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PERIODIC MOTION OF PUNCTURES ON DISKS

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ABSTRACT. In this article, we show that for any positive integer $r \geq 3$ there is a pseudo Anosov homeomorphism $\varphi : \mathbf{D}_r \to \mathbf{D}_r$ on an r times punctured disk satisfying the following conditions :
1) its associated invariant unstable foliation $\mathcal{F}^u$ has no inner singular points.
2) r punctures form a periodic orbit of period r under φ.

1 Introduction Let $\mathbf{D}_r$ be a closed disk with $r \geq 3$ points deleted. In this paper we deal with pseudo Anosov homeomorphisms φ on $\mathbf{D}_r$. Generally speaking, for an orientable surface S with negative Euler characteristic, a pseudo Anosov homeomorphism φ on S is a canonical homeomorphism in its isotopy class. For φ, there exist two foliations $\mathcal{F}^s$, $\mathcal{F}^u$, called stable and unstable respectively, such that they have finite number of common singular points, and their leaves are transverse to each other except at singular points, and they are preserved by φ. Furthermore they have transverse measures μ^s, μ^u under which the lengths of transverse intervals to each foliation are expanded and contracted by the action of φ respectively (see [3], [4] for details).

Let us assume that for an orientable surface S, a set of data which consists of the number of singular points and the numbers of prongs at them is given. Then there is an already solved interesting question that asks whether this set of numerical data can be realized as the data for the associated foliations of a pseudo Anosov $\varphi : S \to S$. The answer given by [5] is that for all orientable surfaces S with $\chi(S) < 0$ except some specified ones, all sets of numerical data which satisfy the Euler-Poincaré formula [4] for S can be realized.

Here we will investigate this realization problem with an additional condition on the dynamical motion of singular points under a pseudo Anosov homeomorphism realizing them. As the first step, we restrict ourselves to the case when 1) surfaces S are punctured disks $\mathbf{D}_r$, 2) pseudo Anosov homeomorphisms have stable and unstable foliations without inner singular points. This case is very simple and elementary, but an important step to the more general situation.

By a generalized pseudo Anosov homeomorphism, we mean a pseudo Anosov homeomorphism admitted to have prong 1 singularities on its associated invariant foliations. Then a pseudo Anosov homeomorphism on $\mathbf{D}_r$, $\varphi : \mathbf{D}_r \to \mathbf{D}_r$, we have a generalized pseudo Anosov homeomorphism $\overline{\varphi}$ on a closed disk $\mathbf{D}$ without punctures as follows. Capping on inner boundary of $\mathbf{D}_r$ by circles and collapsing these boundary circles to single points, we have a closed disk $\mathbf{D}$ without punctures. Then naturally φ induces a generalized pseudo Anosov $\overline{\varphi} : \mathbf{D} \to \mathbf{D}$.

We have the following theorem.

Theorem. *For any integer $r \geq 3$, there exists a pseudo Anosov homeomorphism φ : $\mathbf{D}_r \to \mathbf{D}_r$ such that*
1) its invariant foliations have no singular points except on the boundary of $\mathbf{D}_r$,

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2) its induced generalized pseudo Anosov $\overline{\varphi}$ has r prong 1 singular points which are the image of inner boundary under collapsing, and the set of these singular points forms a periodic orbit of period r under $\overline{\varphi}$.

Remark. In this paper, by a pseudo Anosov homeomorphism we mean an orientation preserving one, and if we deal with orientation reversing ones, explicitly we call them orientation reversing pseudo Anosov homeomorphisms.

In [5], Masur and Smillie use technique on quadratic differentials on Riemann surfaces and the result by [7], but we employ the technique that, from 2 dimensional objects, *i.e.* surfaces and homeomorphisms on them, constructs 1 dimensional objects, *i.e.* graphs and Markov maps on them, which keep essential property of dynamics on surfaces, and this technique is useful for investigating the dynamical property.

In §2, we give a brief exposition on this technique reducing 2 dimensinal objects to 1 dimensional ones. §3 and §4 are devoted to prove the theorem.

2 Preliminaries There are several theories which reduce homeomorphisms on surfaces to Markov maps on graphs [1], [2], [6], and they have much common structure, but have each special feature also. Among them we adopt the one developed by Franks and Misiurewicz, which is the most suitable to the investigation on the dynamics on disks. In this section we will give a brief exposition on this technique.

As described in §1, for a homeomorphism $f : \mathbf{D}_r \to \mathbf{D}_r$, there exists the blow down homeomorphism $\overline{f} : \mathbf{D} \to \mathbf{D}$ on a closed disk without punctures, and we have the set of specified r points q_i which are the images of attached boundary components. To construct a graph, *i.e.* a 1 dimensional simplicial complex, $\mathcal{T}$ from $\mathbf{D}$, we will decompose $\mathbf{D}$ into s closed disks B_j, $s \geq r$, and q rectangles R_k. Each rectangle has specified two boundary edges which are not adjacent to each other and called vertical edges, and the rest two boundary edges are called horizontal. R_k are naturally equipped with vertical and horizontal foliations according to the specification of edges. Let p_j be the center of B_j, and we assume that $p_j = q_j$ for $1 \leq j \leq r$.

