GLOBAL EXISTENCE VERSUS BLOW-UP IN SUPERLINEAR INDEFINITE PARABOLIC PROBLEMS

Julián López-Gómez

Received June 1, 2004; revised October 8, 2004

Abstract. The asymptotic behaviour of all positive solutions with small initial data of a parabolic semi-linear equation of indefinite type is analyzed. Though in same parameter ranges, the solutions stabilize to a positive steady-state, in others, the solutions blow-up in a finite time and their limiting profiles, after the blow-up time, are described through the *metasolutions* of the associated sub-linear problem. As a result, metasolutions are shown to play a crucial role in describing the dynamics of parabolic equations in the presence of spatial heterogeneities.

1 Introduction In this paper we study the asymptotic behavior of the solutions of

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u + a(x)u^p & \text{in } \ \Omega \times (0,\infty) \\ u(x,t) = 0 & \text{on } \ \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 > 0 & \text{in } \ \Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$, e.g., of class $\mathcal{C}^3$, $\lambda \in \mathbb{R}$, $p \in (1,\infty)$, and $a \in \mathcal{C}^1(\bar\Omega)$, $a \neq 0$, is a function for which

$$\Omega_+ := \{\, x \in \Omega \ : \ a(x) > 0 \,\} \qquad \text{and} \qquad \Omega_- := \{\, x \in \Omega \ : \ a(x) < 0 \,\}$$

are two non-empty subdomains of Ω of class $\mathcal{C}^3$ with $\bar\Omega_+ \cup \bar\Omega_- \subset \Omega$ such that

$$\Omega_0 := \Omega \setminus \left(\bar\Omega_+ \cup \bar\Omega_-\right)$$

is connected. Figure 1 represents a typical situation satisfying all these assumptions. Note that

$$\Gamma = \partial\Omega\,, \qquad \Gamma_1 = \partial\Omega_0 \setminus \partial\Omega\,, \qquad \Gamma_2 := \partial\Omega_+\,, \qquad \partial\Omega_- = \Gamma_1 \cup \Gamma_2\,.$$

Although the class of weight functions $a(x)$ for which the theory developed in this paper applies is wider, throughout this paper it will be assumed that $a(x)$ fits the patterns described by Figure 1 and denote

$$a_+ := \max\{a(x), 0\}\,, \qquad a_- := \max\{-a(x), 0\}\,.$$

Then,

$$\Omega_+ = \operatorname{Int}\operatorname{supp} a_+\,, \qquad \Omega_- = \operatorname{Int}\operatorname{supp} a_-\,, \qquad a = a_+ - a_-\,.$$

Subsequently, given a regular subdomain D of Ω and $V \in C(\bar D)$, $\sigma[-\Delta + V, D]$ will stand for the principal eigenvalue of $-\Delta + V$ in D under homogeneous Dirichlet boundary conditions. In this paper, it will be throughout assumed that

$$(1.2)\qquad \sigma_1 := \sigma[-\Delta, \Omega_0] < \sigma_2 := \sigma[-\Delta, \Omega_+]\,,$$

2000 *Mathematics Subject Classification.* 35B45, 35J65.
Key words and phrases. indefinite super-linear parabolic equations, classical solutions, large solutions, metasolutions, asymptotic behaviour.

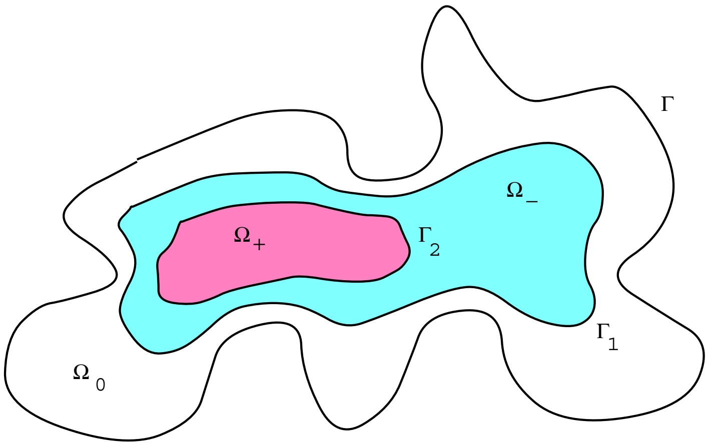

Figure 1: The nodal behavior of $a(x)$.

though most of its mathematical analysis can be easily adapted to cover the most general case when (1.2) fails. Thus, setting

$$\sigma_0 := \sigma[-\Delta, \Omega]\,,$$

the monotonicity of $\sigma[\cdot, D]$ with respect to D together with (1.2) shows that

$$\sigma_0 < \sigma_1 < \sigma_2\,.$$

Thanks to Faber-Krahn inequality, (1.2) holds if the Lebesgue measure of Ω_+ is sufficiently small (e.g., [18, Section 5]). Actually, one might think of (1.2) as a hierarchical order size between Ω_0 and Ω_+, establishing that Ω_0 is larger than Ω_+, however the principal eigenvalue $\sigma[-\Delta, D]$ also depends upon some hidden geometrical properties of D, not merely its size.

Under these assumptions, there exists a maximal existence time, $T := T_{\max}(u_0) \in (0, \infty]$ and a unique smooth solution $u_{[\lambda,a,\Omega]}(x, t; u_0)$ of (1.1) defined in $[0, T)$ such that

$$\lim_{t\uparrow T} \|u_{[\lambda,a,\Omega]}(\cdot, t; u_0)\|_{L^\infty(\Omega)} = \infty \qquad \text{if} \quad T < \infty\,.$$

Our main goal is ascertaining the behavior of the solutions of (1.1) as time passes by, and, particularly, finding out the limiting behavior

$$\lim_{t\uparrow T} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \tag{1.3}$$

if such a limit exists, according to each of the values of the several parameters in the setting of (1.1), as well as the limiting behaviour of the *very weak extension* $\bar{u}$, in the sense of Baras-Cohen [5], of $u_{[\lambda,a,\Omega]}$ after the blow-up time T (cf. Section 4.2 for the precise definition of $\bar{u}$) if $T < \infty$.

In analyzing this problem, it is imperative to study the classical positive steady-states of (1.1), i.e. the positive solutions of

$$\begin{cases} -\Delta u = \lambda u + a(x) u^p & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.4}$$

as well as the classical solutions and *metasolutions* (cf. Section 3 for further details) of the auxiliary problem

$$\begin{cases} -\Delta u = \lambda u - a_-(x)u^p & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{1.5}$$

since the positive solutions of (1.4) and the metasolutions of (1.5) provide us with the asymptotic behavior of the solution of (1.1), within the adequate parameter ranges, in a great variety of circumstances. Note that the metasolutions of (1.5) provide us with the limiting profiles of the positive solutions of the auxiliary problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a_-(x)u^p & \text{in } \ \Omega \times (0,\infty) \\ u(x,t) = 0 & \text{on } \ \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 > 0 & \text{in } \ \Omega \end{cases} \tag{1.6}$$

(cf. [21]) whose solutions are subsolutions of (1.1), since $a \geq -a_-$. Throughout this paper, we shall denote by $u_{[\lambda,a_-,\Omega]}(x,t;u_0)$ the unique solution of (1.6); it is globally defined in time, i.e., $T(u_0) = \infty$, since $a_- \geq 0$. Also, given any function $v \in \mathcal{C}(\Omega)$ it is said that $v > 0$ if $v(x) \geq 0$ for each $x \in \Omega$ and $v \neq 0$, and, given $v \in \mathcal{C}^1(\bar\Omega)$, it is said that $v \gg 0$ if $v(x) > 0$ for each $x \in \Omega$ and $\frac{\partial v}{\partial n}(x) < 0$ for each $x \in \partial\Omega \cap v^{-1}(0)$, where n stands for the outward unit normal to Ω at $x \in \partial\Omega$. Note that $u_{[\lambda,b,\Omega]} = 0$ if $u_0 = 0$, while $u_{[\lambda,b,\Omega]}(\cdot,t;u_0) \gg 0$ for each $t \in (0,T)$ if $u_0 > 0$, where $b \in \{a, -a_-\}$.

If u_0 is *sufficiently large*, then one can adapt very well known techniques to show that $u_{[\lambda,a,\Omega]}$ blows-up in $L^\infty(\Omega_+)$ in a finite time and, hence, $T < \infty$. In such case, some further sufficient conditions on the size of the exponent p can be given so that either $u_{[\lambda,a,\Omega]}$ exhibits complete blow-up in Ω_+, or it can admit some extension in a weak sense after the blow-up time (cf. [22] and the references there in). In this paper we focus our attention into the —possibly— most interesting case when $u_0 > 0$ is a strong subsolution of (1.5). So, we must restrict ourselves to deal with the special case when $\lambda > \sigma_0$, which is the range of values of λ where $u = 0$ is a linearly unstable solution of (1.1). Among the main findings in this work, we list the following:

1. If $\lambda \in (\sigma_0, \sigma_1)$ and a_+ is sufficiently small, then $T = \infty$ and $\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}$ equals the minimal positive solution of (1.4), which is the unique linearly stable steady-state of (1.1).

2. $T < \infty$ if $p > 1$, $\lambda \in (\sigma_0,\sigma_1)$ and a_+ is sufficiently large. Moreover, if $p-1>0$ is sufficiently small and the set of blow-up points of $u_{[\lambda,a,\Omega]}$, $B(u_0)$, satisfies $B(u_0) \subset \Omega_+$, then, $u_{[\lambda,a,\Omega]}$ blows-up completely in Ω_+ at time T, while $\lim_{t\uparrow\infty} \bar u(\cdot,t;u_0)$ (cf. Section 4.2) equals the minimal *large solution* of

$$-\Delta u = \lambda u - a_- u^p \tag{1.7}$$

   in $\Omega \setminus \bar\Omega_+$.

3. If $\lambda \in [\sigma_1, \sigma_2)$ and a_+ is sufficiently small with sufficiently fast decay to zero on $\partial\Omega_+$, then $T = \infty$ and

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]} = \begin{cases} \infty & \text{in } \ \Omega_0\,, \\ \mathfrak{L} & \text{in } \ \Omega \setminus \bar\Omega_0\,, \end{cases}$$

   where $\mathfrak{L}$ stands for the minimal large solution of (1.7) in $\Omega \setminus \bar\Omega_0$.

4. $T < \infty$ if $\lambda \in [\sigma_1, \sigma_2)$, $p > 1$ and a_+ is sufficiently large. Moreover, if $p - 1 > 0$ is sufficiently small and $B(u_0) \subset \Omega_+$, then, $u_{[\lambda,a,\Omega]}$ blows-up completely in Ω_+ at time T, while
$$\lim_{t\uparrow\infty} \bar{u}_{[\lambda,a,\Omega]} = \begin{cases} \infty & \text{in } \Omega_0\,, \\ \mathfrak{L} & \text{in } \Omega_-\,, \end{cases}$$
where $\mathfrak{L}$ stands for the minimal large solution of (1.7) in Ω_-.

