\log \frac{1+c}{2c} = \frac{1}{c} - \frac{1}{2} - \frac{(c-b)(b+c-2bc)}{4b^2c^2}\,. \tag{2}$$

Furthermore it is shown that the equation

$$\frac{1-x}{x(1+2x-x^2)} - \int_x^b \frac{dt}{t^2(1+2t-t^2)} = \frac{(1-b)^2 - b^2(1+b)^2}{4b^2(1-b)^2} \tag{3}$$

has a unique root in the interval $(0, b)$. We denote by $a$ the unique root of the equation (3). The value of $a$ is nearly equal to 0.2025.

**Lemma 1.** *Set*

$$\alpha = 1 - \frac{a(1+2a-a^2)(c^2-b^2)}{4(1-a)b^2c^2} \ (= 0.4153),$$

$$f_1(x) = \frac{k_1}{x^2(1+2x-x^2)}\ , \qquad a \leqq x \leqq b,$$

$$f_2(x) = k_2 x^{-3}\ , \qquad b \leqq x \leqq c,$$

$$g(y) = \frac{k_3}{y^2(1+y)}\ , \qquad c \leqq y \leqq 1$$

*and*

$$\beta = \frac{k_3}{2}\ ,$$

*where*

$$k_1 = \frac{a(1+2a-a^2)}{\alpha(1-a)}\ ,$$

$$k_2 = \frac{2b^2c^2}{c^2-b^2}$$

*and*

$$k_3 = \frac{4b^2c^2}{(c-b)(b+c-2bc)}\ .$$

*Then the following equations hold*:

(i) $\displaystyle\int_a^b f_1(x)\,dx = \int_b^c f_2(x)\,dx = \int_c^1 g(y)\,dy + \beta = 1,$

(ii) $\displaystyle(1-\alpha)\left\{\int_c^1 (1+y)g(y)\,dy + \int_b^c xf_2(x)\,dx \int_c^1 (1-y)g(y)\,dy + 2\beta\right\} = \frac{\alpha k_1}{b},$

(iii) $\displaystyle\int_b^c (1+x)f_2(x)\,dx + \left\{\int_b^c (1-x)f_2(x)\,dx\right\}\left\{\int_c^1 yg(y)\,dy + \beta\right\} = \frac{2k_2}{b}.$

*Proof.* (i) Since $a$ is the root of the equation (3), we have

$$\begin{aligned}\int_a^b \frac{dx}{x^2(1+2x-x^2)} &= \frac{1-a}{a(1+2a-a^2)} - \frac{c^2-b^2}{4b^2c^2}\\ &= \frac{1-a}{a(1+2a-a^2)}\left\{1 - \frac{a(1+2a-a^2)(c^2-b^2)}{4(1-a)b^2c^2}\right\} = \frac{\alpha(1-a)}{a(1+2a-a^2)} = \frac{1}{k_1}\ .\end{aligned}$$

Thus we get

$$\int_a^b f_1(x)\,dx = 1.$$

We can directly show

$$\int_b^c f_2(x)\,dx = 1.$$

Further, we have

$$\int_c^1 g(y)\,dy + \beta = k_3\Big\{-\log\frac{1+c}{2c} - \frac{1}{2} + \frac{1}{c}\Big\}.$$

Thus, from (2), it follows that

$$\int_c^1 g(y)\,dy + \beta = \frac{k_3(c-b)(b+c-2bc)}{4b^2c^2} = 1.$$

(ii) We get

$$\alpha k_1 = \frac{a(1+2a-a^2)}{1-a} \tag{4}$$

and

$$(1-\alpha)k_3 = \frac{a(1+2a-a^2)(b+c)}{(1-a)(b+c-2bc)}\,. \tag{5}$$

We further get

$$\int_c^1 (1+y)g(y)\,dy + \int_b^c xf_2(x)\,dx\int_c^1(1-y)g(y)\,dy + 2\beta = k_3\Big[\frac{1}{c} + \frac{2bc}{b+c}\Big\{-2\log\frac{1+c}{2c} - 1 + \frac{1}{c}\Big\}\Big].$$

Thus by (2), (4) and (5), we have

$$(1-\alpha)\Big\{\int_c^1(1+y)g(y)\,dy + \int_b^c xf_2(x)\,dx\int_c^1(1-y)g(y)\,dy + 2\beta\Big\} = \frac{k_3(1-\alpha)(b+c-2bc)}{b(b+c)} = \frac{\alpha k_1}{b}\,.$$