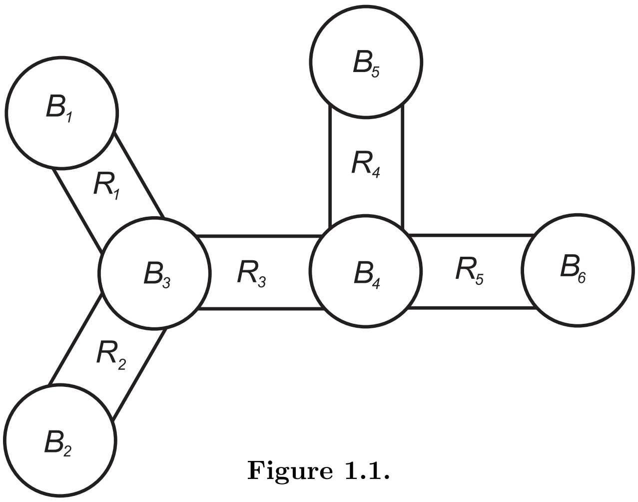

Figure 1.1.

In order to show the way of decomposition of $\mathbf{D}$, we will show the way of reconstruction of $\mathbf{D}$ from B_j and R_k by gluing them. Along all vertical edges of R_k and part of the boundary of B_j, they are glued such that the resulting space is homeomorphic to a disk, and in this disk all vertical edges are disjoint to each other (see Figure 1.1). We denote this reconstructed disk by $\mathbf{T}$, and identify it with $\mathbf{D}$ preserving the identification $q_j = p_j$, $1 \leq j \leq r$. This decomposition of $\mathbf{D}$ is called a *thick tree structure.*

Now we will construct $\mathcal{T}$ from this thick tree structure $\mathbf{D} = \mathbf{T}$. Let us collapse each B_j to a single point, and R_k to a closed segment by collapsing each vertical leaf to a single point. We give the quotient set the quotient topology, and obtain a desired graph $\mathcal{T}$. Obviously $\mathcal{T}$ is contractible, and thus called a *tree.* Let v_j be the vertex which is the image of B_j, and e_k the edge which is the image of R_k, and $\pi_{\mathbf{T}} : \mathbf{T} \to \mathcal{T}$ denotes the projection. Let V be the set of vertices of $\mathcal{T}$.

Next we will construct from $\overline{f} : \mathbf{D} = \mathbf{T} \to \mathbf{D} = \mathbf{T}$ a Markov map $f_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ which inherits the dynamical property of f. As preparation for this construction, we will induce an embedding $f_{\mathbf{T}}$ from $\overline{f}$ as follows. Let $Q = \{p_j | 1 \leq j \leq r\}$. Deforming $\overline{f} : \mathbf{T} \to \mathbf{T}$ through embeddings of $\mathbf{T}$ into itself leaving Q fixed, we may have an embedding $f_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$ such that
0) $f_{\mathbf{T}}(\mathbf{T}) \subset \operatorname{int} \mathbf{T}$,
1) if $\overline{f}(q_j) = q_{\sigma(j)}$, $1 \leq j \leq r$, then $f_{\mathbf{T}}(B_j) \subset \operatorname{int} B_{\sigma(j)}$, where $\sigma(j)$ are integers and satisfy that $1 \leq \sigma(j) \leq r$ and furthermore

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & r \\ \sigma(1) & \sigma(2) & \cdots & \sigma(r) \end{pmatrix}$$

is an element of the symmetric group of degree r,
2) for $j \geq r+1$, $f_{\mathbf{T}}(B_j)$ lie in $\operatorname{int} B_{j'}$ for some $1 \leq j' \leq s$,
3) for any vertical leaf l_v in any R_k, $f_{\mathbf{T}}(l_v)$ is included in a single vertical leaf of some $R_{k'}$ or in a single disk B_j,
4) for any horizontal leaf l_h in any R_k, there exist a finite number of horizontal leaves l_α of R_{k_α} and disks B_{j_β} such that $f_{\mathbf{T}}(l_h) \subset \left(\cup_\beta B_{j_\beta}\right) \cup (\cup_\alpha l_\alpha)$.
We call an embedding of $\mathbf{T}$ into itself which satisfies the above conditions a *thick tree map* on $\mathbf{T}$.

Then $f_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$ determines a Markov map $f_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ which satisfies $\pi_{\mathbf{T}} \circ f_{\mathbf{T}} = f_{\mathcal{T}} \circ \pi_{\mathbf{T}}$ up to homotopy relative to Q and V, where we mean by a Markov map a simplicial map. Remark that we have a "section" $s_{\mathcal{T}} : \mathcal{T} \to \mathbf{T}$ and a "lift" $\widetilde{f}_{\mathcal{T}} : \mathcal{T} \to \mathbf{T}$ as follows. There exist injective continuous maps $s_{\mathcal{T}} : \mathcal{T} \to \mathbf{T}$ and $\widetilde{f}_{\mathcal{T}} : \mathcal{T} \to \mathbf{T}$ such that $\pi_{\mathbf{T}} \circ s_{\mathcal{T}}$ and $\pi_{\mathbf{T}} \circ \widetilde{f}_{\mathcal{T}}$ are homotopic to the identity and $f_{\mathcal{T}}$ relatively to V respectively, and furthermore they are strictly monotone on each connected component of $\mathcal{T} - s_{\mathcal{T}}^{-1}(\cup D_j)$ and $\mathcal{T} - \widetilde{f}_{\mathcal{T}}^{-1}(\cup D_j)$ respectively.

When we construct an embedding $f_{\mathbf{T}}$, we may choose it from vast amount of candidates, because there is only restriction that $f_{\mathbf{T}}$ is connected with $\overline{f}$ by a path of embeddings leaving Q fixed. Therefore we may have the huge number of Markov maps $f_{\mathcal{T}}$. However starting from any Markov maps, by only finitely many times of deformations which reduce the dynamical complexity, we can obtain a reduced map $f^*_{\mathcal{T}} : \mathcal{T}^* \to \mathcal{T}^*$, which has the minimum dynamical complexity, *e.g.* the Perron-Frobenius eigenvalue of its transition matrix is the minimum, and is called *irreducible.* To prove our theorem we do not need the detail of this deformation process, and thus we will not give its exposition. Readers, for this algorithmic process, refer the references, especially [1] and [6].