5. If $\lambda \geq \sigma_2$, then Item 4 always occurs, independently on the size of a_+.

The organization of this paper is as follows. In Section 2 we recall the main existence result concerning the existence and the stability of the positive steady-states of (1.1), and in Section 3 we collect the main features concerning the dynamics of the associated sublinear parabolic problem (1.6). In Section 4 we prove Items 1 and 2, while the proofs of Items 3-5 are given in Sections 5-7, respectively.

Although there is a huge amount of literature dealing with superlinear indefinite elliptic problems (e.g., Ouyang [27], Alama and Tarantello [1], [2], Berestycki et al. [6], [7], Amann and López-Gómez [3], Gómez-Reñasco and López-Gómez [14], [15], M. Gaudenzi et al. [12], [13]), sublinear degenerate parabolic problems (e.g., Brezis and Oswald [9], Ouyang [26], Bandle and Marcus [4], Lazer and McKenna [17], Fraile et al. [11], Marcus and Véron [24], Gómez-Reñasco and López-Gómez [16], Du and Huang [10], López-Gómez [19], [20], [21]), and even superlinear indefinite singular elliptic problems (e.g., Mawhin et al. [25]), the unique available papers addressing the general problem of the asymptotic behavior of the solutions of (1.1) seem to be Gómez-Reñasco and López-Gómez [14], where the uniqueness of the linearly stable positive solution of (1.4) was shown, and the very recent paper López-Gómez and Quittner [22], where some sufficient conditions for complete blow-up were given for sufficiently large initial data u_0. In the very special case when $\Omega_+ = \Omega$, there are hundreds of papers on (1.1), of course (e.g., see the list of references of [22], as well as the list of references in each of them), but it should not be forgotten that we are dealing with the most general case when $a(x)$ changes sign, where live is much harder, as the model might simultaneously exhibit several kind of different behaviours according to the region of Ω where attention is focused. It should be noted that the uniqueness theorem obtained in [14] and [16] (cf. Theorem 2.1 here in) entails that all peak and multi-peak solutions of (1.4) constructed in the literature by means of the mountain-pass theorem and variants of it are unstable, and, therefore, they cannot be detected in real world models. In this paper we are ascertaining all possible stable limiting profiles of the solutions of (1.1), though we were not able to give complete proofs in all cases, but exclusively in some special circumstances of great interest.

2 Positive solutions of (1.4) The next theorem, going back to [14] and [15], collects the main existence, stability and multiplicity results concerning the classical positive solutions of (1.4). By a linearly stable solution, it is meant a solution such that the principal eigenvalue of its linearization is non-negative.

Theorem 2.1 *Problem* (1.4) *possesses a linearly stable positive solution for some* $\lambda \in \mathbb{R}$ *if, and only if,*

$$\int_\Omega a\varphi^{p+1} > 0\,, \tag{2.1}$$

where $\varphi \gg 0$ *denotes any principal eigenfunction associated with* σ_0. *Moreover, in such case, the following properties are satisfied:*

(a) *Let* Λ *denote the set of* $\lambda \in \mathbb{R}$ *for which* (1.4) *possesses a linearly stable positive solution. Then, there exists* $\lambda^* \in (\sigma_0, \sigma_1)$ *such that*

$$\Lambda \in \{\, (\sigma_0, \lambda^*), (\sigma_0, \lambda^*]\,\}$$

and (1.4) *does not admit a positive solution if* $\lambda \in (\sigma_0, \infty) \setminus \Lambda$.

(b) *For each* $\lambda \in \Lambda$*, the minimal positive solution of* (1.4)*, denoted by* $\theta_{[\lambda,a,\Omega]}$*, provides us with the unique linearly stable positive solution of* (1.4)*. Actually,* $\theta_{[\lambda,a,\Omega]}$ *is linearly asymptotically stable if* $\lambda \in \operatorname{Int}\Lambda$*, whereas* $\theta_{[\lambda^*,a,\Omega]}$ *is linearly neutrally stable if* $\lambda^* \in \Lambda$*, and, for each* $\lambda \in \Lambda$*,* $\theta_{[\lambda,a,\Omega]}$ *attracts all solutions of* (1.1) *with* $0 < u_0 \leq \theta_{[\lambda,a,\Omega]}$.

(c) *The solution curve*

$$\begin{array}{ccc} (0, \lambda^*) & \longrightarrow & \mathcal{C}(\bar{\Omega}) \\ \lambda & \mapsto & \theta_{[\lambda,a,\Omega]} \end{array}$$

is real analytic, strongly point-wise increasing, and it satisfies

1. $\lim_{\lambda \downarrow \sigma_0} \theta_{[\lambda,a,\Omega]} = 0$.
2. $\lim_{\lambda \uparrow \lambda^*} \theta_{[\lambda,a,\Omega]} = \theta_{[\lambda^*,a,\Omega]}$ *if* $\Lambda = (\sigma_0, \lambda^*]$.
3. $\lim_{\lambda \uparrow \lambda^*} \|\theta_{[\lambda,a,\Omega]}\|_{L^\infty(\Omega)} = \infty$ *if* $\Lambda = (\sigma_0, \lambda^*)$.

If, instead of (2.1)*, the following estimate is satisfied*

$$\int_\Omega a\varphi^{p+1} \leq 0\,, \tag{2.2}$$

then, (1.4) *cannot admit a positive solution if* $\lambda \geq \sigma_0$*. Moreover, any positive solution of* (1.4) *must be linearly unstable.*

In the presence of uniform a priori bounds in $L^\infty(\Omega)$ for the positive solutions of (1.4) on compact subsets of $\lambda \in \mathbb{R}$, necessarily $\Lambda = (\sigma_0, \lambda^*]$. Actually, in such case, (1.4) possesses at least two positive solutions for each $\lambda \in (\sigma_0, \lambda^*)$ (cf. [3]). Thanks to Theorem 2.1, except $\theta_{[\lambda,a,\Omega]}$, all of them must be linearly unstable. Note that (1.4) might exhibit an arbitrarily large number of positive solutions for any $\lambda \in (\sigma_0, \lambda^*)$ (cf. e.g., [14] and the references therein).

3 Dynamics of (1.6) This section collects the main results concerning the dynamics of (1.6). All those results go back to [16], [23], [19] and [21]. Hence, exclusively statements will be given. They are needed to carry out the mathematical analysis of the subsequent sections.

3.1 Positive classical solutions of (1.5) The following result characterizes the existence of positive solutions for (1.5).

Theorem 3.1 *Problem* (1.5) *has a positive solution if, and only if,* $\sigma_0 < \lambda < \sigma_1$*. Moreover, it is unique if it exists and if we denote it by* $\theta_{[\lambda,-a_-,\Omega]}$*, then the curve*

$$\begin{array}{ccc} (\sigma_0, \sigma_1) & \longrightarrow & \mathcal{C}(\bar{\Omega}) \\ \lambda & \mapsto & \theta_{[\lambda,-a_-,\Omega]} \end{array}$$

is real analytic, strongly point-wise increasing in Ω*, and it satisfies*

$$\lim_{\lambda \downarrow \sigma_0} \theta_{[\lambda,-a_-,\Omega]} = 0\,, \qquad \lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,-a_-,\Omega]} = \infty \quad \textit{unif. in compact sets of } \bar{\Omega}_0 \setminus \partial\Omega\,.$$

3.2 Large solutions of (1.5) Given $D \in \{\Omega \setminus \bar{\Omega}_+, \Omega \setminus \bar{\Omega}_0, \Omega_-\}$, any positive strong solution of the singular problem

$$\begin{cases} -\Delta u = \lambda u - a_- u^p & \text{in} \quad D \\ u = \infty & \text{on} \quad \partial D \setminus \partial\Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

is said to be a *large solution* of

$$-\Delta u = \lambda u - a_- u^p \tag{3.2}$$

in D. A function u is said to be a solution of (3.1) if it solves (3.2) in D and

$$\lim_{\substack{x\in D \\ \text{dist}\,(x,\partial\Omega)\downarrow 0}} u(x) = 0\,, \qquad \lim_{\substack{x\in D \\ \text{dist}\,(x,\partial D\setminus\partial\Omega)\downarrow 0}} u(x) = \infty\,.$$

The following result characterizes the existence of large solutions of (3.2) in each of these D's.

Theorem 3.2 *The following assertions are true:*

a) Equation (3.2) *has a large solution in $\Omega \setminus \bar{\Omega}_+$ if, and only if, $\lambda < \sigma_1$. Moreover, in such case, there are a minimal and a maximal large solution. The minimal (resp. maximal) large solution will be denoted by $\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]}$ (resp. $\mathfrak{L}^{\max}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]}$).*

b) Equation (3.2) *has a large solution in $\Omega \setminus \bar{\Omega}_0$ if, and only if, $\lambda < \sigma_2$. Moreover, in such case, there are a minimal and a maximal large solution. The minimal (resp. maximal) large solution will be denoted by $\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$ (resp. $\mathfrak{L}^{\max}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$).*

c) For each $\lambda \in \mathbb{R}$, (3.2) *has a large solution in Ω_-. Actually, it has a minimal and a maximal large solution, denoted by $\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]}$ and $\mathfrak{L}^{\max}_{[\lambda,-a_-,\Omega_-]}$, respectively.*

Subsequently, for any $M > 0$, $D \in \{\Omega \setminus \bar{\Omega}_+, \Omega \setminus \bar{\Omega}_0, \Omega_-\}$, and $u_0 \in \mathcal{C}(\bar{D})$, we denote by

$$u_{[\lambda,-a_-,D,M]}(x,t;u_0)$$

the unique solution of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a_- u^p & \text{in} \quad D \times (0,\infty) \\ u = M & \text{on} \quad (\partial D \setminus \partial\Omega) \times (0,\infty) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 \geq 0 & \text{in} \quad D \end{cases} \tag{3.3}$$

which is globally defined in time, since $a_- \geq 0$, and smooth, by parabolic regularity. Also, we will consider the associated elliptic problem

$$\begin{cases} -\Delta u = \lambda u - a_- u^p & \text{in} \quad D \\ u = M & \text{on} \quad \partial D \setminus \partial\Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.4}$$

Then, the next result is satisfied.

