THE FDS-PROPERTY AND SPACES IN WHICH COMPACT SETS ARE CLOSED

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Received March 4, 2003; revised October 18, 2004

Abstract. A space in which every infinite set contains an infinite subset with only a finite number of accumulation points is said to have the finite derived set property. We study this property in the class of spaces in which compact sets are closed – the KC-spaces – and apply our results to show that among hereditarily Lindelöf spaces, minimal KC-spaces are compact. This result generalizes a theorem of [2] and gives a partial answer to a question of R. Larson.

1 Which spaces have the FDS-property?

A space X is said to have the *finite derived set property* (hereafter abbreviated as the *FDS-property*) if each infinite subset $A \subseteq X$ contains an infinite subset with only finitely many accumulation points (in X). This concept was introduced in [8] in order to study properties of the lattice of T_1-topologies on a set X. In a subsequent paper [2], we studied the class of KC-spaces, that is to say the class of spaces in which all compact subsets are closed; such spaces are clearly T_1 and every Hausdorff space is KC. The KC-spaces have also been called T_B-spaces (for instance in [6]). A problem ascribed to R. Larson in [6] is whether a space is maximal with respect to being compact if and only if it is minimal with respect to being KC. In [2], it was shown that in the class of KC-spaces, each countable space has the FDS-property and this result was used to prove that every countable minimal KC-space is compact, thus giving a (very) partial answer to the above-mentioned question of Larson. A KC-space (X, τ) is said to be *Katětov-KC* if there is a minimal KC-topology $\sigma \subseteq \tau$. In [6], Fleissner showed that not every KC-space is Katětov-KC, but no characterization of Katětov-KC spaces is known.

In the first section of this paper we continue our study of those spaces which have the FDS-property, while in Section 2, we apply our results to show that in some fairly wide classes of KC-spaces, including all hereditarily Lindelöf spaces, minimal KC implies compact. We also prove that certain classes of KC-spaces are Katětov-KC. All spaces considered here are (at least) T_1 and all undefined notation and terminology can be found in [5], but note that the symbol $\subset$ is used exclusively to denote proper containment. The following result is obvious:

Remark 1.1 *If X is a KC-space, then no infinite subspace of X can have the cofinite topology.*

Lemma 1.2 *Each infinite subspace of a KC-space contains an infinite discrete subspace.*

Proof: Let (X, τ) be a KC-space and suppose $A \subseteq X$ is infinite; since A does not have the cofinite topology, there is some open set U_0 in X such that $A \cap U_0 \neq \emptyset$ and $A \setminus U_0$ is

2000 *Mathematics Subject Classification.* Primary 54D10, 54D25 Secondary 54D99.

Key words and phrases. KC-space, hereditarily Lindelöf space, sequential space, finite derived set property.

[1]Research supported by Consejo Nacional de Ciencia y Tecnología (México), grant 28411E

[2]Research supported by Fundação de Amparo a Pesquisa do Estado de São Paulo (Brasil)

infinite; choose $x_0 \in A \cap U_0$. Having chosen open sets $U_0, \ldots, U_{n-1}$ and points $x_0, \ldots, x_{n-1}$ in such a way that for all $m \in \{0, \ldots, n-1\}$,

i) $x_m \in A \cap U_m$;

ii) $x_k \notin U_m$ if $k \neq m$; and

iii) $A_{n-1} = A \setminus (\bigcup\{U_m : 0 \leq m \leq n-1\})$ is infinite.

Then, by Remark 1.1, A_{n-1} does not have the cofinite topology and so we can find $U_n \in \tau$ such that $x_m \notin U_n$ for each $m < n$, $U_n \cap A_{n-1} \neq \emptyset$ and $A_{n-1} \setminus U_n$ is infinite; we then choose $x_n \in U_n \cap A_{n-1}$. This completes our inductive construction; let $D = \{x_n : n \in \omega\}$. Clearly D is infinite and is discrete since $U_n \cap D = \{x_n\}$ for all $n \in \omega$.