Finally we will make a brief review for a reconstruction process of a Thurston canonical form $f^* : \mathbf{D}_r \to \mathbf{D}_r$ in the isotopy class of f from an irreducible Markov map $f^*_{\mathcal{T}} : \mathcal{T}^* \to \mathcal{T}^*$. Since, in this paper, we deal with only pseudo Anosov cases, and more restrictively with two special cases only, we give its exposition only for the following two cases. The graphs $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ for this two cases are as shown in Figure 1.2. Let $f^*_{\mathcal{T}_1}$ and $f^*_{\mathcal{T}_2}$ be Markov maps on $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ corresponding to pseudo Anosov homeomorphisms which satisfy the condition of Theorem in the cases when r are odd and even respectively. Remark that the reducing procedure keeps the correspondence of 1 dimensional objects to 2 dimensional objects, *i.e.* for a deformed Markov map $f_{\mathcal{T}'} : \mathcal{T}' \to \mathcal{T}'$ there exist disk $\mathbf{T}'$ with a thick tree structure and a thick tree map $f_{\mathbf{T}'} : \mathbf{T}' \to \mathbf{T}'$ such that $f_{\mathbf{T}'}$ and $f_{\mathcal{T}'}$ have a lift $\tilde{f}_{\mathcal{T}'}$.

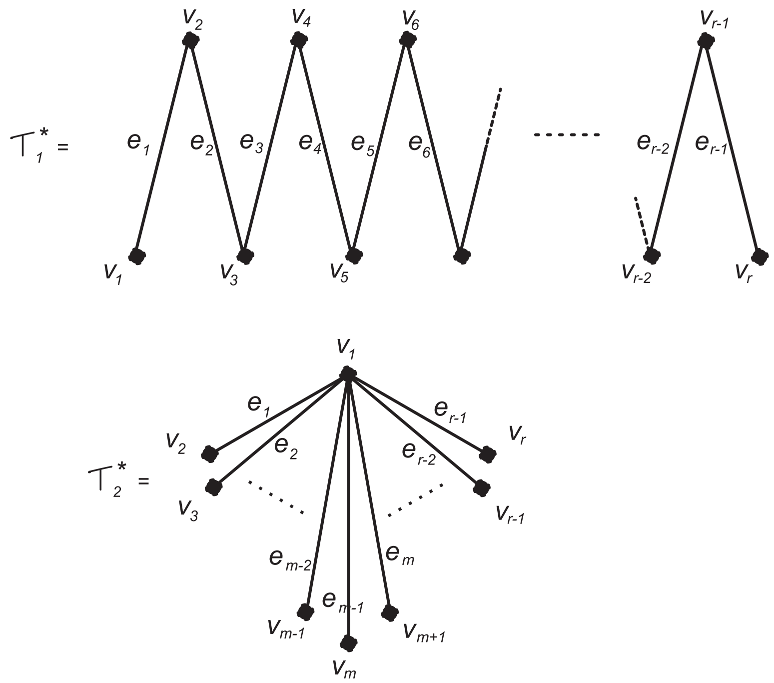

Figure 1.2.

The Markov map $f^*_{\mathcal{T}_\epsilon} : \mathcal{T}^*_\epsilon \to \mathcal{T}^*_\epsilon$, $\epsilon = 1, 2$, defines a transition matrix M_ϵ by assigning the number of times of the image $f^*_{\mathcal{T}_\epsilon}(e_i)$ of an edge e_i under $f^*_{\mathcal{T}_\epsilon}$ passing through e_j as its (i, j) entry a^ϵ_{ij}. Generally speaking, from a transition matrix N for an irreducible Markov map we can judge the type of a canonical form of an original homeomorphism $g : \mathbf{D}_r \to \mathbf{D}_r$, and for example if N is indecomposable with the Perron-Frobenius eigenvalue > 1, then a canonical form of g is pseudo Anosov. Under our assumption, the square matricise M_ϵ of order $r-1$ are indecomposable and have Perron-Frobenius eigenvalues > 1. Let λ^ϵ be the Perron-Frobenius eigenvalue of M_ϵ, and $l^\epsilon = (l^\epsilon_1, l^\epsilon_2, \cdots, l^\epsilon_{r-1})$ and $w^\epsilon = (w^\epsilon_1, w^\epsilon_2, \cdots, w^\epsilon_{r-1})$ be eigenvectors of λ^ϵ for M_ϵ and ${}^tM_\epsilon$ respectively, *i.e.* $\lambda^\epsilon l^\epsilon = M_\epsilon l^\epsilon$ and $\lambda^\epsilon w^\epsilon = {}^tM_\epsilon w^\epsilon$. Since l^ϵ and w^ϵ are strictly positive, to rectangles R_j, corresponding to edges e_j, of the thick tree structure $\mathbf{T}^*_\epsilon$ corresponding to $\mathcal{T}^*_\epsilon$, we give a geometric structure by w^ϵ and l^ϵ , *i.e.* we define the width and length of R_j by w^ϵ_j and l^ϵ_j.