Theorem 3.3 *The following assertions are true:*

(a) *Suppose* $D = \Omega \setminus \bar{\Omega}_+$. *Then,* (3.4) *has a positive solution if, and only if,* $\lambda < \sigma_1$, *and it is unique if it exists. Let* $\theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+,M]}$ *denote it. Then, for each* $\lambda < \sigma_1$, *we have that*

$$\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]} := \lim_{M\uparrow\infty} \theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+,M]}\,,$$

and, for any $u_0 > 0$,

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+,M]}(\cdot,t;u_0) = \theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+,M]} \quad \textit{unif. in} \quad \bar{\Omega}\setminus\Omega_+\,.$$

(b) *Suppose* $D = \Omega \setminus \bar{\Omega}_0$. *Then,* (3.4) *has a positive solution if, and only if,* $\lambda < \sigma_2$, *and it is unique if it exists. Moreover, if we denote it by* $\theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0,M]}$, *then, for each* $\lambda < \sigma_2$, *we have that*

$$\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} := \lim_{M\uparrow\infty} \theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0,M]}\,,$$

and, for any $u_0 > 0$,

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0,M]}(\cdot,t;u_0) = \theta_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0,M]} \quad \textit{unif. in} \ \Omega\setminus\Omega_0\,.$$

(c) *Suppose* $D = \Omega_-$. *Then, for each* $\lambda \in \mathbb{R}$, (3.4) *has a unique positive solution. Moreover, if we denote it by* $\theta_{[\lambda,-a_-,\Omega_-,M]}$, *then*

$$\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]} := \lim_{M\uparrow\infty} \theta_{[\lambda,-a_-,\Omega_-,M]}\,,$$

and, for any $u_0 > 0$,

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega_-,M]}(\cdot,t;u_0) = \theta_{[\lambda,-a_-,\Omega_-,M]} \quad \textit{unif. in} \ \bar{\Omega}_-\,.$$

Furthermore, for each $D \in \{\Omega\setminus\bar{\Omega}_+, \Omega\setminus\bar{\Omega}_0, \Omega_-\}$, *the mapping* $M \mapsto \theta_{[\lambda,-a_-,D,M]}$ *is increasing, and, if* $\underline{u}$ *(resp.* $\overline{u}$*) is a strict subsolution (resp. supersolution) of* (3.4), *then* $\underline{u} \ll \theta_{[\lambda,-a_-,D,M]}$ *(resp.* $\overline{u} \gg \theta_{[\lambda,-a_-,D,M]}$*).*

Remark 3.4 Theorem 3.3 is also true changing $D \in \{\Omega\setminus\bar{\Omega}_+, \Omega\setminus\bar{\Omega}_0, \Omega_-\}$ by

$$D^\delta := \{\, x \in D \ : \ \mathrm{dist}\,(x, \partial D \cap \Omega) > \delta \,\}$$

provided $\delta > 0$ is sufficiently small.

3.3 Metasolutions of (3.2) Given $D \in \{\Omega\setminus\bar{\Omega}_+, \Omega\setminus\bar{\Omega}_0, \Omega_-\}$, a function

$$\mathfrak{M}_{[\lambda,-a_-,D]} \ : \ \Omega \to [0,\infty]$$

is said to be a *metasolution of* (3.2) *supported in* D if

$$\mathfrak{M}_{[\lambda,-a_-,D]} = \begin{cases} \infty & \text{in } \ \Omega\setminus\bar{D}\,, \\ \mathfrak{L}_{[\lambda,-a_-,D]} & \text{in } \ D\,, \end{cases}$$

for some large solution $\mathfrak{L}_{[\lambda,-a_-,D]}$ of (3.2) in D. In other words, metasolutions are extensions by ∞ to the totality of Ω of large solutions in D. As an immediate consequence from Theorem 3.3, the next result holds.

Theorem 3.5 *The following assertions are true:*

(a) (3.2) *has a metasolution supported in* $\Omega \setminus \bar{\Omega}_+$ *if, and only if,* $\lambda < \sigma_1$. *Moreover, in such case, there are a minimal and a maximal metasolution supported in* $\Omega \setminus \bar{\Omega}_+$*; subsequently denoted by* $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]}$ *and* $\mathfrak{M}^{\max}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]}$*, respectively.*

(b) (3.2) *has a metasolution supported in* $\Omega \setminus \bar{\Omega}_0$ *if, and only if,* $\lambda < \sigma_2$. *Moreover, in such case, there are a minimal and a maximal metasolution supported in* $\Omega \setminus \bar{\Omega}_0$*; subsequently denoted by* $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$ *and* $\mathfrak{M}^{\max}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$*, respectively.*

(c) *For each* $\lambda \in \mathbb{R}$, (3.2) *has a metasolution supported in* Ω_-. *Actually, there are a minimal and a maximal metasolution supported in* Ω_-*; subsequently denoted by* $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega_-]}$ *and* $\mathfrak{M}^{\max}_{[\lambda,-a_-,\Omega_-]}$*, respectively.*

Moreover, the following relations are satisfied

$$\begin{aligned} \lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,-a_-,\Omega]} &= \mathfrak{M}^{\min}_{[\sigma_1,-a_-,\Omega\setminus\bar{\Omega}_0]}\,, \\ \lim_{\lambda\uparrow\sigma_2} \mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} &= \mathfrak{M}^{\min}_{[\sigma_2,-a_-,\Omega_-]}\,, \\ \lim_{\lambda\uparrow\sigma_1} \mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]} &= \mathfrak{M}^{\min}_{[\sigma_1,-a_-,\Omega_-]}\,, \end{aligned} \tag{3.5}$$

the first limit being uniform in compact subsets of $\bar{\Omega}_0 \setminus \partial\Omega$ *and* $\Omega \setminus \bar{\Omega}_0$*; the second one in compact subsets of* $\Omega \setminus \Omega_-$ *and* Ω_-*, and the third one in compact subsets of* $\bar{\Omega}_0 \setminus \partial\Omega$ *and* Ω_-.

3.4 Dynamics of (1.6) The following result provides us with the dynamics of (1.6) according to the size of $\lambda \in \mathbb{R}$.

Theorem 3.6 *Suppose* $u_0 \in \mathcal{C}(\bar{\Omega})$, $u_0 > 0$. *Then:*

(a) $\lim_{t\uparrow\infty} \|u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0)\|_{\mathcal{C}(\bar{\Omega})} = 0$ *if* $\lambda \le \sigma_0$.

(b) $\lim_{t\uparrow\infty} \|u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) - \theta_{[\lambda,-a_-,\Omega]}\|_{\mathcal{C}(\bar{\Omega})} = 0$ *if* $\sigma_0 < \lambda < \sigma_1$.

(c) *Suppose* $\sigma_1 \le \lambda < \sigma_2$. *Then,*

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) = \infty \quad \textit{unif. on compact sets of } \bar{\Omega}_0 \setminus \partial\Omega\,,$$

whereas, in $\Omega \setminus \bar{\Omega}_0$,

$$\begin{aligned} \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} &\le \liminf_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \\ &\le \limsup_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \le \mathfrak{L}^{\max}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}\,. \end{aligned}$$

If, in addition, u_0 *is a subsolution of* (1.5)*, then*

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} \qquad \textit{in } \Omega\,. \tag{3.6}$$

(d) *Suppose* $\lambda \ge \sigma_2$. *Then,*

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) = \infty \quad \textit{unif. on compact sets of } \Omega \setminus \Omega_-\,,$$

whereas, in Ω_-,

$$\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]} \leq \liminf_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \leq \limsup_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \leq \mathfrak{L}^{\max}_{[\lambda,-a_-,\Omega_-]}\,.$$

If, in addition, u_0 *is a subsolution of* (1.5)*, then*

$$\lim_{t\uparrow\infty} u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega_-]} \qquad \text{in } \Omega\,. \tag{3.7}$$

Theorem 3.6 establishes that the maximal non-negative solution of (1.5) provides us with the asymptotic behavior of the solutions of (1.6) if $\lambda < \sigma_1$, whereas the dynamics of (1.6) is governed by the metasolutions of (1.5) supported in $\Omega\setminus\bar{\Omega}_0$ if $\sigma_1 \leq \lambda < \sigma_2$, and the metasolutions of (1.5) supported in Ω_- if $\lambda \geq \sigma_2$.

In general, large solutions —and, hence, metasolutions— supported in $\Omega\setminus\bar{\Omega}_0$ and Ω_- are not necessarily unique. Thus, (3.6) and/or (3.7) might fail when u_0 is not a subsolution of (1.5). Some uniqueness results, that will be used later, can be found in [20].

Figure 2 represents all possible limiting profiles of the solutions of (1.6) according to the size of λ. In all cases we have represented a one-dimensional slice of the limiting profile. To discuss the diagram we will assume that u_0 is a subsolution of (1.5). When $\lambda \leq \sigma_0$, all solutions approach zero. When $\sigma_0 < \lambda < \sigma_1$ all solutions approach the unique positive steady state. As $\lambda \uparrow \sigma_1$, the steady states approach $\mathfrak{M}^{\min}_{[\sigma_1,-a_-,\Omega\setminus\bar{\Omega}_0]}$. Rather naturally, for each $\sigma_1 \leq \lambda < \sigma_2$, $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$ provides us with the limiting profiles of the solutions of (1.6). As $\lambda \uparrow \sigma_2$, $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$ approximates $\mathfrak{M}^{\min}_{[\sigma_2,-a_-,\Omega_-]}$, which is the limiting profile of all positive solutions of (1.6), with $\lambda = \sigma_2$, as $t \uparrow \infty$. Actually, $\lambda \geq \sigma_2$, $\mathfrak{M}^{\min}_{[\lambda,-a_-,\Omega_-]}$ provides us with the asymptotic behaviors of all positive solutions of (1.6).

4 Dynamics of (1.1) for $\lambda \in (\sigma_0,\sigma_1)$

4.1 Global existence versus blow-up in finite time The next result shows that the behaviour of (1.1) is strongly based on the relative size of a_+ with respect to λ.

Theorem 4.1 *Suppose* $\lambda \in (\sigma_0,\sigma_1)$ *and* u_0 *is a positive strict subsolution of* (1.5)*. Then:*

(a) *There exists* $\varepsilon > 0$ *such that* (1.4) *has a positive solution if* $\|a_+\|_\infty \leq \varepsilon$*. Moreover,*

$$\lim_{t\uparrow\infty} \|u_{[\lambda,a,\Omega]}(\cdot,t;u_0) - \theta_{[\lambda,a,\Omega]}\|_{\mathcal{C}(\bar{\Omega})} = 0\,, \tag{4.1}$$

where $\theta_{[\lambda,a,\Omega]}$ *stands for the minimal positive solution of* (1.4).

(b) *Suppose there is a smooth subdomain* $D \subset \Omega_+$*, with* $\bar{D} \subset \Omega_+$*, such that*

$$A := \min_{\bar{D}} a_+ > \frac{\omega}{p}\left[\frac{(p-1)\omega}{p\,\alpha}\right]^{p-1}, \tag{4.2}$$

where

$$\begin{aligned} \omega &= \omega(\lambda,D) := \sigma[-\Delta,D] - \lambda\,,\\ \alpha &= \alpha(\lambda,D) := -\int_{\partial D} \frac{\partial\varphi}{\partial n}\,\theta_{[\lambda,-a_-,\Omega]}\,d\sigma\,, \end{aligned} \tag{4.3}$$

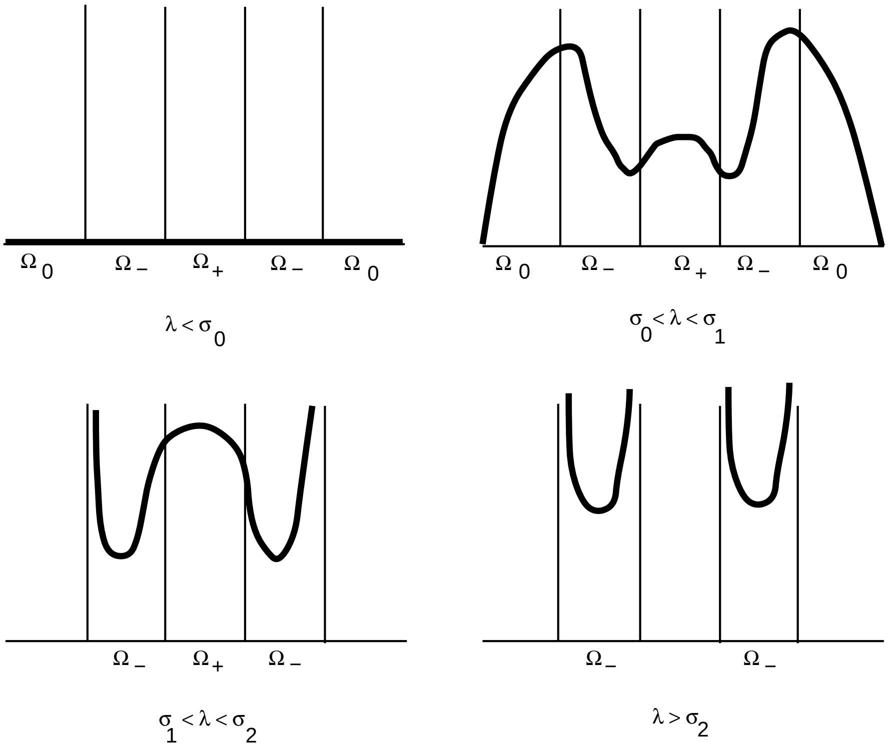

Figure 2: The asymptotic profiles as $t \uparrow \infty$ of the solutions of (1.6).