A family $\mathcal{A} = \{A_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$ (the set of of infinite subsets of ω) has the *strong finite intersection property* if the intersection of any finite subfamily of $\mathcal{A}$ is infinite. If $A, B \in [\omega]^\omega$ and $A \setminus B$ is finite, then we write $A \subseteq^* B$. A set B is a *pseudointersection* of the family $\mathcal{A}$ if $B \subseteq^* A_\alpha$ for all $\alpha \in \kappa$. Recall that p is the smallest cardinal such that there exists a family of infinite subsets $\{A_\alpha : \alpha < \mathrm{p}\}$ of ω with the strong finite intersection property but no infinite pseudointersection. It is known that $\omega_1 \leq \mathrm{p} \leq \mathrm{c}$.

Theorem 1.3 *A KC-space with the property that $\chi(p, X) < p$ for each $p \in X$ has the FDS-property.*

Proof: Let (X, τ) be a KC-space such that $\chi(p, X) < \mathrm{p}$ for all $p \in X$ and let $A \subseteq X$ be infinite. By Lemma 1.2, we can find a countably infinite discrete subspace $D \subseteq A$. If D is closed in X then we are done. If not, then let x be an accumulation point of D and let $\{U_\alpha : \alpha < \lambda\}$ be a local base of open sets at x, where $\lambda = \chi(x, X) < \mathrm{p}$. Then the family $\{U_\alpha \cap D : \alpha < \lambda\}$ is a family of subsets of D with the strong finite intersection property. By the definition of p, there is an infinite subset $S \subseteq D$ such that $S \setminus U_\alpha$ is finite for each $\alpha < \lambda$. Clearly then, the countably infinite set S converges to x and so $S \cup \{x\}$ is compact and hence closed in X. Therefore $|\mathrm{cl}(S) \setminus S| = 1$.

Remark 1.4 *A similar argument can be used to show that if X is a KC-space such that $\chi(x, X) < p$ for each $x \in X$, then every separable subspace of X is Hausdorff.*

If (X, τ) is a Hausdorff space, then for each $p \in X$, set $\psi_c(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau$, $p \in U$ for all $U \in \mathcal{U}$ and $\bigcap\{\overline{U} : U \in \mathcal{U}\} = \{p\}$; as in [7], we can then define the *closed pseudocharacter* of X, $\psi_c(X) = \sup\{\psi_c(p, X) : p \in X\}$.

If in the above theorem, the space (X, τ) is Hausdorff, then in order to conclude that x is the only possible accumulation point of S, it suffices that there exist $\lambda < \mathrm{p}$ such that $\bigcap\{\overline{U_\alpha} : \alpha < \lambda\} = \{x\}$. Hence we have proved:

Theorem 1.5 *Each Hausdorff space X such that $\psi_c(x, X) < p$ for any $x \in X$ has the FDS-property.*

Bounding the size of X also allows us to prove that a space has the FDS-property in case it is either Tychonoff or compact and KC.

Theorem 1.6 *Every Tychonoff space of cardinality less than c has the FDS-property.*

Proof: Suppose to the contrary that X is a Tychonoff space of cardinality less than c in which there is an infinite subset $A \subseteq X$ (which we may suppose to be discrete) such that every infinite subset of A has infinitely many accumulation points. Clearly $Y = \mathrm{cl}_X(A)$ is pseudocompact and hence if $f : Y \to \mathbb{R}$ is continuous, then $f[Y]$ is a compact subset of $\mathbb{R}$ of cardinality less than c; thus $f[Y]$ must be countable. Let $f^\beta : \beta Y \to \mathbb{R}$ be the extension of f to βY; clearly $f^\beta[\beta Y] = f[Y]$ and so every real-valued continuous image of βY is countable. Thus by a result of Shapirovski, (see [9]), βY, and hence Y, is scattered. But then, if x is an isolated point of $Y \setminus A$, it follows by the regularity of Y that there is a closed neighbourhood U of x in Y contained in $A \cup \{x\}$. Since U is compact, there is a sequence in A converging to x, contradicting our hypothesis regarding the set A.

Before our next result, we note that a countable compact KC-space with only a finite number of accumulation points is either finite or a topological union of convergent sequences. Thus a countably compact KC-space has the FDS-property if and only if it is sequentially compact.