(iii) By (2), we get

$$\int_c^1 yg(y)\,dy + \beta = k_3\Big\{\log\frac{1+c}{2c} + \frac{1}{2}\Big\} = \frac{(b+c)(b-c+2bc)}{(c-b)(b+c-2bc)}\,.$$

Since

$$\int_b^c (1+x)f_2(x)\,dx = \frac{b+c+2bc}{b+c}$$

and

$$\int_b^c (1-x)f_2(x)\,dx = \frac{b+c-2bc}{b+c}\,,$$

we get

$$\int_b^c (1+x)f_2(x)\,dx + \Big\{\int_b^c(1-x)f_2(x)\,dx\Big\}\Big\{\int_c^1 yg(y)\,dy + \beta\Big\}$$
$$= \frac{b+c+2bc}{b+c} + \frac{b-c+2bc}{c-b} = \frac{4bc^2}{c^2-b^2} = \frac{2k_2}{b}\,.$$

This completes our proof.

**Lemma 2.** *Set*

$$h(z)=\begin{cases}\dfrac{2(1+2a-a^2)(1-z)}{z(1+2z-z^2)^2}, & a\leqq z\leqq b,\\ k_4z^{-3}, & b\leqq z\leqq c,\\ \dfrac{k_5}{z(1+z)^2}, & c\leqq z\leqq 1,\end{cases}$$

*where*

$$k_4=\frac{b(1-b)(1+2a-a^2)}{2(1+2b-b^2)}$$

*and*

$$k_5=\frac{(1+b)(1+2a-a^2)}{1+2b-b^2}\ .$$

*Then the following statements hold*:

(i) $\int_a^1 h(z)\,dz=1$,

(ii) $\int_a^c zh(z)\,dz=1-\dfrac{k_5}{1+c}$,

(iii) *For all* $x$ *in* $[b,\ c]$,

$$(1+c)\int_x^c zh(z)\,dz\leqq(c-x)\left\{1-\int_a^x zh(z)\,dz\right\}.$$

*Proof.* (i) It suffices to show that

$$2\int_a^b\frac{1-z}{z(1+2z-z^2)^2}\,dz+\frac{b(1-b)}{2(1+2b-b^2)}\int_b^c\frac{dz}{z^3}+\frac{1+b}{1+2b-b^2}\int_c^1\frac{dz}{z(1+z)^2}=\frac{1}{1+2a-a^2}\ .$$

Since $a$ is the root of the equation (3), we have

$$\begin{aligned}2\int_a^b\frac{1-z}{z(1+2z-z^2)^2}\,dz&=\frac{1}{a(1+2a-a^2)}-\frac{1}{b(1+2b-b^2)}-\int_a^b\frac{dz}{z^2(1+2z-z^2)}\\&=\frac{1}{1+2a-a^2}-\frac{1}{b(1+2b-b^2)}+\frac{1-2b-2b^3-b^4}{4b^2(1-b)^2}\end{aligned}\tag{6}$$

and we get

$$\frac{b(1-b)}{2(1+2b-b^2)}\int_b^c\frac{dz}{z^3}=\frac{1-2b-2b^3-b^4}{4b(1-b)(1+2b-b^2)}\ .\tag{7}$$

Further, from (2), it follows that

$$\begin{aligned}\frac{1+b}{1+2b-b^2}\int_c^1\frac{dz}{z(1+z)^2}&=\frac{1+b}{1+2b-b^2}\left\{\log\frac{1+c}{2c}+\frac12-\frac{1}{1+c}\right\}\\&=\frac{(1+b)(-1+4b-4b^2+6b^3+b^4-2b^5)}{4b^2(1-b)^2(1+2b-b^2)}\ .\end{aligned}\tag{8}$$

By (6), (7) and (8), we obtain the desired result.
(ii) We directly get

$$\int_a^b zh(z)\,dz = (1+2a-a^2)\left\{\frac{1}{1+2a-a^2} - \frac{1}{1+2b-b^2}\right\}$$

and

$$\int_b^c zh(z)\,dz = \frac{(1+2a-a^2)(1-2b-b^2)}{2(1+2b-b^2)}\ .$$

Accordingly, we have

$$\int_a^c zh(z)\,dz = 1 - \frac{(1+2a-a^2)(1+b)^2}{2(1+2b-b^2)} = 1 - \frac{k_5}{1+c}\ .$$