$\theta_{[\lambda,-a_-,\Omega]}$ *is the unique positive solution of* (1.5), $\varphi \gg 0$ *is the unique principal eigenfunction of* $\sigma[-\Delta, D]$ *satisfying* $\int_D \varphi = 1$, *and* n *stands for the outward unit normal to* D. *Then,* $u_{[\lambda,a,\Omega]}(x,t;u_0)$ *blows-up in* $L^\infty(\Omega)$ *at some time* $T^b = T^b(u_0) > 0$.

Proof: Fix $\lambda \in (\sigma_0, \sigma_1)$ and let $u_0 > 0$ be a strict subsolution of (1.5). Such subsolution exists, since $\lambda > \sigma_0$; one can take a sufficiently small positive multiple of a principal eigenfunction of $-\Delta$ in Ω. Then, by the uniqueness of the positive solution of (1.5),

$$u_0 \ll \theta_{[\lambda,-a_-,\Omega]} \,. \tag{4.4}$$

Also, u_0 is a subsolution of (1.4) and, hence, the mapping $t \mapsto u_{[\lambda,a,\Omega]}(\cdot,t;u_0)$ is increasing. Moreover, for each $t > 0$,

$$u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \qquad \text{in } \Omega \,,$$

and, hence, thanks to Theorem 3.6(b), if $T = \infty$, then

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq \theta_{[\lambda,-a_-,\Omega]} \,. \tag{4.5}$$

Now, we shall prove Part (a). Since

$$\sigma[-\Delta + pa_-\theta^{p-1}_{[\lambda,-a_-,\Omega]} - \lambda, \Omega] > \sigma[-\Delta + a_-\theta^{p-1}_{[\lambda,-a_-,\Omega]} - \lambda, \Omega] = 0 \,,$$

it follows from the implicit function theorem that there is $\delta_0 > 0$ such that, for each $\delta \in [0, \delta_0)$, the problem

$$\begin{cases} -\Delta u = \lambda u + [\delta a_+ - a_-]u^p & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

has a positive solution in a neighborhood of $\theta_{[\lambda,-a_-,\Omega]}$. Thus, there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that (1.4) possesses a positive solution if $\|a_+\|_\infty < \varepsilon$. Thanks to Theorem 2.1, the minimal positive solution $\theta_{[\lambda,a,\Omega]}$ is the unique linearly stable non-negative solution of (1.4), Moreover, it is apparent, from the maximum principle, that

$$\theta_{[\lambda,-a_-,\Omega]} \ll \theta_{[\lambda,a,\Omega]}\,,$$

since $\theta_{[\lambda,a,\Omega]}$ is a positive strict supersolution of (1.5). Therefore, thanks to (4.4), we have that

$$u_0 \ll \theta_{[\lambda,a,\Omega]}\,,$$

and (4.1) follows readily from Theorem 2.1. This concludes the proof of Part (a).

Now, we will prove Part (b). Since

$$\lambda < \sigma_1 = \sigma[-\Delta,\Omega_0] < \sigma_2 = \sigma[-\Delta,\Omega_+] < \sigma[-\Delta,D]\,,$$

we have that

$$\omega = \sigma[-\Delta,D] - \lambda > 0\,.$$

Also,

$$\alpha = -\int_{\partial D} \frac{\partial\varphi}{\partial n}\,\theta_{[\lambda,-a_-,\Omega]}\,d\sigma > 0\,,$$

since $\frac{\partial\varphi}{\partial n} \ll 0$ on ∂D. Consequently, the constant in the right hand side of (4.2),

$$A_c(\lambda,D) := \frac{\omega(\lambda,D)}{p}\left[\frac{(p-1)\omega(\lambda,D)}{p\,\alpha(\lambda,D)}\right]^{p-1}, \tag{4.6}$$

is positive. Suppose (4.2) and $u_{[\lambda,a,\Omega]}(x,t;u_0)$ is defined for all $t > 0$, and set, for each $t > 0$,

$$I(t) := \int_D u_{[\lambda,a,\Omega]}(x,t;u_0)\varphi(x)\,dx \in (0,\infty)\,.$$

Note that $I(t)$ is increasing, since u_0 is a subsolution of (1.4). Thus,

$$L := \lim_{t\uparrow\infty} I(t) \in (0,\infty] \tag{4.7}$$

is well defined.

On the other hand, setting $u := u_{[\lambda,a,\Omega]}(x,t;u_0)$, multiplying by φ the u-equation of (1.1), integrating in D, and applying the formula of integration by parts gives

$$\begin{aligned} I'(t) &= \int_D \varphi\Delta u\,dx + \lambda\int_D \varphi u\,dx + \int_D a u^p\varphi\,dx \\ &\geq -\int_{\partial D} u\frac{\partial\varphi}{\partial n}\,d\sigma + (\lambda - \sigma[-\Delta,D])\,I(t) + A\int_D u^p\varphi\,dx\,, \end{aligned}$$

where $' := \frac{d}{dt}$ and $A := \min_{\bar D} a$. Moreover,

$$\int_D u\varphi\,dx = \int_D \varphi^{1-\frac{1}{p}}\varphi^{\frac{1}{p}}u\,dx \leq \left(\int_D \varphi\right)^{1-\frac{1}{p}}\left(\int_D \varphi u^p\,dx\right)^{\frac{1}{p}} = \left(\int_D u^p\varphi\,dx\right)^{\frac{1}{p}},$$

since $\int_D \varphi = 1$, and, hence,

$$\int_D u^p \varphi \, dx \geq \left(\int_D u \varphi \, dx \right)^p .$$

Consequently, for each $t > 0$,

$$I'(t) \geq - \int_{\partial D} u \frac{\partial \varphi}{\partial n} \, d\sigma - \omega \, I(t) + A \, I^p(t) \, . \tag{4.8}$$

Suppose $L < \infty$. Then,

$$\lim_{t \uparrow \infty} I'(t) = 0 \, ,$$

and, passing to the limit as $t \uparrow \infty$ in (4.8), we find from (4.5) that

$$0 \geq \alpha - \omega \, L + A \, L^p \, , \tag{4.9}$$

where α is the constant given by (4.3). Note that introducing the function f defined by

$$f(x) := A x^p - \omega x + \alpha \, , \qquad x > 0 \, ,$$

(4.9) can be equivalently written as

$$f(L) \leq 0 \, . \tag{4.10}$$

The function f satisfies

$$f(0) = \alpha > 0 \, , \quad \lim_{x \uparrow \infty} f(x) = \infty \, , \quad \text{and} \;\; f'(x) = pAx^{p-1} - \omega \quad \forall \, x > 0 \, .$$

Thus,

$$f'(x) = 0 \quad \text{if and only if} \quad x = x_L := \left(\frac{\omega}{pA} \right)^{\frac{1}{p-1}}$$

and, due to (4.10), necessarily

$$f(x_L) = A \left(\frac{\omega}{pA} \right)^{\frac{p}{p-1}} - \omega \left(\frac{\omega}{pA} \right)^{\frac{1}{p-1}} + \alpha \leq 0 \, .$$

Equivalently,

$$A \leq \frac{\omega}{p} \left[\frac{(p-1)\omega}{p \, \alpha} \right]^{p-1} ,$$

which contradicts (4.2). Therefore, $L = \infty$. Note that (4.8) gives

$$I'(t) \geq -\omega \, I(t) + A \, I^p(t) \, , \qquad \forall \, t > 0 \, . \tag{4.11}$$

Thanks to (4.7), there exists $t_0 > 0$ such that

$$I(t_0) > \left(\frac{\omega}{A} \right)^{\frac{1}{p-1}} \, . \tag{4.12}$$

Moreover, the change of variable

$$I(t) = e^{-\omega(t - t_0)} J(t) \, , \qquad t \geq t_0 > 0 \, ,$$

transforms (4.11), (4.12) into

$$
\left\{
\begin{array}{ll}
J'(t) \geq A e^{-\omega(p-1)(t-t_0)} J^p(t)\,, & t \geq t_0 > 0\,,\\
J(t_0) = I(t_0) > \left(\frac{\omega}{A}\right)^{\frac{1}{p-1}}\,. &
\end{array}
\right.
\tag{4.13}
$$

Thus, integrating the differential inequality of (4.13) gives

$$
\frac{1}{1-p}\left[J^{1-p}(t) - J^{1-p}(t_0)\right] \geq \frac{-A}{\omega(p-1)}\left[e^{-\omega(p-1)(t-t_0)} - 1\right] \qquad \forall\, t > t_0\,,
$$

and, hence,

$$
J^{1-p}(t) \leq J^{1-p}(t_0) - \frac{A}{\omega}\left[1 - e^{-\omega(p-1)(t-t_0)}\right] \qquad \forall\, t > t_0\,,
\tag{4.14}
$$

since $p > 1$. Consequently, passing to the limit as $t \uparrow \infty$ in (4.14), we find from (4.7) —with $L = \infty$— that

$$
0 \leq J^{1-p}(t_0) - \frac{A}{\omega}\,,
$$

since $1 - p < 0$. Equivalently,

$$
I(t_0) = J(t_0) \leq \left(\frac{\omega}{A}\right)^{\frac{1}{p-1}}\,,
$$

which contradicts (4.12). Therefore, in either case we reach a contradiction; coming from the assumption that u is globally defined in time, which concludes the proof. $\square$

Note that $\lambda \mapsto \omega(\lambda, D)$ is decreasing and $\lambda \mapsto \alpha(\lambda, D)$ is increasing, since $\lambda \mapsto \theta_{[\lambda,-a_-,\Omega]}|_D$ is point-wise increasing. Thus, the mapping $\lambda \mapsto A_c(\lambda, D)$ is decreasing; a rather natural feature establishing that as bigger is λ as smaller can be taken $\min_{\bar{D}} a$ for $u_{[\lambda,a,\Omega]}$ to blow-up in L^∞.