We first deal with the case of $f^*_{\mathcal{T}_1} : \mathcal{T}^*_1 \to \mathcal{T}^*_1$. For simplicity on notation, we drop off the suffix 1. We will reconstruct the disk $\mathbf{D}$ using only rectangles R_j, and thus we will glue R_j with R_{j+1} as follows. Put R_j and R_{j+1} in the plane such that they have as the only common point their single vertices, which are the end points of vertical edges $b^v_{j,j+1}$, $b^v_{j+1,j+1}$ lying on ∂B_{j+1} in $\mathbf{T}^*$, and these edges form one smooth edge $b_{j+1} = b^v_{j,j+1} \cup b^v_{j+1,j+1}$ of length $w_j + w_{j+1}$ as shown in Figure 1.3. Then let us choose the middle point m_{j+1} of b_{j+1}, and bend R_j or R_{j+1} at m_{j+1} and glue them by identifying each two points on two halves of b_{j+1} devided by m_{j+1} and with the same distance from m_{j+1} as shown in Figure 1.3. For the vertical edges of R_1 and R_{r-1} glued with ∂B_1 and ∂B_r in $\mathbf{T}^*$, let us also choose the middle points m_1 and m_r, and bend R_1 and R_{r-1} at them respectively, and glue each of them with itself.

From this gluing procedure, we obtain a topological disk $\mathbf{D}_0$, and by thickening $f_{\mathcal{T}}^*$ we can define a continuous map $f_0 : \mathbf{D}_0 \to \mathbf{D}_0$ since $f_{\mathcal{T}}^*$ has a lift $\widetilde{f}_{\mathcal{T}}^* : \mathcal{T}^* \to \mathbf{T}^*$. Furthermore by our way of defining width and length of R_j, f_0 can be constructed as to be linear with respect to the coordinate on R_j, precisely to say, to be linearly contracting and expanding with multiplier $1/\lambda$ and λ in the vertical and horizontal direction of any R_j, and to be surjective, precisely to say, to cover $\mathbf{D}_0$ totally by tiling it with 'tiles' $f_0(R_j)$.

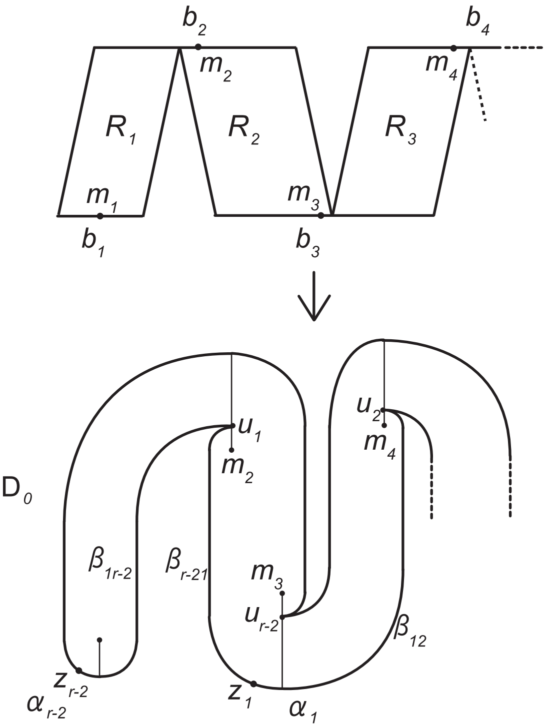

Figure 1.3.

Since the image of horizontal edges consists of the set of double points of f_0, f_0 is not injective, and thus not homeomorphic. To overcome this defect, we will construct a sphere S_0 from $\mathbf{D}_0$. On the boundary of $\mathbf{D}_0$, u_i, $i = 1, 2, \cdots, r-2$, are cusp points, and they devide $\partial\mathbf{D}_0$ into $r-2$ smooth arcs α_i with end points u_i and u_{i+1}, where u_{r-1} is equal to u_1. Then there is an integer $n > 0$ such that $f_0^n(\alpha_i) \supset \alpha_i$ for any i, and thus we can find exactly one fixed point z_i on each α_i. By definition of l, two arcs β_{ii+1} and β_{i+1i+1} in α_i and α_{i+1} bounded by z_i and u_{i+1}, and u_{i+1} and z_{i+1} have the same length. Therefore we glue $\partial\mathbf{D}_0$ with itself by identifying points on β_{ii+1} and β_{i+1i+1} with the same distance from u_{i+1}, and we obtain a sphere and the naturally induced map $\overline{f}_0$, which is homeomorphic. Finally after suitably smoothing $\overline{f}_0$ around the single point w_0 which is the image of all z_i, we blow up $\overline{f}_0$ at w_0, and by puncturing $\mathbf{D}_0$ at $\{m_j\}$, obtain a desired homeomorphism $f^* : \mathbf{D}_r \to \mathbf{D}_r$.

Remark that the vertical and horizontal foliations on R_k form the associated stable and unstable foliations for f^*, and that m_j come to be prong 1 singular points for the generalized pseudo Anosov $\overline{f}^* : \mathbf{D} \to \mathbf{D}$.

For $f_{\mathcal{T}_2}^* : \mathcal{T}_2^* \to \mathcal{T}_2^*$, the reconstruction procedure of $f_2^* : \mathbf{D}_r \to \mathbf{D}_r$ is the almost same as for $f_{\mathcal{T}_1}^*$. The only different point is the gluing procedure of R_j. In this case, the vertical edges of all R_j lying in ∂B_1 are connected to form a single smooth edge, and at the center point of this single edge we bend $\cup_j R_j$, and perform the same process on other all 'free' vertical edges as on R_1 and R_{r-1} in the case of $f^*_{\mathcal{T}_1}$, and obtain a topological disk with $r-2$ cusps on its boundary.

3 Proof of Theorem (1) In this section we will prove the theorem in the case of r odd. We will construct an example which satisfies the property asserted in the theorem. Set $r = 2m+1$, $m \geq 1$.