(iii) It is seen that

$$x\left\{\frac{2}{c(1-c)} + \frac{1}{x} - \frac{1}{c}\right\} \leqq \frac{2}{1-c} - \frac{1}{x} + \frac{1}{c}$$

for every $x$ in $[b, c]$, thus we get

$$x\left\{\frac{k_5}{1+c} + k_4\left(\frac{1}{x} - \frac{1}{c}\right)\right\} \leqq \frac{ck_5}{1+c} - k_4\left(\frac{1}{x} - \frac{1}{c}\right).$$

Therefore, by (ii) in Lemma 2, we obtain

$$x\left\{1 - \int_a^c zh(z)\,dz + \int_x^c zh(z)\,dz\right\} \leqq c\left\{1 - \int_a^c zh(z)\,dz\right\} - \int_x^c zh(z)\,dz,$$

*i.e.*,

$$(1+c)\int_x^c zh(z)\,dz \leqq (c-x)\left\{1 - \int_a^x zh(z)\,dz\right\}$$

for all $x$ in $[b,\ c]$. This completes our proof.

## 4. Strategies in the game $G$

In what follows, we denote by $v_1(z)$ the expected payoff of the game $G$ when player I applies the strategy $\{\alpha, f_1(x), f_2(x), g(y), \beta\}$ given in Lemma 1 and player II fires his bullet when player I is at the point $z$ in $[0, 1]$. Similarly, we denote by $v_2(x, y)$ the expected payoff of the game $G$ when player II applies the strategy $\{h(z)\}$ given in Lemma 2 and player I fires his first bullet and second when he is at the points $x$ and $y$, respectively.

**Lemma 3.** *For all $z$ in $[a,\ 1)$, $v_1(z) = 2a - a^2$ $(= 0.3640)$.*

*Proof.* For all $z$ in $[a,\ b]$, we have

$$\begin{aligned} v_1(z) &= \alpha\int_a^z \{x + (1-x)x\} f_1(x)\,dx + \alpha\int_z^b \{-z + (1-z)(2x-x^2)\} f_1(x)\,dx \\ &\quad + (1-\alpha)\int_b^c\int_c^1 \{-z + (1-z)x + (1-z)(1-x)y\} g(y) f_2(x)\,dydx \\ &\quad + (1-\alpha)\beta\int_b^c \{-z + (1-z)x + (1-z)(1-x)\} f_2(x)\,dx \\ &= -1 + \alpha\int_a^z (1+2x-x^2) f_1(x)\,dx + \alpha(1-z)\int_z^b (1+2x-x^2) f_1(x)dx \\ &\quad + \ (1-\alpha)(1-z)\left\{\int_b^c\int_c^1 \{1 + x + (1-x)y\} g(y) f_2(x)\,dydx + 2\beta\right\}. \end{aligned}$$

Thus by (ii) in Lemma 1, we get

$$v_1(z) = -1 + \alpha \int_a^z (1 + 2x - x^2) f_1(x)\, dx + \alpha(1 - z) \int_z^b (1 + 2x - x^2) f_1(x) dx + \frac{\alpha(1 - z)k_1}{b}.$$

Therefore we obtain

$$v_1(z) = 2a - a^2$$

for all $z$ in $[a, b]$. For every $z$ in $[b, c]$, we get

$$\begin{aligned} v_1(z) = \alpha & \int_a^b (2x - x^2) f_1(x)\, dx + (1 - \alpha) \int_b^z \int_c^1 \{x - (1 - x)z + (1 - x)(1 - z)y\} g(y) f_2(x)\, dydx \\ & + (1 - \alpha)\beta \int_b^z \{x - (1 - x)z + (1 - x)(1 - z)\} f_2(x)\, dx \\ & + (1 - \alpha) \int_z^c \int_c^1 \{-z + (1 - z)x + (1 - z)(1 - x)y\} g(y) f_2(x)\, dydx \\ & + (1 - \alpha)\beta \int_z^c \{-z + (1 - z)x + (1 - z)(1 - x)\} f_2(x)\, dx \\ = & -1 + \alpha \int_a^b (1 + 2x - x^2) f_1(x)\, dx + (1 - \alpha)(1 - z) \left\{ \int_b^c (1 - x) f_2(x)\, dx \right\} \left\{ \int_c^1 yg(y) dy + \beta \right\} \\ & + (1 - \alpha) \int_b^z \{1 + x - (1 - x)z\} f_2(x)\, dx + (1 - \alpha)(1 - z) \int_z^c (1 + x) f_2(x)\, dx. \end{aligned}$$