Theorem 1.7 *A compact KC-space of cardinality less than c has the FDS-property (and hence is sequentially compact).*

Proof: Suppose that (X, τ) is a compact KC-space and $|X| <$ c. We assume to the contrary that X does not have the FDS-property and so there is some countably infinite subset $A \subseteq X$ such that every infinite subset of A has infinitely many accumulation points in X. By Lemma 1.2, without loss of generality, we can assume that A is discrete and that $\mathrm{cl}_X(A) = X$. Let $x \in X \setminus A$; if every neighbourhood U of x is such that $A \setminus U$ is finite, then $A \cup \{x\}$ is compact, hence closed in X and so x is the unique accumulation point of A, a contradiction. Thus we can choose an open neighbourhood V of x, such that $A \setminus V$ is infinite. The closed subspace $X \setminus V$ of X is compact and A is countable, and hence Lindelöf, and so $A \cup (X \setminus V)$ is Lindelöf. This in its turn implies that $A \cup (X \setminus V)$ is not countably compact, for otherwise it would be compact, but being a proper dense subspace of X, it is not closed in X, a contradiction. Thus there is a discrete set $D \subseteq A \cup (X \setminus V)$ which is closed in $A \cup (X \setminus V)$. However, since $X \setminus V$ is compact, only a finite number of points of D lie in $X \setminus V$ and hence $D \cap V$ is infinite and all its accumulation points (by our hypothesis, an infinite number in X) must lie in V; that is to say, $\mathrm{cl}_X(D \cap V) \subseteq V$. Thus we have constructed two infinite sets $D \cap V$ and $A \setminus V$ whose closures are disjoint in X. Since $D \cap V \subseteq A$, each of these sets has the property that every infinite subset has an infinite number of accumulation points and the above argument can be repeated using $D \cap V$ and $A \setminus V$ in place of A. A standard binary tree argument can now be used to show that $|X| \geq$ c.

Remark 1.8 *Since any topology stronger than a topology with the FDS-property has the FDS-property, it follows from Theorem 1.7 that in the search for a KC-topology (respectively Hausdorff topology) with no weaker compact KC-topology, assuming $\omega_1 <$ c, it suffices to find a KC-topology (respectively, Hausdorff topology) on a set of size ω_1 without the FDS-property.*

Note that if $\omega_1 =$ c then there is a countably compact subspace of $\beta\omega$ of cardinality ω_1 which does not have the FDS-property. On the other hand, if $\omega_1 <$ p, and τ is any Hausdorff topology on ω_1, then (ω_1, τ) can be condensed onto a Hausdorff topology of weight ω_1, which, by Theorem 1.5, has the FDS-property, implying in its turn that τ has the FDS-property.

Question 1.9 *If p $= \omega_1 <$ c, does there exist a KC (or even a Hausdorff) topology on ω_1, without the FDS-property?*

Again using a tree argument, it is easy to show, under $\omega_1 <$ c, that a countably compact Hausdorff topology on ω_1 without the FDS-property cannot be Urysohn or even weakly regular (each non-empty open set contains the closure of a non-empty open set).

Theorem 1.10 *A hereditarily Lindelöf KC-space has the FDS-property.*

Proof: Suppose to the contrary that (X,τ) is a hereditarily Lindelöf KC-space which does not have the FDS-property. Then there is some infinite subset $D \subseteq X$, which by Lemma 1.2 we may assume to be discrete, such that every infinite subset of D has infinitely many accumulation points. As a consequence, for all infinite $C \subseteq D$, and $x \in \mathrm{cl}_\tau(C) \setminus C$, there is an open neighbourhood U_x of x such that $C \setminus U_x$ is infinite, for otherwise, $C \cup \{x\}$ is compact, hence closed in X, contradicting our assumption regarding D. Clearly we can assume also that D is countable. We will construct recursively a strictly increasing nested family of open sets in a subspace of X of length ω_1, contradicting the fact that X is hereditarily Lindelöf.

To this end, let $D = D_0$ and choose $x_0 \in \mathrm{cl}(D) \setminus D$ and an open neighbourhood U_0 of x_0 such that $D_1 = D_0 \setminus U_0$ is infinite.