Thick tree structure. To give a thick tree structure on $\mathbf{D}$, let us choose just r disks B_j, $j = 1, 2, \cdots, r$, and $r-1$ rectangles R_j, and glue them such that each R_j is glued with B_j and B_{j+1}. This reconstruction of $\mathbf{D}$ is also denoted by $\mathbf{T}$, and from $\mathbf{T}$ we obtain a very simple tree, a segment, $\mathcal{T}$, which is shown in Figure 1.2 as $\mathcal{T}_1^*$.

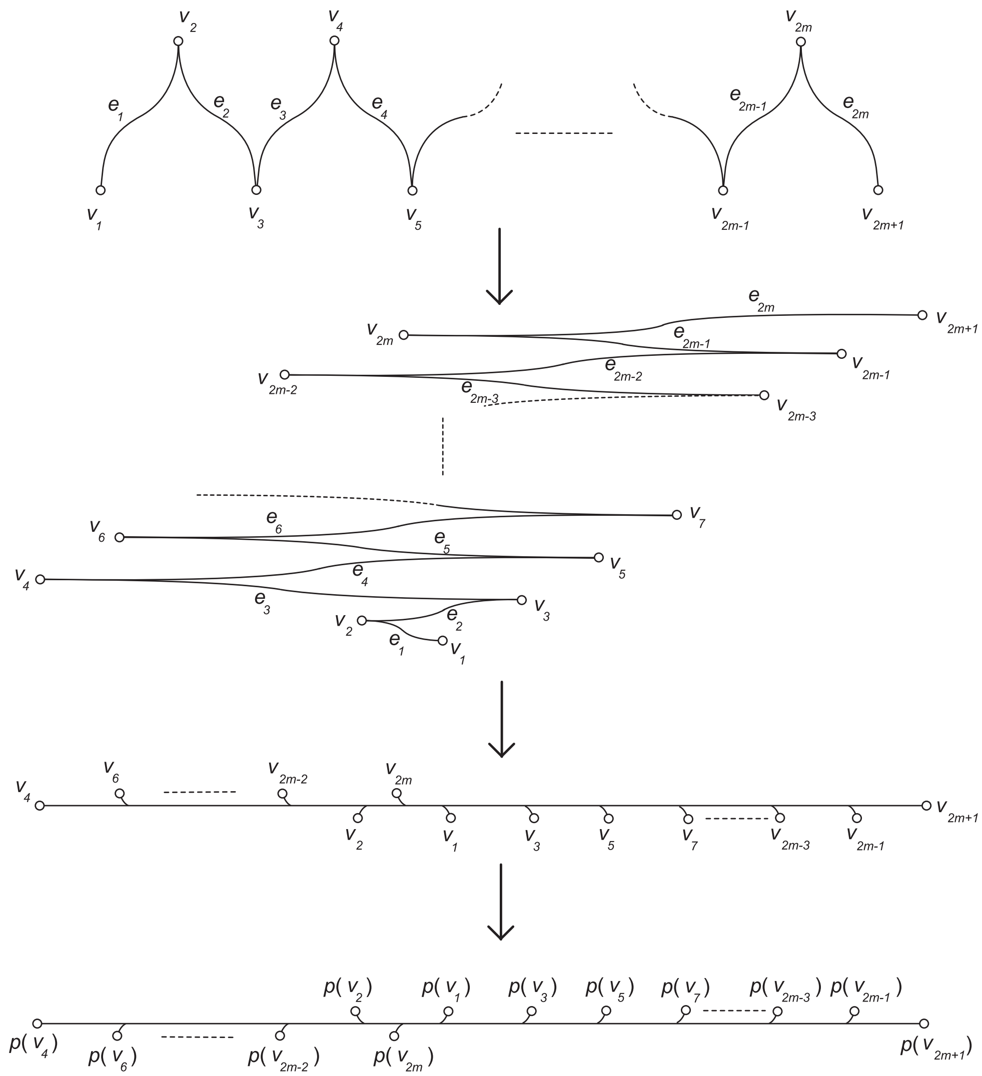

Figure 2.1.

Construction of Markov map. Here we will not trace the algorithmic reducing process starting from a reducible Markov map, but we will construct an already irreducible

Markov map $f_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ such that its corresponding canonical form $f^* : \mathbf{D}_r \to \mathbf{D}_r$ satisfies the desired property, *i.e.* f^* is an example of φ. Therefore each vertex of $\mathcal{T}$ will correspond to a prong 1 singular point, and according to this property of $\mathcal{T}$ we add an additional structure to $\mathcal{T}$. At each vertex v_j we give $\mathcal{T}$ cusped form as shown in Figure 2.1.

Note that there are many other cusped structures on $\mathcal{T}$ if we do not care about whether such structures support Markov maps with desired property or not, but we choose the structure shown as below, like bellows of an accordion, for our construction.

We will construct an irreducible Markov map $f'_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ on the cusped tree $\mathcal{T}$ such that it defines an orientation reversing pseudo Anosov f'^*, and $f_{\mathcal{T}} = f'_{\mathcal{T}} \circ f'_{\mathcal{T}}$ and $f^* = f'^* \circ f'^*$ are a desired Markov map and a desired pseudo Anosov.