Thus, we have

$$v_1(z) = -1 + \alpha \int_a^b (1 + 2x - x^2) f_1(x)\, dx + (1 - \alpha)\varphi_1(z),$$

where

$$\begin{aligned} \varphi_1(z) = 2z & \int_b^z x f_2(x)\, dx + (1 - z) \int_b^c (1 + x) f_2(x)\, dx \\ & + (1 - z) \left\{ \int_b^c (1 - x) f_2(x) dx \right\} \left\{ \int_c^1 yg(y)\, dy + \beta \right\}. \end{aligned}$$

Consequently, by (iii) in Lemma 1, we get

$$\varphi_1(z) = 2z \int_b^z x f_2(x)\, dx + \frac{2(1 - z)k_2}{b} = \frac{2k_2(1 - b)}{b}.$$

Therefore by (4) and

$$(1 - \alpha)k_2 = \frac{a(1 + 2a - a^2)}{2(1 - a)}, \tag{9}$$

we obtain

$$v_1(z) = 2a - a^2$$

for every $z$ in $[b,\ c]$. For all $z$ in $[c, 1)$, we have

$$\begin{aligned} v_1(z) &= \alpha \int_a^b (2x - x^2) f_1(x)\,dx + (1-\alpha) \int_b^c \int_c^z \{x + (1-x)y\} g(y) f_2(x)\,dydx \\ &\quad + (1-\alpha) \int_b^c \int_z^1 \{x - (1-x)z + (1-x)(1-z)y\} g(y) f_2(x)\,dydx \\ &\quad + (1-\alpha)\beta \int_b^c \{x - (1-x)z + (1-x)(1-z)\} f_2(x)\,dx \\ &= -1 + \alpha \int_a^b (1 + 2x - x^2) f_1(x)\,dx \\ &\quad + (1-\alpha) \int_b^c (1+x) f_2(x)\,dx + (1-\alpha)\varphi_2(z) \int_b^c (1-x) f_2(x)\,dx, \end{aligned}$$

where

$$\begin{aligned} \varphi_2(z) &= \int_c^z y g(y)\,dy + \int_z^1 \{-z + (1-z)y\} g(y)\,dy + \beta(1-2z) \\ &= \int_c^z (1+y) g(y)\,dy + (1-z) \int_z^1 (1+y) g(y)\,dy + 2\beta(1-z) - 1. \end{aligned}$$

It is seen that

$$\varphi_2(z) = \frac{k_3}{c} - k_3 - 1$$

for all $z$ in $[c, 1)$, and thus,

$$\begin{aligned} v_1(z) &= -1 + \alpha \int_a^b (1 + 2x - x^2) f_1(x)\,dx \\ &\quad + \frac{k_3(1-\alpha)(1-c)}{c} \int_b^c f_2(x)\,dx + (1-\alpha)\left\{2 - \frac{k_3(1-c)}{c}\right\} \int_b^c x f_2(x)\,dx \\ &= -1 + \alpha k_1 \left(\frac{1}{a} - \frac{1}{b}\right) + \frac{k_3(1-\alpha)(1-c)}{c} + 2k_2(1-\alpha)\left(\frac{1}{b} - \frac{1}{c}\right) - \frac{2bk_3(1-\alpha)(1-c)}{b+c}. \end{aligned}$$

Thus, by (4), (5) and (9), we obtain

$$v_1(z) = 2a - a^2$$

for all $z$ in $[c, 1)$. This completes our proof.

We get, for all $x$ and $y$ such that $a \leqq x \leqq y \leqq 1$,

$$\begin{aligned} v_2(x,y) &= \int_a^x \{-z + (1-z)x + (1-z)(1-x)y\} h(z)\,dz \\ &\quad + \int_x^y \{x - (1-x)z + (1-x)(1-z)y\} h(z)\,dz + \int_y^1 \{x + (1-x)y\} h(z)\,dz \\ &= -1 + \{1 + x + (1-x)y\}\left\{1 - \int_a^x z h(z)\,dz\right\} - (1-x)(1+y) \int_x^y z h(z)\,dz. \end{aligned} \tag{10}$$

**Lemma 4.** *The following statements hold*:
(i) *For all $x$ in $[a,\ b]$, $v_2(x,x) = 2a - a^2$,*
(ii) *For all $x$ and $y$ such that $b \leqq x \leqq y \leqq 1$, $v_2(x,y) \leqq 2a - a^2$.*