4.2 Complete blow-up in Ω_+ Subsequently, we denote by $T^b = T^b(u_0)$ the $L^\infty(\Omega)$-blow-up-time of the solution of Theorem 4.1(b) to show that it does not admit a weak continuation for $t > T^b$ if $p - 1 > 0$ is sufficiently small and the blow-up set at time T^b lyes within Ω_+. To carry out this analysis, we introduce the following approximating functions

$$
f_k(x, u) := \left\{
\begin{array}{ll}
\lambda u + a(x)\min\{u^p, k\} & \text{if } x \in \Omega_+ \cup \Omega_0,\\
\lambda u + a(x) u^p & \text{if } x \in \Omega_-,
\end{array}
\right.
\qquad k \in \mathbb{N}\,,
\tag{4.15}
$$

as well as the associated approximating problems

$$
\left\{
\begin{array}{ll}
u_t - \Delta u = f_k(x, u), & x \in \Omega,\ t > 0,\\
u(x, t) = 0, & x \in \partial\Omega,\ t > 0,\\
u(x, 0) = u_0(x), & x \in \Omega,
\end{array}
\right.
\qquad k \in \mathbb{N}.
\tag{4.16}
$$

For each $k \in \mathbb{N}$, let $u_k := u_k(x, t; u_0)$ denote the solution of (4.16). Clearly, u_k is globally defined in time and it satisfies $u_k \leq u_{k+1}$, since $f_k \leq f_{k+1}$. Thus, the limit

$$
\bar{u}(x, t) = \bar{u}(x, t; u_0) := \lim_{k\to\infty} u_k(x, t; u_0) \in (0, \infty]\,, \qquad (x, t) \in \Omega \times (0, \infty),
\tag{4.17}
$$

is well defined. Note that $\bar{u}(x, t) = u(x, t; u_0)$ for any $x \in \bar{\Omega}$ and $t < T^b$, though the problem of finding out $\bar{u}(x, t)$ for $t \geq T^b$ might be extremely involved. Also, we denote by

$$
B(u_0) := \{x \in \bar{\Omega}\ :\ (\exists x_k \to x)(\exists t_k \uparrow T^b)\ \text{ such that } \lim_{k\to\infty} u_{[\lambda,a,\Omega]}(x_k, t_k; u_0) = \infty\}
$$

the *blow-up set* of $u_{[\lambda,a,\Omega]}(x,t;u_0)$, and set

$$u(x,t;u_0) := u_{[\lambda,a,\Omega]}(x,t;u_0)\,, \qquad L_{u_0}(t) := E(u(\cdot,t;u_0)),$$

where

$$E(w) := \int_\Omega \Big(\frac{1}{2}|\nabla w|^2 - \frac{\lambda}{2}w^2 - \frac{a}{p+1}|w|^{p+1}\Big)\,dx, \qquad w\in H^1_0(\Omega),$$

is the associated *energy functional.* A direct calculation shows that

$$\frac{dL_{u_0}}{dt}(t) = -\int_\Omega u_t^2(x,t;u_0)\,dx \le 0$$

and, hence, L_{u_0} is decreasing in $[0,T^b)$.

The following result shows that, under the general assumptions of Theorem 4.1(b), $\bar u(x,t)=\infty$ for any $x\in\Omega_+$ and $t>T^b$ if $B(u_0)\subset\Omega_+$ and $p-1>0$ is sufficiently small. Therefore, under these assumptions, the solution $u_{[\lambda,a,\Omega]}(x,t;u_0)$ blows-up completely in Ω_+ at time T^b. Consequently, it does not admit any weak extension in Ω_+ after time T^b (cf. [22], and the references therein, for a further detailed discussion).

Theorem 4.2 *Suppose $\lambda\in(\sigma_0,\sigma_1)$, $u_0>0$ is a strict subsolution of* (1.5)*, and* (4.2) *is satisfied for some smooth domain D with $\bar D\subset\Omega_+$. Then, $u(x,t;u_0):=u_{[\lambda,a,\Omega]}(x,t;u_0)$ blows-up in $L^\infty(\Omega)$ in a finite time T^b, and*

$$\bar u(\cdot,t;u_0) = u(\cdot,t;u_0) \le \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar\Omega_+]} \qquad \text{in } \Omega\setminus\bar\Omega_+\,.$$

In particular, $B(u_0)\subset\bar\Omega_+$. Moreover, if either

$$\text{(4.18)}\qquad p< p_{CL} := \begin{cases} +\infty, & \text{if } n=1,\\ (3n+8)/(3n-4), & \text{if } n>1,\end{cases}$$

or

$$\text{(4.19)}\qquad a_+(x) = \alpha_+(x)[\operatorname{dist}(x,\partial\Omega_+)]^\gamma \quad \text{for } x\in\Omega_+ \text{ near } \partial\Omega_+,$$

where α_+ is a positive continuous function and $\gamma>0$ is a constant satisfying

$$\text{(4.20)}\qquad p<\min\{(n+1+\gamma)/(n-1)\,,\,(n+2)/(n-2)\} \qquad \text{if} \quad n\ge 3\,,$$

then,

$$\text{(4.21)}\qquad \lim_{t\uparrow T^b} L_{u_0}(t) = -\infty\,.$$

Further, if, in addition, $B(u_0)\subset\Omega_+$, then

$$\text{(4.22)}\qquad \bar u(x,t;u_0)=+\infty \quad \text{for any } (x,t)\in\Omega_+\times(T^b,\infty)\,,$$

i.e., $u(x,t;u_0)$ blows-up completely in Ω_+.

Proof: The fact that u blows-up in $L^\infty(\Omega)$ in a finite time T^b follows from Theorem 4.1(b). Note that, thanks to Theorem 3.2(a), $\mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar\Omega_+]}$ is well defined, since $\lambda<\sigma_1$. Moreover, since u_0 is a subsolution of (1.4), for each $t>0$ the restriction $u(\cdot,t;u_0)|_{\Omega\setminus\bar\Omega_+}$ provides us with a subsolution of (3.1) (with $D=\Omega\setminus\bar\Omega_+$) and, hence,

$$u(\cdot,t;u_0)\le \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar\Omega_+]} \qquad \text{in} \quad \Omega\setminus\bar\Omega_+\,.$$

Thus, for each $t > 0$,

$$\bar{u}(x,t;u_0) = \lim_{k\to\infty} u_k(x,t;u_0) = u(x,t;u_0)$$

uniformly in compact subsets of $\Omega \setminus \bar{\Omega}_+$. In particular, $B(u_0) \subset \bar{\Omega}_+$. Now, suppose that either (4.18), or (4.19) and (4.20) are satisfied. Then, (4.21) is a consequence from the results of [22, Section 5]. Therefore, assuming $B(u_0) \subset \Omega_+$, (4.22) is an immediate consequence from [22, Theorem 1.1] applied in the domain $\Omega_\delta := \{\, x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega_+) < \delta \,\}$, for a sufficiently small $\delta > 0$, instead of in Ω, where the associated Ω_0 is empty. This concludes the proof. $\square$

It should be noted that the point-wise limit

$$\mathfrak{M}_{[\lambda,a,\Omega]}(x;u_0) := \begin{cases} +\infty & \text{if } x \in \Omega_+\,, \\ \lim_{t\uparrow\infty} \bar{u}(x,t;u_0) & \text{if } x \in \Omega \setminus \Omega_+\,, \end{cases}$$

is well defined. Actually, it provides us with the minimal positive strong solution of

$$\begin{cases} -\Delta w = \lambda w - a_- w^p & \text{in } \ \Omega \setminus \bar{\Omega}_+\,, \\ w = \lim_{t\uparrow\infty} \bar{u} & \text{on } \ \partial\Omega_+\,, \\ w = 0 & \text{on } \ \partial\Omega\,. \end{cases}$$

Also, note that the well known fact that initial data u_0 which are subsolutions to (1.4) give rise to non-decreasing in time solutions of (1.1) as long as the solution is classical remains valid as well for the very weak extension $\bar{u}$ in $(0,\infty)$, since u_0 is a subsolution of (4.16) whenever $k > \|u_0\|_\infty$, and, hence, u_k is non-decreasing in time on $(0,\infty)$. Consequently, the property is inherited by its limit $\bar{u}$, and, therefore, $\lim_{t\uparrow\infty} \bar{u}$ is well defined in Ω.

In the radially symmetric case, Theorem 4.2 can be sharpened up to obtain the following result.

Theorem 4.3 *Suppose* $0 < R_1 < R_2 < R$*,*

$$\Omega = B_R := \{x \in \mathbb{R}^n \ : \ |x| < R\}\,, \quad \Omega_+ = B_{R_1}\,, \quad \Omega_- = B_{R_2} \setminus \bar{B}_{R_1}\,, \quad \Omega_0 = B_R \setminus \bar{B}_{R_2}\,,$$

$$\sigma_0 := \sigma[-\Delta, B_R] < \lambda < \sigma_1 := \sigma[-\Delta, \Omega_0] < \sigma_2 := \sigma[-\Delta, \Omega_+]\,,$$

$a(x) = a(|x|)$, $a \in \mathcal{C}^1([0,R])$, *satisfies* (4.2) *for some smooth domain* D *with* $\bar{D} \subset \Omega_+$, *and* $u_0(x) = u_0(|x|)$, $u_0 \in \mathcal{C}^1([0,R])$, *is a positive strict subsolution of* (1.5) *in* B_R. *Then, the solution* $u(x,t;u_0) := u_{[\lambda,a,\Omega]}(x,t;u_0)$ *of*

$$\begin{cases} u_t - \Delta u = \lambda u + a(|x|)u^p, & x \in B_R,\ t > 0, \\ u(x,t) = 0, & x \in \partial B_R,\ t > 0, \\ u(x,0) = u_0(|x|), & x \in B_R, \end{cases} \tag{4.23}$$

blows up in a finite time $T^b := T^b(u_0) < \infty$ *in* L^∞. *Suppose, in addition, the following:*

(A1) $1 < p < (n+2)/(n-2)$ *if* $n \geq 3$.