Suppose that for some ordinal $\alpha \in \omega_1$ we have chosen points $\{x_\beta : \beta \in \alpha\}$, infinite subsets $\{D_\beta : \beta \in \alpha\}$ of D and open sets $\{U_\beta : \beta \in \alpha\}$, such that

i) $x_\beta \in U_\beta$ for all $\beta \in \alpha$,

ii) $x_\beta \in (\mathrm{cl}(D_\beta) \setminus D_\beta) \setminus U_\gamma$ for all $\gamma < \beta < \alpha$,

iii) $D_\beta \subseteq^* D_\gamma \setminus U_\gamma$ for all $\gamma < \beta < \alpha$, and

iv) $D_\beta \setminus U_\beta$ is infinite for all $\beta \in \alpha$,

we proceed to choose x_α, D_α and U_α as follows:

By *iii)* and *iv)*, $D_\beta \cap D_\gamma$ is infinite for each $\gamma \in \beta \in \alpha$ and, since $|\alpha| = \omega < \mathrm{p}$, there is some infinite set $D_\alpha \subseteq^* D_\beta$ for all $\beta \in \alpha$. Again by *iii)*, we have $D_\alpha \subseteq^* D_\beta \subseteq^* D_\gamma \setminus U_\gamma$ for all $\gamma \in \beta \in \alpha$ whence it follows that $D_\alpha \subseteq^* D \setminus U_\gamma$ for all $\gamma \in \alpha$ and hence all accumulation points of D_α lie outside U_γ for each $\gamma \in \alpha$. Choose $x_\alpha \in \mathrm{cl}(D_\alpha) \setminus D_\alpha$ and an open neighbourhood U_α of x_α such that $D_\alpha \setminus U_\alpha$ is infinite. It is clear that $\{x_\beta : \beta \leq \alpha\}$, $\{D_\beta : \beta \leq \alpha\}$ and $\{U_\beta : \beta \leq \alpha\}$ satisfy *i)- iv)* above.

Let $L = \{x_\alpha : \alpha \in \omega_1\}$; by construction, each $x_\alpha \in L$ has an open neighbourhood $U_\alpha \cap L$ contained in $\{x_\beta : \beta \leq \alpha\}$; that is to say, for each $\alpha \in \omega_1$, $\{x_\beta : \beta \in \alpha\}$ is open in L and the result follows.

We note that the above result can be somewhat improved since the recursive construction can be continued as far as any cardinal $\mu < \mathrm{p}$. Thus we have actually proved:

Corollary 1.11 *If X is a KC-space with $hL(X) < p$, then X has the FDS-property.*

Since consistently $\mathrm{p} = \mathrm{c}$ and c is regular we have:

Corollary 1.12 *It is consistent that every KC-space X with $hL(X) < c$ has the FDS-property.*

A space X is said to be *weakly discretely generated* if whenever $A \subset X$ is not closed, then there is some discrete subset $D \subseteq A$ such that $\mathrm{cl}(D) \setminus A \neq \emptyset$. It was shown in Proposition 3.1 of [4] that every compact Hausdorff space is weakly discretely generated and a similar proof applying Lemma 2.3 of [1] can be used for compact KC-spaces.

Of course, a space with a countable network is hereditarily Lindelöf and so it is worth noting that a Hausdorff space with σ-discrete network need not have the FDS-property. The Katětov extension, $\kappa\omega$ of ω (see [5, 3.12.6]) is strongly σ-discrete, hence has a σ-discrete network, but lacks the FDS-property. A modification of the topology of the Stone-Čech compactification βX of van Douwen's countable maximal space X (see [3]) obtained by declaring X and each of its supersets to be open is an H-closed space with a σ-discrete network which is neither weakly discretely generated nor has the FDS-property.

2 Properties of minimal KC-spaces.

Our first lemma in this section generalizes Theorem 10 of [2] and gives a partial answer to the question of R. Larson mentioned in the first paragraph of Section 1.

Lemma 2.1 *A hereditarily Lindelöf, minimal KC-space is compact.*

Proof: Suppose that (X, σ) is a hereditarily Lindelöf minimal KC-space; by Theorem 1.10, X has the FDS-property. If (X, σ) is not compact then since it is Lindelöf, it is not countably compact and hence there is some countably infinite closed discrete subspace $D = \{d_n : n \in \omega\} \subseteq X$. Fix $p \in X$ and a free ultrafilter $\mathcal{G} \in \beta\omega \setminus \omega$ and define a new topology μ on X as follows:

(i) If $p \notin U$, then $U \in \mu$ if and only if $U \in \sigma$,

and

(ii) If $p \in U$, then $U \in \mu$ if and only if $U \in \sigma$ and $\{n \in \omega : d_n \in U\} \in \mathcal{G}$.

