SMARANDACHE DISJOINT IN BCK/D-ALGEBRAS

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Received September 24, 2004

Abstract. In this paper we include several new families of Smarandache-type P-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

Let $(X, *)$ be a binary system/algebra. Then $(X, *)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y, *)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y, *)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

In this paper we include several new families of Smarandache-type P-algebras and we study some of their properties in relation to the properties of previously defined Smarandache-types.

A *d-algebra* ([4]) $(X, *, 0)$ is an algebra satisfying the following axioms: (i) $x * x = 0$ for all $x \in X$; (ii) $0 * x = 0$ for all $x \in X$; (iii) $x * y = y * x = 0$ if and only if $x = y$.

If $X = [0, \infty) = \{x \in R \,|\, x \geq 0\}$, where R is the collection of real numbers, and if $x * y = \max\{0, x - y\}$, then $(X, *, 0)$ is a d-algebra. d-algebras are quite common and occur in many situations as the example above indicates.

Instead of asking whether a d-algebra can be a semigroup using the same operation, we can instead ask the much wider question: Can a d-algebra be a Smarandache-type semigroup ?

Theorem 1. *If $(X, *, 0)$ is a d-algebra, then it cannot be a Smarandache-type semigroup.*

Proof. Suppose that it is in fact a Smarandache-type semigroup. Let $|Y| \geq 2$, where $(Y, *)$ is a subalgebra of $(X, *)$, which is also a semigroup. Thus, if $y \in Y$, then $y * y = 0 \in Y$ as well. Therefore $(y * y) * y = 0 * y = 0 = y * (y * y) = y * 0$. But $y * 0 = 0 * y = 0$ by condition (iii) for d-algebras implies $y = 0$, so that in fact $Y = \{0\}$ and $|Y| = 1$, a contradiction. □

Corollary 2. *If $(X, *, 0)$ is a d-algebra, then it cannot be a Smarandache-type group.*

We can say the following however:

Theorem 3. *If $(X, *)$ is a semigroup, then it cannot be a Smarandache-type d-algebra.*

2000 *Mathematics Subject Classification.* 06F35.
Key words and phrases. BCK/d-algebras, Smarandache-type, Smarandache disjoint.

Proof. Suppose that $(Y, *, 0)$ is a non-trivial sub-d-algebra of $(X, *)$. Then if $y \in Y$, we have $y * y = 0$ and $0 \in Y$ and $(y * y) * y = 0 * y = 0 = y * (y * y) = y * 0$, so that $y = 0$ and $|Y| = 1$, $Y = \{0\}$. Hence, the condition $|Y| \geq 2$ is impossible, i.e., $(X, *)$ is not a Smarandache-type d-algebra. □

Given a d-algebra and abelian group, we can construct an algebra which is both a Smarandache-type d-algebra and a Smarandache-type abelian group. Let $(Y, *, 0)$ be a d-algebra and $(Z, +)$ be the additive abelian group of natural numbers. Let $(X, \otimes)$ be defined for $X = Y \times Z = \{(a, m) \mid a \in Y, m \in Z\}$ as follows: $(a, m) \otimes (b, n) := (a * b, m + n)$. Then $(a, 0) \otimes (b, 0) = (a * b, 0)$, and thus $(Y \times \{0\}, \otimes, (0, 0))$ is a subalgebra of $(X, \otimes)$ which is isomorphic to the d-algebra $(Y, *, 0)$. On the other hand, $(\{0\} \times Z, \otimes)$ has a product $(0, m) \otimes (0, n) = (0 * 0, m + n) = (0, m + n)$ so that $(\{0\} \times Z, \otimes)$ is an example of an algebra which is a Smarandache-type abelian group. Hence $(X, *)$ is both a Smarandache-type d-algebra and Smarandache-type abelian group.

Given algebra types $(X, *)$ (type-P_1) and $(X, \circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

(A) If $(X, *)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X, \circ)$;

(B) If $(X, \circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X, *)$.

This condition does not exclude the existence of algebras $(X, \diamond)$ which are both Smarandache-type-P_1-algebras and Smarandache-type-P_2-algebras. In fact we have already produced an example of such an algebra where $P_1 \equiv$ 'semigroup' and $P_2 \equiv$ 'd-algebra'.

If $(X, *, 0)$ is a d-algebra which also satisfy the following conditions:

(iv) $(x * (x * y)) * y = 0$ for all $x, y \in X$;

(v) $((x * y) * (x * z)) * (z * y) = 0$ for all $x, y, z \in X$,

then it is a BCK-algebra (see [1, 3]). Since BCK-algebras are d-algebras it follows that:

Theorem 4. *Semigroups and d-algebras are Smarandache disjoint.*

Corollary 5. *Semigroups and BCK-algebras are Smarandache disjoint.*

Corollary 6. *Groups and d-algebras are Smarandache disjoint.*

Of course, since groups are semigroups we have:

Corollary 7. *Groups and semigroups are not Smarandache disjoint.*

Consider the collection of left semigroups, i.e., semigroups $(X, *)$ with an associative product $x * y = x$. If $(Y, *)$ is a subgroup of $(X, *)$, then $x * y = x$ means that y is the multiplicative identity of $(Y, *)$ and since $y \in Y$ is arbitrary, it follows that $|Y| = 1$ as well. Hence we show that:

Theorem 8. *If $(X, *)$ is a left semigroup, then $(X, *)$ cannot be a Smarandache-type group.*

If $(X, *, e)$ is a group and if $(Y, *)$ is a left semigroup, then being closed, we have for $x, y \in Y$, $x * y = x$, whence from the group structure of $(X, *)$ we find $y = e$, and since $y \in Y$

is arbitrary, $|Y| = 1$ as well. Thus $(X, *)$ cannot be a Smarandache-type left semigroup and hence we have shown that:

Theorem 9. *Left semigroups and groups are Smarandache disjoint.*

Corollary 10. *If $(X, *)$ with $x * y = y$ is a right semigroup, then it follows that right semigroups and groups are Smarandache disjoint.*

The notion of Smarandache disjointness illustrated here appears to be novel and of interest as well.

Question. *Give an examples of a special class of d-algebras which is Smarandache disjoint from the class of BCK-algebras.*

References

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