(A2) *There exists* $\rho \in (R_1, R_2)$ *such that* $a'(r) \leq 0$ *for each* $r \in [0,\rho]$, *and*

$$\int_0^\rho a(r) r^{n-1}\, dr > 0\,, \qquad \sigma[-\Delta, B_\rho] < \lambda < \sigma_1\,. \tag{4.24}$$

(A3) $u_0(r)$ *is non-increasing.*

Then, there exists $T^c \geq T^b$ such that

$$\bar{u}(x,t;u_0) = +\infty \quad \textit{for any } (x,t) \in \bar{\Omega}_+ \times (T^c, \infty)\,, \tag{4.25}$$

while

$$\bar{u}(\cdot,t;u_0) = u(\cdot,t;u_0) \leq \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]} \qquad \textit{in } \Omega\setminus\bar{\Omega}_+ \tag{4.26}$$

is a classical solution for each $t>0$. Moreover, if $\lim_{t\uparrow T^b} u(R_1,t;u_0) = \infty$, then

$$\lim_{t\uparrow\infty} u(\cdot,t;u_0) = \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]} \qquad \textit{in } \Omega\setminus\bar{\Omega}_+\,, \tag{4.27}$$

uniformly on compact sets of $\Omega\setminus\bar{\Omega}_+$, and, hence,

$$\lim_{t\uparrow\infty} \bar{u}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,a,\Omega]}(\cdot;u_0) := \begin{cases} +\infty & \textit{in } \bar{\Omega}_+\,,\\ \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_+]} & \textit{in } \Omega\setminus\bar{\Omega}_+\,. \end{cases}$$

Consequently, in such case, the asymptotic behaviour of $u_{[\lambda,a,\Omega]}$ is governed by the minimal metasolution of (3.2) *supported in $\Omega\setminus\bar{\Omega}_+$.*

To construct examples satisfying the requirements of the statement, one can proceed as follows. First, fix $p>1$ satisfying $p<(n+2)/(n-2)$ if $n\geq 3$, and $R>0$. Then, choose $R_2\in(0,R)$ sufficiently close to R so that

$$\sigma[-\Delta,B_{R_2}] < \sigma[-\Delta,B_R\setminus\bar{B}_{R_2}] = \sigma[-\Delta,\Omega_0]\,. \tag{4.28}$$

Note that, thanks to Faber-Krahn inequality and the continuous dependence of the principal eigenvalue with respect to the domain, we have that

$$\lim_{R_2\uparrow R}\sigma[-\Delta,B_{R_2}] = \sigma[-\Delta,B_R] \quad \text{and} \quad \lim_{R_2\uparrow R}\sigma[-\Delta,B_R\setminus\bar{B}_{R_2}] = \infty\,,$$

since $|B_R\setminus\bar{B}_{R_2}|\downarrow 0$ if $R_2\uparrow R$ (e.g., [18]). Thus, (4.28) can be easily reached. Once obtained (4.28) one has to choose a sufficiently small $R_1\in(0,R_2)$ such that

$$\sigma[-\Delta,B_R\setminus\bar{B}_{R_2}] = \sigma[-\Delta,\Omega_0] < \sigma[-\Delta,B_{R_1}] = \sigma[-\Delta,\Omega_+]\,,$$

which is possible, since

$$\lim_{R_1\downarrow 0}\sigma[-\Delta,B_{R_1}] = \infty\,.$$

Now, pick λ satisfying

$$\sigma[-\Delta,B_{R_2}] < \lambda < \sigma[-\Delta,B_R\setminus\bar{B}_{R_2}]$$

and $\rho\in(R_1,R_2)$ sufficiently close to R_2 so that

$$\sigma[-\Delta,B_\rho] < \lambda < \sigma[-\Delta,B_R\setminus\bar{B}_{R_2}]\,.$$

Finally, by choosing an adequate $a(r)$, all the requirements of the theorem are fulfilled.

Proof of Theorem 4.3: The fact that u blows-up in $L^\infty(\Omega)$ in a finite time T^b and (4.26) follow from Theorem 4.2(b). Subsequently, we suppose (A1)-(A3). In particular,

$$a(0)>0>a(\rho)\,, \qquad \int_0^\rho a(r)r^{n-1}\,dr > 0\,.$$

Pick $t_0 \in (0, T^b)$ and consider the auxiliary problem

$$\begin{cases} v_t - \Delta v = \lambda v + a(|x|)v^p, & x \in B_\rho,\ t > t_0, \\ v(x,t) = 0, & x \in \partial B_\rho,\ t > t_0, \\ v(x,t_0) = \delta\varphi, & x \in B_\rho, \end{cases} \tag{4.29}$$

where $\varphi > 0$ stands for a principal eigenfunction associated with $\sigma[-\Delta, B_\rho]$ and $\delta > 0$ is sufficiently small so that $\delta\varphi$ be a strict subsolution of $-\Delta v = \lambda v - a_- v^p$ in B_ρ (it can be accomplished since $\lambda > \sigma[-\Delta, B_\rho]$) satisfying

$$\delta\varphi < u(\cdot, t_0; u_0) \qquad \text{in } \bar{B}_\delta\,. \tag{4.30}$$

Let $v(x,t;\delta\varphi)$ denote the solution of (4.29) and $\bar{v}(x,t;\delta\varphi)$ its associated function through the approximating process (4.16), (4.17). Thanks to (4.30), we find from the parabolic maximum principle that, for each $t > 0$ and $x \in B_\rho$,

$$\bar{v}(x,t;\delta\varphi) \le \bar{u}(x,t+t_0;u_0)\,. \tag{4.31}$$

Moreover, thanks to Theorem 4.1, $v(x,t;\delta\varphi)$ blows-up in $L^\infty(B_\rho)$ in a finite time $\tilde{T}^b \ge T^b - t_0$. Thus, thanks to [22, Theorem 1.3],

$$\bar{v}(x,t;\delta v) = +\infty \qquad \text{for each} \quad (x,t) \in \bar{B}_{R_1} \times (\tilde{T}^b, \infty)\,.$$

Therefore, setting $T^c := \tilde{T}^b + t_0 \ge T^b$, we find from (4.31) that

$$\bar{u}(x,t;u_0) = +\infty \qquad \text{for each} \quad (x,t) \in \bar{B}_{R_1} \times (T^c, \infty)\,,$$

which concludes the proof of (4.25).

Now, set $u = u(r,t)$, for each $r \in [0,R]$ and $t \in [0,T^b)$, and suppose that

$$\lim_{k\to\infty} u(R_1, t_k; u_0) = \infty \tag{4.32}$$

for some sequence $t_k \uparrow T^b$. The auxiliary function

$$w(r,t) := r^{n-1}u_r(r,t), \qquad (r,t) \in [0,R] \times [0,T^b),$$

satisfies

$$w(0,t) = 0, \quad w(R,t) \le 0, \quad w(r,0) \le 0,$$

and, differentiating with respect to r the u-equation, multiplying by r^{n-1} the resulting identity, and rearranging terms gives

$$\begin{aligned} w_t &= w_{rr} - \frac{n-1}{r} w_r + \lambda w + apu^{p-1}w + a_r r^{n-1}u^p \\ &\le w_{rr} - \frac{n-1}{r} w_r + (\lambda + apu^{p-1})w \qquad \text{in } (0,R) \times (0,T^b). \end{aligned}$$

Thus, $w \le 0$ and, hence, $u_r \le 0$ in $(0,R) \times (0,T^b)$. Therefore, it follows from (4.32) that

$$\lim_{t\uparrow T^b} u(r,t;u_0) = +\infty \qquad \text{uniformly in} \quad [0,R_1]\,,$$

since $t \mapsto u(r,t;u_0)$ is increasing. The remaining assertions of the theorem follow straight ahead from the fact that

$$\mathfrak{L}(x) := \lim_{t\uparrow\infty} \bar{u}(x,t;u_0)\,, \qquad x \in \Omega \setminus \bar{\Omega}_+\,,$$

provides us with a strong solution of (3.1) in $D := \Omega \setminus \bar{\Omega}_+$. This concludes the proof. $\square$

In the contrary case when there exists a constant $C > 0$ such that

$$u(R_1, t; u_0) \leq C \qquad \text{for each } \; t > 0\,, \tag{4.33}$$

then, $g(R_1) := \lim_{t\uparrow\infty} u(R_1, t; u_0)$ is well defined and the limit $\mathfrak{L} := \lim_{t\uparrow\infty} u(\cdot, t; u_0)$ in $\Omega \setminus \bar{\Omega}_+$ provides us with a positive strong solution of

$$\begin{cases} -\Delta w = \lambda w - a_- w^p & \text{in} \quad \Omega \setminus \bar{\Omega}_+\,, \\ w = g & \text{on} \quad \partial\Omega_+\,, \\ w = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

Therefore, it is of the greatest interest to ascertain whether or not condition (4.33) holds true. In Figure 3 we have represented the two possible limiting profiles of $\bar{u}(x, t; u_0)$, as $t \uparrow \infty$, according to each of the cases (4.32) —(a)— and (4.33) —(b)—. It should be noted that $\bar{u} = \infty$ in Ω_+ for sufficiently large $t > 0$. As a consequence from the third identity of

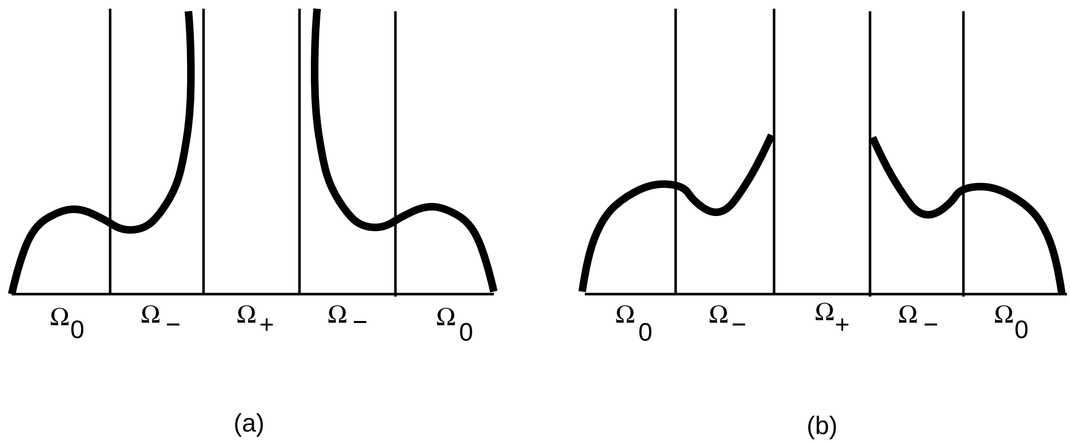

Figure 3: The asymptotic profiles of $\bar{u}$ in cases (4.32) and (4.33).

(3.5), the profile of $\mathfrak{L}^{\min}_{[\lambda, a, \Omega\setminus\bar{\Omega}_+]}$ for λ sufficiently close to σ_1 looks much like shows Figure 3. Therefore, it cannot be reached at time T^b, since $u_r \leq 0$ for each $t < T^b$. Therefore, the metasolution cannot be reached in a finite time. Such possibility cannot be a priori avoided if λ is separated away from σ_1. By comparing the gradients from both sides (from the interior of Ω_+ and $\Omega \setminus \bar{\Omega}_+$, respectively) it should be possible to prove that the case (b) of Figure 3 cannot occur, but this sharper analysis will appear elsewhere.

5 Dynamics of (1.1) **for** $\lambda \in [\sigma_1, \sigma_2)$ **and** a_+ **small** The following result is a consequence from the main theorem of [20].