If f^* satisfies the desired property, $f'_{\mathcal{T}}$ is factored into the composition of continuous maps $p_{\mathcal{T}} : \mathcal{T} \to I$ and $q_{\mathcal{T}} : I \to \mathcal{T}$, where I denotes an interval, in other words, any two edges with the same vertex their common end point are glued with each other, or precisely saying, one of them is glued with a part of the other. This corresponds to the necessity that $f'_{\mathcal{T}}$ is the projection of a thick tree map $f'_{\mathbf{T}}; \mathbf{T} \to \mathbf{T}$, or $f'_{\mathcal{T}}$ has a lift $\widetilde{f'_{\mathcal{T}}} : \mathcal{T} \to \mathbf{T}$, because any vertex of $\mathcal{T}$ will correspond to a prong 1 singular point and $\mathcal{T}$ has no vertex of valence $k \geq 3$, and the contracting property of the transverse measure on $\mathcal{F}^u$ requires that any two rectangles adjacent to each other are glued and mapped into the same rectangle. Therefore to construct $f'_{\mathcal{T}}$, we will construct two continuous map $p_{\mathcal{T}} : \mathcal{T} \to I$, which irons out the cusped graph $\mathcal{T}$ into the flat graph I, and $q_{\mathcal{T}} : I \to \mathcal{T}$, which winds the flat graph around $\mathcal{T}$.

First we will construct $p_{\mathcal{T}}$. The image of a vertex v under $p_{\mathcal{T}}$, needless to say, lies on I, but in order to reconstruct $f'_{\mathbf{T}}$ or to construct $\widetilde{f'_{\mathcal{T}}}$, it is necessary to determine which side of the image $p_{\mathcal{T}}(e)$ of an edge e, with $p_{\mathcal{T}}(e) \ni p_{\mathcal{T}}(v)$, $p_{\mathcal{T}}(v)$ lies on virtually. Then to indicate the virtual position of $p_{\mathcal{T}}(v)$ explicitly, we draw the images of vertices such that they pop up slightly from I. Let us define $p_{\mathcal{T}}$ by showing its image in Figure 2.1. In order to show how $\mathcal{T}$ is stretched on I, we draw the process from the top to the bottom of Figure 2.1. When we thicken I to a disk, then the third arrow will be the projection of an orientation reversing homeomorphism on a disk, and in fact it is simply identity of I if the information on the virtual position of vertices is neglected. We orient I such that going from left to right is according to its orientation. Then going through I in the positive direction, $p_{\mathcal{T}}(v_4)$ is the left end, $p_{\mathcal{T}}(v_{2k})$, $k = 3, 4, , \cdots, m-1$, lie virtually on the right hand side in this order, and then we meet $p_{\mathcal{T}}(v_2)$ on the left and $p_{\mathcal{T}}(v_{2m})$ on the right, and then $p_{\mathcal{T}}(v_{2k-1})$, $k = 1, 2, \cdots, m$, lie on the left hand side in this order, and finally we meet the right end $p_{\mathcal{T}}(v_{2m+1})$.

To complete the definition of $f'_{\mathcal{T}}$ we will describe $q_{\mathcal{T}} : I \to \mathcal{T}$. Let us show the image of $q_{\mathcal{T}}$ in Figure 2.2. The image of $q_{\mathcal{T}}$ is, needless to say, on $\mathcal{T}$, but in order to show how $q_{\mathcal{T}}$ is lifted to a planar map, we draw it in offsetting way from $\mathcal{T}$. Thus the complete $q_{\mathcal{T}}$ is the one which succeedingly pushes the offset image onto $\mathcal{T}$. In Figure 2.2 we denote $p(v_i)$ by v'_i.

First we assign $p_{\mathcal{T}}(v_4)$ to v_2, and we will map I such that the image goes for the right of $\mathcal{T}$. Note that it is impossible to assign $p_{\mathcal{T}}(v_{2j})$, $j = 3, 4, \cdots, m-1$, to v_{2k+1} for any k before the image turns around the right end point v_{2m+1}, because $p_{\mathcal{T}}(v_{2j})$ pop up to the right hand side, and along the image we must turn to the left at cusped vertices v_{2k+1}. We pass through v_3 of $\mathcal{T}$ and put $p_{\mathcal{T}}(v_6)$ on v_4 and succeesively pass through v_{2j-3} and put $p_{\mathcal{T}}(v_{2j})$ on v_{2j-2}, $j = 4, 5, \cdots, m-1$, respectively in this order.

After that we turn around the right end point v_{2m+1} and go back to the left end point v_1 without putting any point $p_{\mathcal{T}}(v_i)$ to any vertex of $\mathcal{T}$, and then put $p_{\mathcal{T}}(v_2)$ to v_1. Turning around v_1 to the left, we again go through until we arrive at the right end point, and set $p_{\mathcal{T}}(v_{2m})$ on v_{2m+1} and then turn around this point to the right. We turn back to the left

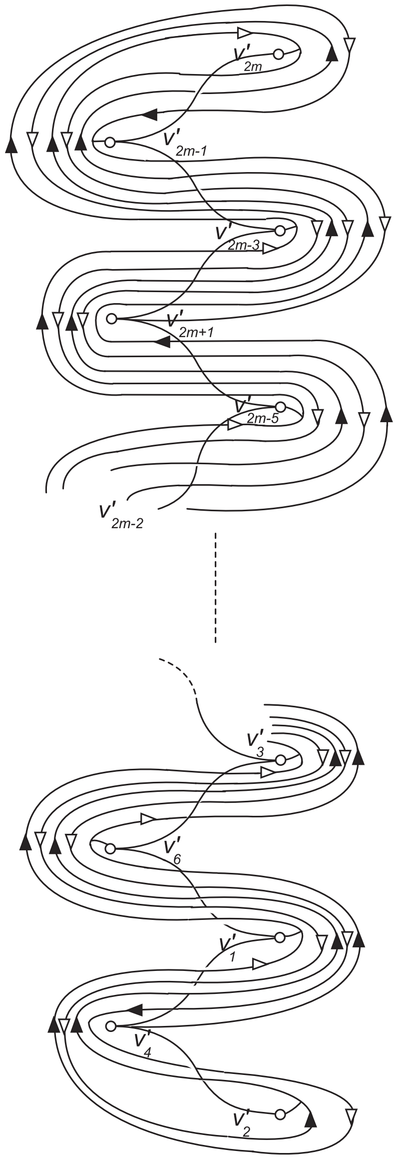

Figure 2.2.