*Proof.* (i) For each $x$ in $[a,\ b]$, we directly get

$$v_2(x,x) = -1 + (1+2x-x^2)\left\{1 - \int_a^x zh(z)\,dz\right\} = 2a - a^2.$$

(ii) For all $x$ and $y$ such that $b \leqq x \leqq c \leqq y \leqq 1$, from (10), we have

$$\begin{aligned} v_2(x,y) = -1 &+ \{1+x+(1-x)y\}\left\{1 - \int_a^c zh(z)\,dz\right\} \\ &+ 2x\int_x^c zh(z)\,dz - (1-x)(1+y)\int_c^y zh(z)\,dz. \end{aligned}$$

Thus, by (ii) in Lemma 2, we get

$$\begin{aligned} v_2(x,y) &= -1 + \frac{k_5\{1+x+(1-x)y\}}{1+c} + \frac{2k_4(c-x)}{c} - k_5(1-x)(1+y)\left(\frac{1}{1+c} - \frac{1}{1+y}\right) \\ &= -1 + 2k_4 + k_5 + x\left(\frac{2k_5}{1+c} - k_5 - \frac{2k_4}{c}\right) = 2a - a^2. \end{aligned}$$

For every $x$ and $y$ with $c \leqq x \leqq y \leqq 1$, from (ii) in Lemma 2 and (10), it follows that

$$v_2(x,y) = -1 + \frac{k_5(1+2x-x^2)}{1+x}.$$

Consequently, $v_2(x,y)$ is decreasing in $x$ for each $y$ in $[c,1]$ since $\sqrt{2}-1 < c \leqq x \leqq y \leqq 1$. Thus we have

$$v_2(x,y) \leqq v_2(c,y) = 2a - a^2$$

for all $x$ and $y$ such that $c \leqq x \leqq y \leqq 1$. For every $x$ and $y$ with $b \leqq x \leqq y \leqq c$, from (10), we get

$$\frac{\partial^2 v_2(x,y)}{\partial y^2} = \frac{2k_4(1-x)}{y^3} > 0.$$

Further from (iii) in Lemma 2, we have

$$\begin{aligned} v_2(x,x) &= -1 + (1+2x-x^2)\left\{1 - \int_a^x zh(z)\,dz\right\} \\ &\leqq -1 + \{1+x+(1-x)c\}\left\{1 - \int_a^x zh(z)\,dz\right\} - (1-x)(1+c)\int_x^c zh(z)\,dz \\ &= v_2(x,c) = 2a - a^2 \end{aligned}$$

for each $x$ in $[b,\ c]$. Therefore we obtain

$$v_2(x,y) \leqq 2a - a^2$$

for all $x$ and $y$ such that $b \leqq x \leqq y \leqq c$. This completes our proof.

## 5. A Theorem

**Theorem 1.** *For the game $G$, the strategy $\{\alpha, f_1(x), f_2(x), g(y), \beta\}$ given in Lemma 1 is optimal for player I, and the strategy $\{h(z)\}$ given in Lemma 2 is optimal for player II. Moreover, the game value of $G$ is $2a - a^2$.*

*Proof.* It suffices to show that

$$v_1(z) \geqq 2a - a^2$$

for all $z$ in [0, 1] and

$$v_2(x, y) \leqq 2a - a^2$$

for all $x$ and $y$ such that $0 \leqq x \leqq y \leqq 1$. From Lemma 3, we have

$$v_1(z) = 2a - a^2$$

for every $z$ in $[a, 1)$. For each $z$ in $[0, a]$, we get

$$\begin{aligned} v_1(z) = \alpha &\int_a^b \{-z + (1-z)(2x - x^2)\} f_1(x)\, dx \\ &+ (1-\alpha) \int_b^c \int_c^1 \{-z + (1-z)x + (1-z)(1-x)y\} g(y) f_2(x)\, dy dx \\ &+ (1-\alpha)\beta \int_b^c \{-z + (1-z)x + (1-z)(1-x)\} f_2(x)\, dx. \end{aligned}$$