Clearly (X, μ) is a T_1-space, $\mu \subset \sigma$ and for each $B \subseteq X$, $\mathrm{cl}_\mu(B) \subseteq \mathrm{cl}_\sigma(B) \cup \{p\}$; since (X, σ) has the FDS-property, it follows that (X, μ) does as well. We proceed to show that (X, μ) is a KC-space. To this end, suppose to the contrary that A is a non-closed, compact subset of (X, μ). Obviously $p \in \mathrm{cl}_\mu(A)$ and there are two cases to consider:

(a) If $p \notin A$, then $\mu|A = \sigma|A$ and so A is compact and hence closed in (X, σ). Thus $U = X \setminus A$ is open and $p \in U$. If $\{n \in \omega : d_n \in A\} \notin \mathcal{G}$, then $\{n \in \omega : d_n \in D \setminus A\} \in \mathcal{G}$ and for each $d \in D \setminus A$, $d \in U$ and so $p \in U \in \mu$ contradicting the fact that $p \in \mathrm{cl}_\mu(A)$. Thus $\{n \in \omega : d_n \in A\} \in \mathcal{G}$ and hence there is some infinite set $S \subset A \cap D$ such that $\{n \in \omega : d_n \in S\} \notin \mathcal{G}$ and S is then an infinite closed discrete subset of A in (X, μ), implying that $(A, \mu|A)$ is not compact, again a contradiction.

(b) If $p \in A$, then $\mathrm{cl}_\mu(A) = \mathrm{cl}_\sigma(A)$, implying that A is not closed in (X, σ). Thus A is not compact and since A is Lindelöf, it is not countably compact in (X, σ). Thus there is a countably infinite, discrete subset $C \subseteq A$ which is closed in $(A, \sigma|A)$. However, C is not closed in $(A, \mu|A)$ and so $\mathrm{cl}_\mu(C) \cap A = C \cup \{p\}$. This implies that $\{n \in \omega : d_n \in \mathrm{cl}_\mu(C)\} \in \mathcal{G}$. If $P = \{n \in \omega : d_n \in C\}$ is infinite, then there is some infinite subset $S \subseteq P$ such that $S \notin \mathcal{G}$ and hence $\{d_n : n \in S\}$ is a closed, discrete subspace of $(A, \mu|A)$, contradicting the compactness of this space. If, on the other hand, P is finite, then since (X, μ) has the FDS-property, there is an infinite subset $B \subseteq C$ with only a finite number of accumulation points in (X, μ). Thus $\{n \in \omega : d_n \in \mathrm{cl}_\mu(B)\} \notin \mathcal{G}$ which implies that B is closed and discrete in $(A, \mu|A)$, implying in its turn that A is not compact in (X, μ).

In fact Lemma 2.1 can be improved.

Theorem 2.2 *A hereditarily Lindelöf minimal KC-space is compact and sequential.*

Proof: Suppose that (X, τ) is a hereditarily Lindelöf minimal KC-space; the previous lemma shows that X is compact and we proceed to show that it is sequential. To this end, suppose that $A \subset X$ is not closed and hence not compact. Since X is hereditarily Lindelöf, A is not countably compact and hence we can find a countable discrete subset $D = \{x_n : n \in \omega\} \subseteq A$ which is closed in A; that is to say, all of the accumulation points of D lie outside of A. By Theorem 1.10, X has the FDS-property, and so there is some countably infinite set $E \subseteq D$ with only a finite number of accumulation points in X, all of which lie in $\mathrm{cl}(A) \setminus A$. Thus $\mathrm{cl}(E)$ is a countable, compact KC-space and it follows from Corollary 3 of [2], $\mathrm{cl}(E)$ is sequential; thus there is a sequence in E converging out of E and hence out of A.

Clearly, we have shown in the previous theorem that a compact, hereditarily Lindelöf KC-space is sequential but need not be first countable as the one-point compactification of the rationals illustrates. It is interesting to note that there are H-closed, hereditarily Lindelöf, Hausdorff spaces which are not sequential. Let μ denote the usual metric topology on $[0,1]$ and consider the topology τ on $[0,1]$ generated by the family of sets of the form

$$\{U \setminus D : U \in \mu \text{ and } D \text{ is closed and discrete in } [0,1] \setminus Q\},$$

where Q denotes the set of rational numbers.