end point again, this is the second time, and turn around it. Going to the right, this is the third time, we put $p_{\mathcal{T}}(v_{2j-1})$ on v_{2j+1}, $j = 1, 2, \cdots, m-1$, in this order, and then turn around the right end and put $p_{\mathcal{T}}(v_{2m-1})$ on v_{2m} and $p_{\mathcal{T}}(v_{2m+1})$ on v_{2m-2}. This completes the definition of $q_{\mathcal{T}}$.

Irreducibility and the transition matrix. We can easily confirm that $f'_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is irreducible, because whenever the image of a lift $\widetilde{f}'^{n}_{\mathcal{T}} = \widetilde{f}'_{\mathcal{T}} \circ f'^{n-1}_{\mathcal{T}} : \mathcal{T} \to \mathbf{T}$ of $f'^{n}_{\mathcal{T}}$ passes through a rectangle $R_{j\pm1}$, enters B_j and then turns back to the same rectangle, it turns around the center point q_j, which corresponds to a puncture of $\mathbf{D}_r$, and therefore we cannot perform any gluing and tightening procedures [6]. It is obvious that $f_{\mathcal{T}} = f'_{\mathcal{T}} \circ f'_{\mathcal{T}}$ is also irreducible. Therefore to show that $f_{\mathcal{T}}$ reproduces a pseudo Anosov homeomorphism

f^*, it is sufficient to show that the transition matrix M for $f'_{\mathcal{T}}$ is indecomposable and has a Perron-Frobenius eigenvalue > 1. By definition of $f'_{\mathcal{T}}$, M is as follows :

$$M = \begin{pmatrix} 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 5 & 5 & 5 & 5 & 5 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 5 & 5 & 5 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 & 4 & \cdots & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 4 & \cdots & 4 & 4 & 4 & 4 & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & & \vdots & \vdots \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \cdots & 4 & 5 & 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \cdots & 4 & 5 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 3 & 3 & 3 \end{pmatrix}.$$

Thus it is clear that M is indecomposable, and Perron-Frobenius eigenvalue > 1.

Periodic motion. Finally we will show the motion of punctures as an element of the symmetric group of degree r. For an orientation reversing pseudo-Anosov f'^* corresponding to $f'_{\mathcal{T}}$, we have the following element :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 2j & 2j+1 & \cdots & 2m-4 & 2m-3 & 2m-2 & 2m-1 & 2m & 2m+1 \\ 3 & 1 & 5 & 2 & 7 & \cdots & 2j-2 & 2j+3 & \cdots & 2m-6 & 2m-1 & 2m-4 & 2m & 2m+1 & 2m-2 \end{pmatrix}$$

It is easy to see that the order of σ is $r = 2m+1$, and thus the order of σ^2 corresponding to $f_{\mathcal{T}} = f'^2_{\mathcal{T}}$ is also r. This completes the proof of Theorem in the case of r odd.

By the above proof, we have the following proposition.

Proposition. *For any odd integer $r \geq 3$, there exists an orientation reversing pseudo Anosov homeomorphism $\varphi : \mathbf{D}_r \to \mathbf{D}_r$ which satisfies the condition 1) and 2) in Theorem.*

4 Proof of Theorem (2) To complete the proof of Theorem, we will deal with the rest case in this section. Thus let us assume that $r = 2m$ for a positive integer $m \geq 2$.

Thick tree structure. We choose the cusped graph $\mathcal{T}$ as shown in Figure 3.1, and choose the corresponding thick tree structure on $\mathbf{D}$. $\mathcal{T}$ has one valence $2m-1$ vertex, and all the other vertices are valence 1.

Markov map. As in the previous case, we will construct a Markov map $f_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ as the composite of two continuous maps $p_{\mathcal{T}} : \mathcal{T} \to I$ and $q_{\mathcal{T}} : I \to \mathcal{T}$, but in this case $q_{\mathcal{T}} \circ p_{\mathcal{T}}$ itself, not its 2 times iteration, is a desired Markov map $f_{\mathcal{T}}$, and therefore $p_{\mathcal{T}}$ will correspond to an orientation preserving map on a 2-disk.

The collapsing map $p_{\mathcal{T}}$ is given as shown in Figure 3.1. To indicate the virtual position of the image of vertices, they are also popped up from I. The definition of $q_{\mathcal{T}}$ is shown by its image in Figure 3.2. In this figure v'_i denote $p_{\mathcal{T}}(v_i)$ again. Also in this case we draw the image as to be offset from $\mathcal{T}$. We start from the right end point of I, which is $p_{\mathcal{T}}(v_{m+1})$, and go to the left along I. First $q_{\mathcal{T}}$ assigns $p_{\mathcal{T}}(v_{m+1})$ to v_1, and then $p_{\mathcal{T}}(v_{m+2})$ to v_{m+1}. At v_{m+1} the image turns to the right virtually, goes back to v_1, and then at v_1 turns to the right. After that $p_{\mathcal{T}}(v_j)$ is set on v_{2m+2-j} for $2 \leq j \leq m$, and on v_{2m+3-j} for $m+3 \leq j \leq 2m$, and at the vertices on the right half of $\mathcal{T}$, the image turns to the left and at the vertices on the left half, it turns to the right. Finally $p_{\mathcal{T}}(v_1)$ is put on v_2.