It is seen that $v_1(z)$ is decreasing in $z$ over $[0, a]$, and thus

$$v_1(z) \geqq v_1(a) = 2a - a^2.$$

Further since

$$\int_b^c (2x-1) f_2(x)\, dx = \frac{k_2(1 - 2b - b^2)(-1 + 4b - 5b^2)}{2b^2(1-b)^2} < 0,$$

we get

$$\begin{aligned} v_1(1) &= \alpha \int_a^b (2x - x^2) f_1(x)\, dx + (1-\alpha) \int_b^c \int_c^1 \{x + (1-x)y\} g(y) f_2(x)\, dy dx \\ &> \alpha \int_a^b (2x - x^2) f_1(x)\, dx + (1-\alpha) \int_b^c \int_c^1 \{x + (1-x)y\} g(y) f_2(x)\, dy dx \\ &\quad + (1-\alpha)\beta \int_b^c (2x-1) f_2(x)\, dx \\ &= 2a - a^2. \end{aligned}$$

Therefore we obtain

$$v_1(z) \geqq 2a - a^2$$

for all $z$ in [0, 1]. Now, from Lemm 4, it follows that

$$v_2(x, y) \leqq 2a - a^2$$

for all $x$ and $y$ such that $b \leqq x \leqq y \leqq 1$ and

$$v_2(x, x) = 2a - a^2$$

for every $x$ in $[a,\, b]$. For all $x$ and $y$ such that $a \leqq x \leqq y \leqq b$, from (10), we get

$$\frac{\partial v_2}{\partial y} = (1-x)\left\{1 - \int_a^y zh(z)\,dz - y(1+y)h(y)\right\}$$
$$= \frac{(1-x)(1+2a-a^2)(-1+2y+y^2)}{(1+2y-y^2)^2} < 0.$$

Thus we have

$$v_2(x,y) \leqq v_2(x,x) = 2a - a^2.$$

Further, for all $x$ and $y$ with $a \leqq x \leqq b$ and $c \leqq y \leqq 1$, we have

$$\frac{\partial v_2}{\partial x} = (1-y)\left\{1 - \int_a^x zh(z)\,dz\right\} - 2x^2h(x) + (1+y)\int_x^y zh(z)\,dz$$
$$= \frac{2(1+2a-a^2)(1+x^2)}{(1+2x-x^2)^2} - \frac{(1+2a-a^2)(1+b)}{1+2b-b^2}.$$

It is shown that $\dfrac{\partial v_2}{\partial x}$ is decreasing in $x$ over $[a,\, b]$ for each $y$ in $[c,\, 1]$. Moreover we can show that

$$\left.\frac{\partial v_2(x,y)}{\partial x}\right|_{x\,=\,b} = \frac{(1+2a-a^2)(1-b)(1-2b-b^2)}{(1+2b-b^2)^2} > 0$$

for every $y$ in $[c,\, 1]$, and thus $v_2(x,y)$ is increasing in $x$ over $[a,\, b]$ for each $y$ in $[c,\, 1]$. Therefore we get

$$v_2(x,y) \leqq v_2(b,y) \leq 2a - a^2$$

for all $x$ in $[a,\, b]$ and $y$ in $[c,\, 1]$. For every $x$ and $y$ such that $a \leqq x \leqq b \leqq y \leqq c$, we have

$$\frac{1}{1-x}\,\frac{\partial v_2}{\partial y} = \frac{k_5}{1+b} - \frac{k_4}{b} - \frac{k_4}{y^2}$$

and hence

$$\frac{\partial^2 v_2}{\partial y^2} > 0.$$

Furthermore since

$$v_2(x,b) \leqq 2a - a^2$$

and

$$v_2(x,c) \leqq 2a - a^2$$

for each $x$ in $[a,\, b]$, we have

$$v_2(x,y) \leqq 2a - a^2$$

for all $x$ and $y$ with $a \leqq x \leqq b \leqq y \leqq c$. It is easily seen that

$$v_2(x,y) \leqq v_2(a,y) \leqq 2a - a^2$$

for every $x$ in $[0,\, a]$ and $y$ in $[x,\, 1]$. Therefore we obtain

$$v_2(x,y) \leqq 2a - a^2$$

for all $x$ and $y$ such that $0 \leqq x \leqq y \leqq 1$. This completes our proof.