The completely Hausdorff space $([0,1],\tau)$ has a countable network, and is H-closed (since its semiregularization is the compact space $([0,1],\mu)$), but is not sequential, because $[0,1]\setminus Q$ is sequentially closed but not closed.

In a first countable non-Hausdorff space there always exists a sequence convergent to two distinct points and hence it is clear that a first countable KC-space is Hausdorff. Hence in Theorem 2.2 we have actually proved:

Corollary 2.3 *A second countable minimal KC-space is compact Hausdorff.*

By way of contrast, second countable, non-compact, minimal Hausdorff spaces are known and the one-point compactification of the rationals is a non-Hausdorff, Fréchet-Urysohn, minimal KC-space. The question then arises as to whether a first countable minimal KC-space is compact. In fact, we can prove a stronger result:

Theorem 2.4 *A sequential minimal KC-space is compact.*

Proof: Let (X,τ) be a non-compact space satisfying the hypothesis of the theorem. Fix $a \in X$ and define a new topology σ on X as follows:

$$\sigma = \{U \in \tau : a \notin U\} \cup \{U \in \tau : a \in U \text{ and } X \setminus U \text{ is compact}\}.$$

Clearly (X,σ) is a compact T_1-space and $\sigma \subset \tau$. Thus to complete the proof, it suffices to show that (X,σ) is a KC-space. To this end, suppose that $S \subseteq X$ is a compact subset of (X,σ). It is clear that $\mathrm{cl}_\sigma(S) \subseteq \mathrm{cl}_\tau(S) \cup \{a\}$ and that if $a \notin S$, then $\sigma|S = \tau|S$. There are then two possibilities:

(*i*) If $a \notin S$, then by the preceding remarks, S is compact, and hence closed, in (X,τ) and so $X \setminus S$ is an open σ-neighbourhood of a. Thus $a \notin \mathrm{cl}_\sigma(S)$ and so $\mathrm{cl}_\sigma(S) = \mathrm{cl}_\tau(S) = S$.

(*ii*) If $a \in S$ then $\mathrm{cl}_\sigma(S) = \mathrm{cl}_\tau(S)$ and so if S is not closed in (X,σ), then it is not closed in (X,τ) either. Thus there is some $x \in \mathrm{cl}_\tau(S) \setminus S$ and a sequence $\{x_n\}_{n\in\omega}$ in S convergent to x. Since $a \neq x$, we may assume that $x_n \neq a$ for all $n \in \omega$. Then $K = \{x_n : n \in \omega\} \cup \{x\}$ is compact in (X,τ), hence closed in (X,τ) and since $a \notin K$, it is closed in (X,σ). Thus $K \cap S = \{x_n : n \in \omega\}$ is a closed subset of the compact space $(S,\sigma|S)$ and thus is compact. Since $a \notin K \cap S$, we have $\sigma|(K\cap S) = \tau|(K \cap S)$ so $K \cap S$ is compact in (X,τ) and hence closed in (X,τ). However, $x \in \mathrm{cl}_\tau(K \cap S) \setminus (K \cap S)$ which is a contradiction.

The next result is an immediate consequence of Theorem 2.4 and the comments preceding Corollary 2.3.

Corollary 2.5 *A first countable KC-space is minimal KC if and only if it is compact Hausdorff.*

Corollary 2.6 *Every sequential KC-space is Katětov-KC.*

Corollary 2.7 *Each countable Hausdorff space is Katětov-KC.*

Proof: If (X, τ) is a countable Hausdorff space, then it can be condensed onto a second countable Hausdorff space. Now apply Corollary 2.6.

A proof very similar to that of Theorem 2.4 can be used to show the following:

Theorem 2.8 *A Hausdorff k-space is minimal KC if and only if it is compact.*

Proof: The sufficiency is clear. For the necessity, let (X, τ) be a non-compact space which satisfies the hypothesis of the theorem. Define σ as in Theorem 2.4. Again, we claim that (X, σ) is a KC-space. If S is a compact subset of (X, σ) and $a \notin S$, then the proof proceeds as in (i) of Theorem 2.4. If on the other hand, $a \in S$, then $\mathrm{cl}_\sigma(S) = \mathrm{cl}_\tau(S)$ and so if S is not closed in (X, σ), then it is not closed in (X, τ) either. Since (X, τ) is a k-space, there is some compact set C in (X, τ) such that $C \cap S$ is not closed in C. Furthermore, if the chosen compact set C has the property that $a \in C$, then since (X, τ) is Hausdorff, given $x \in \mathrm{cl}_\tau(C \cap S) \setminus (C \cap S)$, we can find disjoint open neighbourhoods U, V of x and a respectively. Then $C \setminus V$ is a compact subset of (X, τ) with the property that $S \cap (C \setminus V)$ is not closed in $C \setminus V$. Hence we have shown that it is possible to choose C so that $a \notin C$. Then $\mathrm{cl}_\tau(C \cap S) \subseteq C$ is a closed, hence compact subset of (X, τ) which does not contain a and hence is also closed in (X, σ). Thus $T = S \cap \mathrm{cl}_\tau(C \cap S)$ is a σ-closed subset of S and hence is compact in (X, σ). However, since $a \notin T$, it follows that $\tau|T = \sigma|T$ and hence T is compact in (X, τ), a contradiction, since $x \in \mathrm{cl}_\tau(T) \setminus T$.

In the proof of the above theorem, we have constructed a compact KC-topology σ on X with $\sigma \subset \tau$. Thus we have also proved:

Corollary 2.9 *A Hausdorff k-space is Katětov-KC.*

Larson's original question remains open but appears to be a difficult problem. However, considering the results obtained above, a number of interesting and possibly more tractable questions remain; below we mention a few of them.

Question 2.10 *Can a non-compact minimal Hausdorff space be minimal KC? Alternatively, is every Hausdorff minimal KC-space compact? Is every minimal Hausdorff space a k-space?*

Question 2.11 *Is a closed subspace of a minimal KC-space, minimal KC?*

Note that a positive answer to the last question implies that a minimal KC-space is countably compact. Furthermore, if X is Hausdorff and every closed subspace is minimal KC, then every closed subspace is H-closed and then by a result of Stone (see [5, 3.12.5]), X is compact. Since a hereditarily Lindelöf Hausdorff space has countable pseudocharacter. we are led to ask:

Question 2.12. Does a Lindelöf KC-space with countable pseudocharacter have the FDS-property?

Question 2.13. Can every KC-space (or each T_2-space) be embedded in a compact KC-space? Is the Wallman compactification of a KC-space KC?

However, there is no way of embedding a KC-space in some power of a compact KC-space since the square of a non-Hausdorff compact KC-space is never KC (the diagonal is compact but not closed). Indeed, it is easy to show that if X is a KC-space then for each $\kappa \geq 2$, X^κ is KC if and only if each compact subspace of X is Hausdorff.

Question 2.14. Does there exist a compact KC-space in which every open set is dense?

If the answer to Question 2.13 is affirmative, then so is the answer to Question 2.14, for if X is the co-countable topology on an uncountable set, then in any T_1-compactification of X, all open sets are dense.

References

[1] Alas, O.T., Tkachuk, V.V. and Wilson, R.G., *Closures of discrete sets often reflect global properties*, Topology Proceedings **25**(2000), 27–44.

[2] Alas, O.T. and Wilson, R.G., *Spaces in which compact subsets are closed and the lattice of T_1-topologies on a set*, to appear in Comment. Math. Univ. Carolinae.

[3] van Douwen, E.K., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.

[4] Dow, A., Tkachenko, M.G., Tkachuk, V.V. and Wilson, R.G. *Topologies generated by discrete subspaces*, Glasnik Mat. Ser. III **37** (57) no. 1 (2002), 187–210.

[5] Engelking, R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] Fleissner, W.G., *A T_B-space which is not Katětov T_B*, Rocky Mountain J. of Mathematics, **10** no. 3 (1980), 661–663.

[7] Juhász, I., *Cardinal Functions in Topology, ten years later*, Math. Centrum, Amsterdam, 1981.

[8] Shakhmatov, D., Tkachenko, M.G. and Wilson, R.G. *Complementation in the lattice $\mathcal{L}_1(X)$*, to appear in Houston J. Math.

[9] Shapirovski, B. E., *On mappings onto Tychonoff cubes* (in Russian), Uspekhi Mat. Nauk, **35** no. 3 (1980), 122–130.

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