ON 0-COMMUTATIVE B-ALGEBRAS

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Abstract. In this paper we show that if X is a 0-commutative B-algebra, then $(x * a) * (y * b) = (b * a) * (y * x)$. Using this property we show that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4], Q. P. Hu and X. Li studied a huge class of abstract algebras, so called BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([14]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras ([15, 16]), i.e., (I) $x * x = 0$, (II) $x * 0 = x$, and (III) $(x * y) * z = x * (z * (0 * y))$ for any x, y, z in a B-algebra X. It is known that the B-algebra is equivalent in some sense to a group. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([12]) introduced a new notion, called a BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I) $x * x = 0$, (II) $x * 0 = x$, and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$ for any x, y in a BH-algebra X. A. Walendziak obtained the another equivalent axioms for a B-algebra ([17]). H. S. Kim and J. Neggers ([11]) introduced the notion of (pre-)Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have the same order 2, i.e., a Boolean group. Recently, C. B. Kim and H. S. Kim ([10]) introduced the notion of a BM-algebra which is a specialization of B-algebras, and they proved the followings: the class of BM-algebras is a proper subclass of B-algebras, and also show that a BM-algebra is equivalent to a 0-commutative B-Algebra. Moreover, they showed that the class of Coxeter algebras is a proper subclass of BM-algebras. In this paper, we show that if X is a 0-commutative B-algebra, then $(x*a)*(y*b) = (b*a)*(y*x)$. Through the use of this property, we show that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras.

2. Preliminaries

A B-*algebra* ([15]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(A1) $x * x = 0$,

(A2) $x * 0 = x$,

(A3) $(x * y) * z = x * (z * (0 * y))$,

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for any $x, y, z \in X$.

Proposition 2.1. ([15]) *If* $(X; *, 0)$ *is a* B*-algebra, then*

(i) $(x * y) * (0 * y) = x$,

(ii) $x * (y * z) = (x * (0 * z)) * y$,

(iii) $x * y = 0$ *implies* $x = y$,

(iv) $0 * (0 * x) = x$,

for any $x, y, z \in X$.

Recently, A. Walendziak obtained the equivalent axiomatizations for B-algebras ([17]), and he proved that the congruence lattice of B-algebras is isomorphic to the lattice of their normal subalgebras ([18]).

Theorem 2.2. ([17]) $(X; *, 0)$ *is a* B*-algebra if and only if it satisfies the following axioms:*

(A1) $x * x = 0$,

(W2) $0 * (0 * x) = x$,

(W3) $(x * z) * (y * z) = x * y$,

for any $x, y, z \in X$.

Lemma 2.3. ([17]) *If* $(X; *, 0)$ *is a* B*-algebra, then* $0 * (x * y) = y * x$ *for any* $x, y \in X$.

A B-algebra $(X; *, 0)$ is said to be 0-*commutative* ([2]) if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$.

Proposition 2.4. ([15]) *If* $(X; *, 0)$ *is a* 0*-commutative* B*-algebra, then* $(0*x)*(0*y) = y*x$, *for any* $x, y \in X$.

Theorem 2.5. ([2, 15]) *In any* B*-algebra, the left and the right cancellation laws hold.*

An algebraic system $(X; *, 0)$ is said to be a BCI*-algebra* ([5, 6]) if it satisfies the following conditions:

(B1) $(x * y) * (x * z) \leq z * y$,

(B2) $x * (x * y) \leq y$,

(B3) $x \leq x$,

(B4) $x \leq y$, $y \leq x$ imply $x = y$,

where $x \leq y$ is defined by $x * y = 0$. A BCI-algebra X is said to be a BCK*-algebra* if $0 \leq x$ for all $x \in X$.

For any BCI-algebra X, the set $X_+ := \{x \in X \mid 0 \leq x\}$ is called a BCK*-part* of X. A BCI-algebra X is said to be p*-semisimple* if $X_+ = \{0\}$ (see [1]).

Theorem 2.6. ([19]) *Let* $(X; *, 0)$ *be a* BCI*-algebra. Then the following are equivalent.*

(1) X *is* p*-semisimple,*

(2) $0 * x = 0$ *implies* $x = 0$,

(3) $x * (x * y) = y$,

(4) $x * (y * z) = z * (y * x)$,

(5) $(x * y) * (z * u) = (x * z) * (y * u)$,

(6) $x * (0 * z) = z * (0 * x)$,

for any $x, y, z, u \in X$.

3. 0-commutative B-algebras and p-semisimple BCI-algebras

From the following theorem, we can obtain an interesting observation for a 0-commutative B-algebra.