## 6. Optimal Strategies

In this section, we examine the game $G^*$ defined at the beginning of section 2. For any $\varepsilon$ such that $0 < \varepsilon < 1$, we define $N$ as the smallest natural number that is larger than $1/\varepsilon$. For the $N$, we define constants $a_i$ $(i = 0, 1, 2, \cdots, n_1 + 1)$ and $c_i$ $(i = 1, 2, \cdots, n_2 + 1)$ as follows:

$$a_0 = a,$$

$$\alpha \int_{a_{i-1}}^{a_i} f_1(x)\, dx = \frac{1}{N}\ , \qquad i = 1, 2, \cdots, n_1,$$

$$a_{n_1+1} = b,$$

$$c_0 = c,$$

$$(1-\alpha) \int_{c_{i-1}}^{c_i} g(y)\, dy = \frac{1}{N}, \quad i = 1, 2, \cdots, n_2,$$

$$c_{n_2+1} = 1,$$

where

$$\alpha \int_a^b f_1(x)\, dx > \alpha \int_a^{a_{n_1}} f_1(x)\, dx \geqq \alpha \int_a^b f_1(x)\, dx - \frac{1}{N}$$

and

$$(1-\alpha) \int_c^1 g(y)\, dy > (1-\alpha) \int_c^{c_{n_2}} g(y)\, dy \geqq (1-\alpha) \int_c^1 g(y)\, dy - \frac{1}{N}.$$

Now we define the strategy $\{\alpha, f_1(x), f_2(x), g(y), \beta\}^\varepsilon$ of player I in the game $G^*$ as follows:
(i) Player I fires both his bullets simultaneously with probability $\alpha$ and he fires his bullets at different moments with probability $1 - \alpha$.
(ii) Player I moves back and forth in the following manner: at first between 0 and $a_1$, then between 0 and $a_2, \cdots$, and then between 0 and $a_{n_1+1}$. At the $i$-th step $(i = 1, 2, \cdots, n_1 + 1)$, he fires both of his bullets simultaneously at random only if he is between $a_{i-1}$ and $a_i$ and goes forward, and he fires them with conditional probability density $f_1(x)$ under the condition that he fires both his bullet simultaneously. After he has fired both of his bullets at the $i$-th step, he reaches the point $a_i$, escapes to 0 and never approaches player II.
(iii) When player I has not fired his bullets in $[a, b]$, he further moves back and forth between 0 and $c$ and he fires his first bullet between $b$ and $c$, only if he goes forward, according to the conditional distriburion with density $f_2(x)$ under the condition that he fires his bullets at different moments. Furthermore player I moves back and forth between 0 and $c_1$, then between 0 and $c_2, \cdots$, and then between 0 and $c_{n_2+1}$. When he moves back and forth between 0 and $c_i$, he fires his second bullet at random only if he is between $c_{i-1}$ and $c_i$ and goes forward, and he fires his bullet with conditional distribution with density part $g(y)$ and mass part $\beta$ at 1, independently of the point where he has fired his first bullet. If he has fired his second bullet between $c_{i-1}$ and $c_i$, he reaches the point $c_i$, escapes to 0 and never approaches player II.

**Theorem 2.** *For the game $G^*$, the strategy $\{\alpha, f_1(x), f_2(x), g(y), \beta\}^\varepsilon$ is $\varepsilon$-optimal for player* I*, and the strategy $\{h(z)\}$ given in Lemma* 2 *is optimal for player* II*. Moreover, the game value of $G^*$ is $2a - a^2$.*

*Proof.* It is seen that if player I fires his bullets at a point, then he has to fire the bullets when he is at the point for the first time. Similarly, if player II fires his bullet when player I is at a point, then player II has to fire his bullet when player I is at the point for the first time. Thus, in what follows, we assume that player I fires at points when he is at these points for the first tme, and player II fires his bullet when player I is at a point for the first time. Now we denote by $v_1^*(z)$ the expected payoff of the game $G^*$ when player I applies the strategy $\{\alpha, f_1(x), f_2(x), g(y), \beta\}^\varepsilon$ and player II fires his bullet when player I is at the point $z$ in [0, 1]. Similarly, we denote by $v_2^*(x,y)$ the expected payoff of the game $G^*$ when player II applies the strategy $\{h(z)\}$ and player I fires his bullets when he is at the points $x$ and $y$. Clearly,

$$v_1^*(z) = v_1(z) \geqq 2a - a^2$$

for all $z$ in $[0, a)$. For every $z$ in $(a_i, a_{i+1}]$ $(i = 0, 1, 2, \cdots, n_1)$, we get