Theorem 5.1 *Suppose $\lambda < \sigma_2$ and there exist $\beta \in \mathcal{C}(\Gamma_1; (0, \infty))$ and $\gamma \in \mathcal{C}(\Gamma_1; [0, \infty))$ such that*

$$\lim_{\substack{x\in\Omega\setminus\bar{\Omega}_0 \\ x\to x_1}} \frac{a_-(x)}{\beta(x_1)[\operatorname{dist}(x, \Gamma_1)]^{\gamma(x_1)}} = 1 \qquad \textit{uniformly in } \; x_1 \in \Gamma_1\,. \tag{5.1}$$

Then, the problem

$$\left\{\begin{array}{lll} -\Delta u = \lambda u - a_- u^p & \text{in} & \Omega \setminus \bar{\Omega}_0\,, \\ u = \infty & \text{on} & \Gamma_1 = \partial(\Omega \setminus \bar{\Omega}_0)\,, \end{array}\right. \tag{5.2}$$

has a unique positive solution, denoted by $\mathfrak{L}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$. *Moreover, for each* $w \in (0,\pi/2)$,

$$\lim_{\substack{x\to x_1\\ x\in C_{x_1,w}}} \frac{\mathfrak{L}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}(x)}{\left[\frac{r(x_1)(r(x_1)+1)}{\beta(x_1)}\right]^{\frac{1}{p-1}} [\mathrm{dist}\,(x,\Gamma_1)]^{-r(x_1)}} = 1 \qquad \text{uniformly in } x_1 \in \Gamma_1\,, \tag{5.3}$$

where, for each $x_1 \in \Gamma_1$,

$$r(x_1) := \frac{\gamma(x_1)+2}{p-1}\,, \qquad C_{x_1,\omega} := \left\{ x \in \Omega \setminus \bar{\Omega}_0 \ : \ \mathrm{angle}\,(x - x_1, -\mathbf{n}_{x_1}) \le \frac{\pi}{2} - \omega \right\}\,,$$

and $\mathbf{n}_{x_1}$ *stands for the outward unit normal to* $\Omega\setminus\bar{\Omega}_0$ *at* $x_1 \in \Gamma_1$. *Therefore, there exists a constant* $C > 0$ *such that*

$$a_-(x)\mathfrak{L}^{p-1}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}(x) \le \frac{C}{d^2(x)}\,, \qquad d(x) := \mathrm{dist}\,(x,\Gamma_1)\,, \qquad x \in \Omega\setminus\bar{\Omega}_0\,. \tag{5.4}$$

Thanks to Theorem 5.1, and using Hardy's inequality, it follows from [8] that the principal eigenvalue of the linearization of (5.2) at $\mathfrak{L}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]}$ is well defined and it satisfies

$$\sigma[-\Delta + p\,a_-\mathfrak{L}^{p-1}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} - \lambda, \Omega\setminus\bar{\Omega}_0] > 0\,, \tag{5.5}$$

since

$$\sigma[-\Delta + a_-\mathfrak{L}^{p-1}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} - \lambda, \Omega\setminus\bar{\Omega}_0] = 0\,.$$

Actually, the *non-degeneracy condition* (5.5) allows us to apply the implicit function theorem in order to get the following existence result.

Theorem 5.2 *Suppose* $\lambda < \sigma_2$ *and there exist* β *and* γ *satisfying* (5.1). *Then, there exists* $\varepsilon > 0$ *such that*

$$\left\{\begin{array}{lll} -\Delta u = \lambda u + a u^p & \text{in} & \Omega \setminus \bar{\Omega}_0\,, \\ u = \infty & \text{on} & \Gamma_1 = \partial(\Omega \setminus \bar{\Omega}_0)\,, \end{array}\right. \tag{5.6}$$

possesses a minimal positive solution if $\|a_+\| \le \varepsilon$.

The technical details of the proof of Theorem 5.2, as well as some other related questions, will appear elsewhere, as they deserve special attention in its own right. Subsequently, the minimal solution of (5.6) will be denoted by $\mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}$, if it exists. The following result ascertains the dynamics of (1.1) when u_0 is a subsolution of (1.5) and $\|a_+\| \le \varepsilon$.

Theorem 5.3 *Suppose* $\lambda \in [\sigma_1, \sigma_2)$, u_0 *is a positive strict subsolution of* (1.5), *and there exist* β *and* γ *satisfying* (5.1). *Let* $\varepsilon > 0$ *be for which Theorem 5.2 is satisfied and assume* $\|a_+\| \le \varepsilon$. *Then,* $u_{[\lambda,a,\Omega]}(x,t;u_0)$ *is globally defined in time, i.e.,* $T = \infty$, *and*

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(x,t;u_0) = \mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}(x) \qquad \text{for each} \quad x \in \Omega\setminus\bar{\Omega}_0\,, \tag{5.7}$$

while

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(x,t;u_0) = \infty \qquad \text{if} \quad x \in \bar{\Omega}_0 \setminus \partial\Omega\,. \tag{5.8}$$

Proof: The subsolution u_0 exists since $\lambda > \sigma_0$. Moreover, u_0 is a subsolution of (1.4), since $-a_- \leq a$. Thus, $t \mapsto u_{[\lambda,a,\Omega]}(\cdot,t;u_0)$ is point-wise increasing, i.e., $u_{[\lambda,a,\Omega]}(\cdot,t;u_0)$ provides us with a subsolution of (1.4) for each $t>0$. Moreover, since u_0 is a positive strict subsolution of (1.5) and $\mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}$ provides us with a strict supersolution of (5.2), one has that

$$u_0 < \theta_{[\lambda,-a_-,\Omega]} < \mathfrak{L}_{[\lambda,-a_-,\Omega\setminus\bar{\Omega}_0]} < \mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]} \quad \text{in} \quad \Omega\setminus\bar{\Omega}_0\,.$$

Thus, the maximum principle implies

$$u_{[\lambda,a,\Omega]}(x,t;u_0) \leq \mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}(x) \qquad \text{for each } \ (x,t)\in(\Omega\setminus\Omega_0)\times(0,T)\,, \tag{5.9}$$

where $T\in(0,\infty]$ stands for the existence time of $u_{[\lambda,a,\Omega]}$. Now, given a sufficiently small $\delta>0$, consider the open set

$$\Omega_+^\delta := \{\, x\in\Omega \ : \ \operatorname{dist}(x,\Omega_+) < \delta \,\}\,;$$

δ must be chosen so that $\partial\Omega_+^\delta \subset \Omega_-$. By (5.9), there exists a constant $M>0$ such that

$$u_{[\lambda,a,\Omega]}(x,t;u_0) \leq M \qquad \text{for each } \ (x,t)\in\partial\Omega_+^\delta\times(0,T)\,.$$

Consequently, the maximum principle implies that

$$u_{[\lambda,a,\Omega]}(x,t;u_0) \leq U(x,t) \qquad \text{for each } \ (x,t)\in(\Omega\setminus\Omega_+^\delta)\times(0,T)\,,$$

where U is the unique solution of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a_- u^p \leq \lambda u & \text{in } \ (\Omega\setminus\Omega_+^\delta)\times(0,\infty) \\ u = 0 & \text{on } \ \partial\Omega\times(0,\infty) \\ u = M & \text{on } \ \partial\Omega_+^\delta\times(0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \ \Omega\setminus\Omega_+^\delta \end{cases}$$

Therefore, $u_{[\lambda,a,\Omega]}$ is bounded above for each $t\in(0,T)$ and, hence, $T=\infty$.

On the other hand, for each $t>0$,

$$u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \qquad \text{in } \ \Omega\,,$$

and, hence, (5.8) follows from Theorem 3.6(c). In particular,

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(x,t;u_0) = \infty \qquad \text{for each } \ (x,t)\in\Gamma_1\times(0,\infty)\,.$$

Therefore, $u_{[\lambda,a,\Omega]}(x,t;u_0)$ must approach a positive strong solution of (5.6) bounded above by $\mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}$. By the minimality of $\mathfrak{L}^{\min}_{[\lambda,a,\Omega\setminus\bar{\Omega}_0]}$, (5.7) holds true. □

The first figure of the second row of Figure 2 shows the limiting profile of $u_{[\lambda,a,\Omega]}$ as $t\uparrow\infty$ in the case described by Theorem 5.3.

6 Dynamics of (1.1) **for** $\lambda\in[\sigma_1,\sigma_2)$ **and** a_+ **large** The following counterpart of Theorem 4.1(b) shows that $u_{[\lambda,a,\Omega]}$ blows-up in $L^\infty(\Omega)$ in a finite time T^b if $u_0>0$ is a strict subsolution of (1.5) and a is sufficiently large in Ω_+.

Theorem 6.1 *Suppose $\lambda \in [\sigma_1, \sigma_2)$, $u_0 > 0$ is a strict subsolution of* (1.5), *and there exists a smooth domain D such that $\bar{D} \subset \Omega_+$ and*

$$A := \min_{\bar{D}} a_+ > \frac{\omega}{p} \left[\frac{(p-1)\omega}{p\,\alpha} \right]^{p-1}, \tag{6.1}$$

where

$$\omega = \omega(\lambda, D) := \sigma[-\Delta, D] - \lambda, \qquad \alpha = \alpha(\lambda, D) := -\int_{\partial D} \frac{\partial \varphi}{\partial n} \mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]} \, d\sigma, \tag{6.2}$$

$\mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]}$ is the minimal positive solution of (5.2), *$\varphi \gg 0$ is the unique principal eigenfunction associated to $\sigma[-\Delta, D]$ normalized so that $\int_D \varphi = 1$, and n stands for the outward unit normal to D on ∂D. Then, $u_{[\lambda,a,\Omega]}(x, t; u_0)$ blows-up in a a finite time $T^b = T^b(u_0)$ in $L^\infty(\Omega_+)$.*

Proof: Fix $\lambda \in [\sigma_1, \sigma_2)$ and let $u_0 > 0$ be a strict subsolution of (1.5); it exists since $\lambda > \sigma_0$. Then, u_0 provides us with a subsolution of (1.4) and, hence, the mapping $t \mapsto u_{[\lambda,a,\Omega]}(\cdot, t; u_0)$ is increasing. Moreover, for each $t > 0$, $u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \geq u_{[\lambda,-a_-,\Omega]}(\cdot, t; u_0)$ in Ω, and, so, thanks to Theorem 3.6(c),

$$\lim_{t \uparrow \infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \geq \mathfrak{M}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]}.$$

In particular,

$$\lim_{t \uparrow \infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \geq \mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]} \qquad \text{in } D \subset \Omega_+ . \tag{6.3}$$

Note that $\lambda < \sigma_2 = \sigma[-\Delta, \Omega_+] < \sigma[-\Delta, D]$, and, hence, $\omega = \sigma[-\Delta, D] - \lambda > 0$. Moreover, since $\frac{\partial \varphi}{\partial n} \ll 0$ on ∂D,

$$\alpha = -\int_{\partial D} \frac{\partial \varphi}{\partial n} \mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]} \, d\sigma > 0 .$$

Consequently, the constant in the right hand side of (6.1) is positive. The remaining of the proof can be easily obtained by adapting the argument given in the proof of Theorem 4.2(b). Now, one should use (6.3), instead of (4.5). $\square$

If the definition of the constant $A_c(\lambda, D)$ introduced in (4.6) is extended to the whole interval $[\sigma_0, \sigma_2)$, by means of (6.2), for each $\lambda \in [\sigma_1, \sigma_2)$, then, the mapping $\lambda \mapsto A_c(\lambda, D)$ is decreasing, since $\lambda \mapsto \omega(\lambda, D)$ is decreasing and $\lambda \mapsto \alpha(\lambda, D)$ is increasing, because, for each $\lambda_1 \in [\sigma_0, \sigma_1)$ and $\lambda_2 \in [\sigma_1, \sigma_2)$, $\theta_{[\lambda_1, -a_-, \Omega]} < \mathfrak{L}^{\min}_{[\lambda_2, -a_-, \Omega \setminus \bar{\Omega}_0]}$ in $D \subset \Omega_+$, and, for any λ, $\mu \in [\sigma_1, \sigma_2)$ with $\lambda < \mu$, $\mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega \setminus \bar{\Omega}_0]} < \mathfrak{L}^{\min}_{[\mu, -a_-, \Omega \setminus \bar{\Omega}_0]}$ in $D \subset \Omega_+$. Thus, (4.2) implies (6.1).