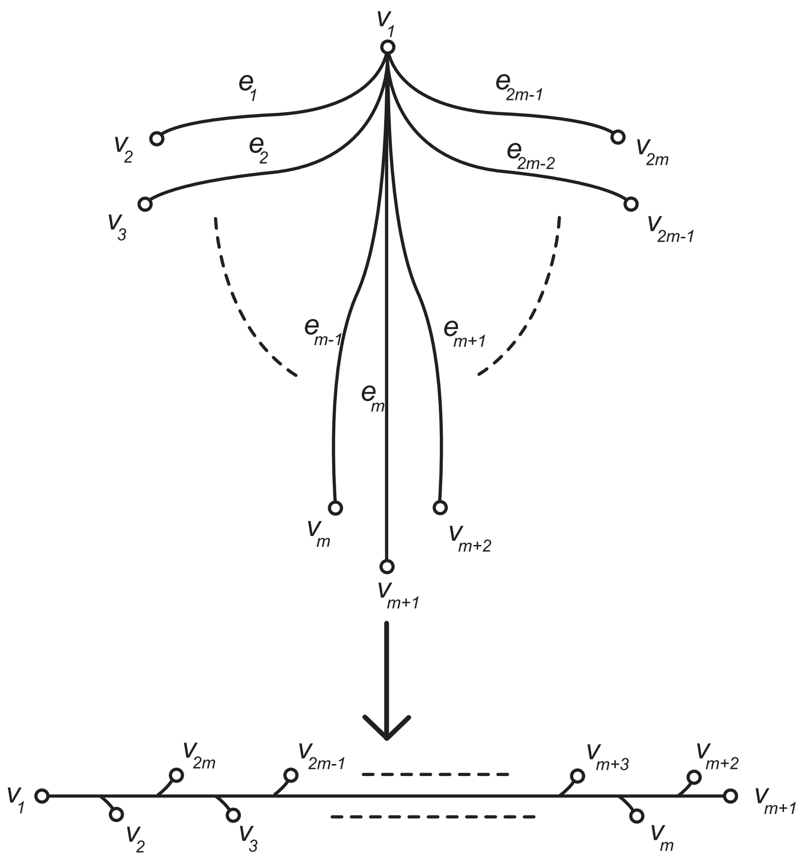

Figure 3.1.

Irreducibility and transition matrix. By definition, $f_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is irreducible. The transition matrix M of $f_{\mathcal{T}}$ is as follows.

$$M = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\
1 & 2 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
1 & 2 & 2 & \cdots & 2 & 0 & 0 & 0 & 1 & \cdots & 2 & 2 & 2 \\
1 & 2 & 2 & \cdots & 2 & 2 & 0 & 1 & 2 & \cdots & 2 & 2 & 2 \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\
1 & 2 & 2 & \cdots & 2 & 2 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\
1 & 2 & 2 & \cdots & 2 & 1 & 0 & 0 & 2 & \cdots & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 2 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 2 & 2 \\
1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 2 \\
1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2
\end{pmatrix}.$$

It is easy to verify that M is indecomposable and has the Perron-Frobenius eigenvalue > 1.

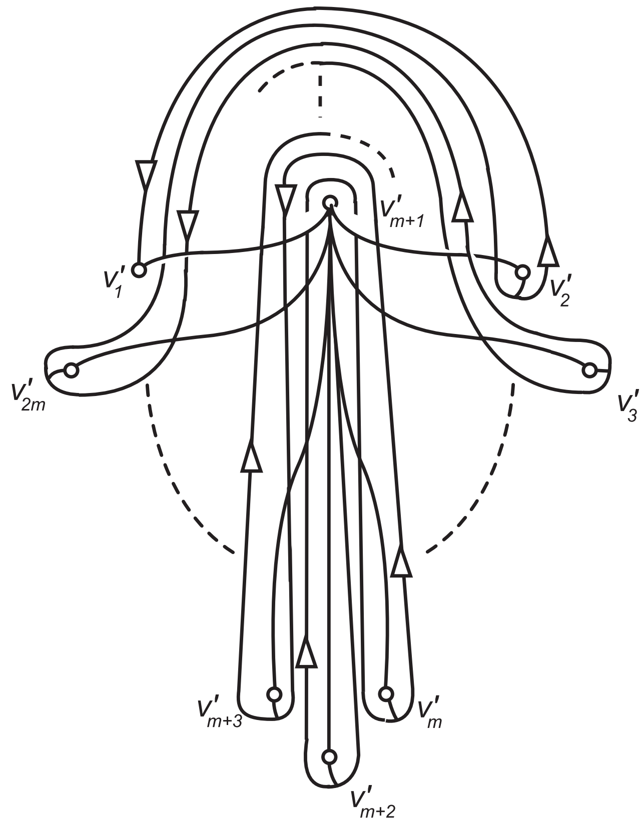

Figure 3.2.

Periodic motion. The element of the symmetric group corresponding to the motion of q_j under a reconstructed planar map f^* is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & m-1 & m & m+1 & m+2 & \cdots & 2m-2 & 2m-1 & 2m \\ 2 & 2m & 2m-1 & 2m-2 & \cdots & m+3 & m+2 & 1 & m+1 & \cdots & 5 & 4 & 3 \end{pmatrix}$$

Clearly the order of σ is $r = 2m$. Therefore f^* is a desired pseudo Anosov homeomorphism. This completes the proof.

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