$$\begin{aligned} v_1^*(z) = \alpha &\int_a^{a_i} \{x + (1-x)x\} f_1(x)\,dx + \alpha \int_{a_i}^{z} \{x + (1-x)x - (1-x)^2 z\} f_1(x)\,dx \\ &+ \alpha \int_z^b \{-z + (1-z)(2x - x^2)\} f_1(x)\,dx \\ &+ (1-\alpha) \int_b^c \int_c^1 \{-z + (1-z)x + (1-z)(1-x)y\} g(y) f_2(x)\,dydx \\ &+ (1-\alpha)\beta \int_b^c \{-z + (1-z)x + (1-z)(1-x)\} f_2(x)\,dx \\ = v_1(z) &- \alpha \int_{a_i}^{z} z(1-x)^2 f_1(x)\,dx \geqq 2a - a^2 - \frac{1}{N} > 2a - a^2 - \varepsilon. \end{aligned}$$

For every $z$ in $[b,\ c]$, we directly get

$$v_1^*(z) = v_1(z) = 2a - a^2.$$

Further, for all $z$ in $(c_i,\ c_{i+1}]$ $(i = 0, 1, 2, \cdots, n_2)$, we have

$$\begin{aligned} v_1^*(z) = \alpha &\int_a^b (2x - x^2) f_1(x)\,dx + (1-\alpha) \int_b^c \int_c^{c_i} \{x + (1-x)y\} g(y) f_2(x)\,dydx \\ &+ (1-\alpha) \int_b^c \int_{c_i}^{z} \{x + (1-x)y - (1-x)(1-y)z\} g(y) f_2(x)\,dydx \\ &+ (1-\alpha) \int_b^c \int_z^1 \{x - (1-x)z + (1-x)(1-z)y\} g(y) f_2(x)\,dydx \\ &+ (1-\alpha)\beta \int_b^c \{x - (1-x)z + (1-x)(1-z)\} f_2(x)\,dx \\ = v_1(z) &- (1-\alpha) \int_b^c \int_{c_i}^{z} (1-x)(1-y) z g(y) f_2(x)\,dydx \\ \geqq v_1(z) &- \frac{1}{N} > 2a - a^2 - \varepsilon. \end{aligned}$$

It is obvious that

$$v_1^*(1) = v_1(1) \geqq 2a - a^2.$$

Therefore we obtain

$$v_1^*(z) \geqq 2a - a^2 - \varepsilon$$

for all $z$ in [0, 1]. It is clear that

$$v_2^*(x, y) = v_2(x, y) \leqq 2a - a^2$$

for all $x$ and $y$ such that $0 \leqq x \leqq y \leqq 1$. We suppose that player I fires his first bullet when he is at a place $x$ in $(0,1]$, and escapes to 0 and stays in 0 forever. We denote such a strategy by $(x,*)$ and we denote by $v_2^*(x,*)$ the expected payoff when players I and II apply the strategies $(x,*)$ and $\{h(z)\}$, respectively. Then we get

$$\begin{aligned} v_2^*(x,*) &= \int_a^x \{-z+(1-z)x\}h(z)\,dz \leqq \int_a^1 \{-z+(1-z)\}h(z)\,dz \\ &= 1-2\int_a^1 zh(z)\,dz = -1+k_5 = -1+\frac{(1+b)(1+2a-a^2)}{1+2b-b^2} \leqq 2a-a^2 \end{aligned}$$

for every $x$ in $[a,\, 1]$ and

$$v_2^*(x,*) = a \leqq 2a-a^2$$

for all $x$ in $[0,\, a]$. Thus we can conclude that, if player II applies the strategy $\{h(z)\}$, the expected payoff is at most $2a-a^2$ whatever stratedy player I may apply. This completes our proof.

In this paper, we have assumed that player I may fire his two bullets at different moments and we have figured out that the game value is $2a-a^2 = 0.3640$. As was mentioned in section 1, Trybula [10] solved an $m$-versus-$n$ silent duel with arbitrary accuracy functions under arbitrary motion. In Trybula's model, player I has to fire his $m$ bullets simultaneously, whereas player II can fire his $n$ bullets at different moments. If we put $m = 2$, $n$ =1 and $p(x) = q(x) = x$ in Trybula's model, then the game value is $2\hat{a}-\hat{a}^2$ $(= 0.3241)$, where $\hat{a} = 0.1779$ is the unique root in (0, 1) of the equation

$$\int_x^1 \frac{dt}{t^2(1+2t-t^2)} + \frac{1}{2} = \frac{1-x}{x(1+2x-x^2)}.$$

Thus, the game value of our model is larger than the game value of the Trybula's model as might be expected.

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