As a consequence from Theorem 6.1, the following counterpart of Theorem 4.2 holds.

Theorem 6.2 *Suppose $\lambda \in [\sigma_1, \sigma_2)$, $u_0 > 0$ is a strict subsolution of* (1.5), *and* (6.1) *is satisfied for some smooth domain D with $\bar{D} \subset \Omega_+$. Then, $u(x, t; u_0) := u_{[\lambda,a,\Omega]}(x, t; u_0)$ blows-up in $L^\infty(\Omega_+)$ in a finite time T^b and the following assertions are true:*

(a) *$\bar{u} = u \leq \mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega_-]}$ in Ω_- is a classical solution for each $t > 0$, where $\mathfrak{L}^{\min}_{[\lambda, -a_-, \Omega_-]}$ stands for the minimal positive solution of*

$$\begin{cases} -\Delta w = \lambda w - a_- w^p & \text{in } \ \Omega_- , \\ w = \infty & \text{on } \ \partial \Omega_- . \end{cases} \tag{6.4}$$

(b) *For each sufficiently small* $\delta > 0$,

$$u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \leq \bar{u}(\cdot,t;u_0) \leq U_\delta(\cdot,t) \qquad in\ \bar{\Omega}_0\,, \tag{6.5}$$

where $U_\delta(x,t)$ *stands for the unique solution of*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a_- u^p & in \quad \Omega_\delta \times (0,\infty) \\ u = 0 & on \quad \partial\Omega \times (0,\infty) \\ u = \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]} & on \quad (\partial\Omega_\delta \cap \Omega_-) \times (0,\infty) \\ u(\cdot,0) = u_0 & in \quad \Omega_\delta \end{cases} \tag{6.6}$$

with $\Omega_\delta := \{\, x \in \Omega \ : \ \operatorname{dist}(x,\Omega_0) < \delta \,\}$.

Thus, $\bar{u}(\cdot,t;u_0)|_{\bar{\Omega}_0}$ *is a classical solution for each* $t > 0$ *and*

$$\lim_{t\uparrow\infty} \bar{u}(\cdot,t;u_0) = \infty \quad uniformly\ on\ compact\ subsets\ of\ \bar{\Omega}_0 \setminus \partial\Omega\,. \tag{6.7}$$

In particular, $B(u_0) \subset \bar{\Omega}_+$. *If one further assumes that either* (4.18), *or* (4.19) *and* (4.20), *are satisfied, then* (4.21) *holds true. If, in addition,* $B(u_0) \subset \Omega_+$, *then* (4.22) *holds as well true and, hence,* $u(x,t;u_0)$ *blows-up completely in* Ω_+.

Proof: The fact that $u(x,t;u_0)$ blows-up in $L^\infty(\Omega_+)$ in a finite time T^b is guaranteed by Theorem 6.1. Part (a) follows from the fact that, for each $t > 0$, $\bar{u}(\cdot,t;u_0)|_{\Omega_-}$ provides us with a subsolution of the singular problem (6.4). Now, choose a sufficiently small $\delta > 0$ such that $\partial\Omega_\delta \setminus \partial\Omega \subset \Omega_-$. Then, thanks to Part (a), $\bar{u}$ provides us with a subsolution of (6.6) and, therefore, the upper estimate of (6.5) is satisfied. The lower estimate follows from the fact that $u_{[\lambda,-a_-,\Omega]}$ is a subsolution of (1.1). Relation (6.7) is a direct consequence from Theorem 3.6. The remaining assertions of the theorem follow straight ahead from the previous features adapting the proof of Theorem 4.2. □

Adapting Theorem 4.3 to the present situation, one can obtain some further sufficient conditions ensuring that $\bar{u}(\cdot,t;u_0) = \infty \quad \text{in} \quad \bar{\Omega}_+ \quad \text{for any} \quad t > \tilde{T}^b$,
while

$$\lim_{t\uparrow\infty} \bar{u}_{[\lambda,a,\Omega]}(\cdot,t;u_0) = \mathfrak{M}_{[\lambda,a,\Omega\setminus\bar{\Omega}_+]}(\cdot;u_0) := \begin{cases} +\infty & \text{in } \bar{\Omega}_0 \setminus \partial\Omega\,, \\ \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]} & \text{in } \Omega_-\,, \end{cases}$$

though we refrain of giving further details here in. In such cases, for each $t > \tilde{T}^b$, $\bar{u}$ looks like shows the left picture on the second row of Figure 2 , approaching a limiting profile like the one shown in the right picture of the second row as $t \uparrow \infty$.

7 Dynamics of (1.1) **for** $\lambda \geq \sigma_2$ For this range of values of λ is not needed any restriction on the size of a_+ in order to get L^∞-blow-up in Ω_+ and, hence, the following result holds.

Theorem 7.1 *Suppose* $\lambda \geq \sigma_2$ *and* $u_0 > 0$ *is a strict subsolution of* (1.5). *Then,* $u_{[\lambda,a,\Omega]}(x,t;u_0)$ *blows-up in* $L^\infty(\Omega_+)$ *in a finite time* T^b *and the following assertions are true:*

1. $\bar{u}_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq \mathfrak{L}^{\min}_{[\lambda,-a_-,\Omega_-]}$ *in* Ω_- *is a classical solution for each* $t > 0$.
2. *For each sufficiently small* $\delta > 0$,

$$u_{[\lambda,-a_-,\Omega]}(\cdot,t;u_0) \leq \bar{u}_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq U_\delta(\cdot,t) \qquad in\ \bar{\Omega}_0\,, \tag{7.1}$$

where $U_\delta(x,t)$ *stands for the unique solution of* (6.6). *Thus,* $\bar{u}(\cdot,t;u_0)|_{\bar{\Omega}_0}$ *is a classical solution for each* $t > 0$ *and* (6.7) *holds true.*

In particular, $B(u_0) \subset \bar{\Omega}_+$. If one further assumes that either (4.18)*, or* (4.19) *and* (4.20)*, are satisfied, then* (4.21) *holds true. If, in addition, $B(u_0) \subset \Omega_+$, then* (4.22) *holds as well true, i.e., $u_{[\lambda,a,\Omega]}(x,t;u_0)$ blows-up completely in Ω_+.*

Acknowledgements: The author thanks to Professor P. Quittner at Bratislava University for a careful reading of a preliminary draft of this manuscript, as well as for detecting some errors whose correction had considerably improved it. Also, thanks are given to the Spanish Ministry of Education and Culture for support under Grant REN2003–00707.

References

[1] S. Alama and G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, *Calc. Var. Part. Diff. Eqns.* **1** (1993), 439–475.

[2] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, *J. Funct. Anal.* **141** (1996), 159–215.

[3] H. Amann and J. López-Gómez, A priori bounds and mltiple solutions for superlinear indefinite elliptic problems, *J. Diff. Eqns.* **146** (1998), 336–374.

[4] C. Bandle and M. Marcus, Large solutions of semilinear ellitpic equations. Existence, Uniqueness and asymptotic behaviour, *J. Anal. Math.* **58** (1991), 9–24.

[5] P. Baras and L. Cohen, Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation, *J. Funct. Anal.* **71** (1987), 142–174.

[6] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems, *Top. Meth. Nonl. Anal.* **4** (1994), 59–78.

[7] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, *Nonl. Diff. Eqns. Appl.* **2** (1995), 553–572.

[8] M. Bertsch and R. Rostamian, The principle of linearized stability for a class of degenerate diffusion equations, *J. Diff. Eqns.* **57** (1985), 373–405.

[9] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonl. Anal. Th. Methd. Appns.* **10** (1986), 55-64.

[10] Y. Du and Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, *SIAM J. Math. Anal.* **31** (1999), 1–18.

[11] J. M. Fraile, P. Koch-Medina, J. López-Gómez and S. Merino, Eeliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear equation, *J. Diff. Eqns.* **127** (1996), 295–319.

[12] M. Gaudenzi, P. Habets and F. Zanolin, Positive solutions of superlinear boundary value problems with singular indefinite weight, *Comm. on Pure and Appl. Anal.* **2** (2003), 403–414.

[13] M. Gaudenzi, P. Habets and F. Zanolin, An example of superlinear problem with multiple positive solutions, *Atti Sem. Fis. Univ. Modena* **LI** (2003), 259–272.

[14] R. Gómez-Reñasco and J. López-Gómez, The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction diffusion equations, *J. Diff. Eqns.* **167** (2000), 36–72.

[15] R. Gómez-Reñasco and J. López-Gómez, The uniqueness of the stable positive solution for a class of superlinear indefinite reaction diffusion equations, *Diff. Int. Eqns.*, **14** (2001), 751–768.

[16] R. Gómez-Reñasco and J. López-Gómez, On the existence and numerical computation of classical and non-classical solutions for a family of elliptic boundary value problems, *Nonl. Anal. Th. Meth. Appl.* **48** (2002), 567–605.

[17] A. C. Lazer and P. L. McKenna, A singular elliptic boundary value problem, *Appl. Math. Comp.* **65** (1994), 183–194.

[18] J. López-Gómez, The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems, *J. Diff. Eqns.* **127** (1996), 263–294.

[19] J. López-Gómez, Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems, *El. J. Diff. Eqns.* **Conf. 05** (2000), 135–171.

[20] J. López-Gómez, The boundary blow-up of large solutions, *J. Diff. Eqns.* **195** (2003), 25–45.

[21] J. López-Gómez, Dynamics of parabolic equations. From classical solutions to metasolutions, *Diff. Int. Eqns.* **16** (2003), 813–818.

[22] J. López-Gómez and P. Quittner, Complete and energy blow-up in indefinite superlinear parabolic problems. *Disc. and Cont. Dyn. Syst.*, to appear.

[23] J. López-Gómez and J. C. Sabina de Lis, First variations of principal eigenvalues with respect to the domain and point-wise growth of positive solutions for problems where bifurcation from infinity occurs, *J. Diff. Eqns.* **148** (1998), 47–64.

[24] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of non-linear elliptic equations, *Ann. Inst. H. Poincaré, ANL.* **14** (1997), 237–274.

[25] J. Mawhin, D. Papini and F. Zanolin, Boundary blow-up for differential equations with indefinite weight, *J. Diff. Eqns.* **188** (2003), 33–51.

[26] T. Ouyang, On positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, *Ind. Univ. Math. J.* **40** (1991), 1083–1141.

[27] T. Ouyang, On positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, *Trans. Amer. Math. Soc.* **331** (1992), 503–527.

Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040-Madrid, Spain
e-mail: Lopez-Gomezmat.ucm.es