Theorem 3.1. *If* $(X; *, 0)$ *is a* 0*-commutative* B*-algebra, then*

$$(x * a) * (y * b) = (b * a) * (y * x) \tag{1}$$

for any $x, y, a, b \in X$.

Proof. For any $x, y, a, b \in X$, we obtain

$$\begin{aligned}
(x * a) * (y * b) &= x * [(y * b) * (0 * a)] && [(A3)] \\
&= x * [y * \{(0 * a) * (0 * b)\}] && [(A3)] \\
&= x * (y * (b * a)) && [\text{Proposition 2.4 }] \\
&= [x * (0 * (b * a))] * y && [\text{Proposition 2.1-(ii)}] \\
&= [(b * a) * (0 * x)] * y && [\text{0-commutative}] \\
&= (b * a) * [y * (0 * (0 * x))] && [(A3)] \\
&= (b * a) * (y * x), && [\text{Proposition 2.1-(iv)}]
\end{aligned}$$

proving the theorem. □

Corollary 3.2. *If* $(X; *, 0)$ *is a* 0*-commutative* B*-algebra, then*

$$(x * z) * (y * z) = x * y$$

for any $x, y, z \in X$.

Proof. If we let $x := a$ in (1), then by Lemma 2.3, $b * y = 0 * (y * b) = (a * a) * (y * b) = (b * a) * (y * a)$. □

Remark. In fact, the condition "0-commutative" need not to be necessary, since $(x * z) * (y * z) = x * ((y * z) * (0 * z)) = x * y$ by Proposition 2.1-(i). A. Walendziak ([17]) gave another proof for it.

Corollary 3.3. *If* $(X; *, 0)$ *is a* 0*-commutative* B*-algebra, then*

$$(z * y) * (z * x) = x * y$$

for any $x, y, z \in X$.

Proof. By applying Theorem 3.1, we obtain $x * y = (x * y) * (z * z) = (z * y) * (z * x)$ for any $x, y, z \in X$. $\square$

Corollary 3.4. *If $(X; *, 0)$ is a 0-commutative B-algebra, then*

$$(x * a) * y = (0 * a) * (y * x)$$

for any $x, y, a \in X$.

Proof. If we let $b := 0$ in (1), then $(x * a) * y = (x * a) * (y * 0) = (0 * a) * (y * x)$. $\square$

Corollary 3.5. *If $(X; *, 0)$ is a 0-commutative B-algebra, then*

$$x * (y * b) = b * (y * x)$$

for any $x, y, b \in X$.

Proof. If we let $a := 0$ in (1), then $x * (y * b) = (x * 0) * (y * b) = (b * 0) * (y * x) = b * (y * x)$. $\square$

Theorem 3.6. *If $(X; *, 0)$ is a 0-commutative B-algebra, then*

$$(x * y) * z = (x * z) * y$$

for any $x, y, z \in X$.

Proof. For any $x, y, z \in X$, we obtain

$$\begin{aligned} (x * y) * z &= x * [z * (0 * y))] && [\text{(A3)}] \\ &= x * [y * (0 * z))] && [\text{0-commutative}] \\ &= (x * z) * y, && [\text{(A3)}] \end{aligned}$$

proving the theorem. $\square$

Proposition 3.7. *Let $(X; *, 0)$ be a 0-commutative B-algebra. Then $(X; \leq)$ is a partially ordered set, where $x \leq y$ if and only if $x * y = 0$.*

Proof. It is straightforward by Proposition 2.1-(iii). $\square$

Theorem 3.8. *If $(X; *, 0)$ is a 0-commutative B-algebra, then*

$$[(x * y) * (x * z)] * (z * y) = 0$$

for any $x, y, z \in X$.

Proof. By applying Theorem 3.1, we obtain

$$\begin{aligned} [(x * y) * (x * z)] * (z * y) &= [(z * y) * (x * x)] * (z * y) \\ &= [(z * y) * 0] * (z * y) \\ &= (z * y) * (z * y) \\ &= 0, \end{aligned}$$

proving the theorem. □

Theorem 3.9. *If* $(X;*,0)$ *is a* 0*-commutative* B*-algebra, then*

$$[x*(x*y)]*y=0$$

for any $x,y,z\in X$*.*

Proof. By applying Theorem 3.6, we obtain $[x*(x*y)]*y=(x*y)*(x*y)=0$. □

Theorem 3.10. *Every* 0*-commutative* B*-algebra is a* BCI*-algebra.*

Proof. It follows from Proposition 3.7 and Theorems 3.8 and 3.9. □

The converse of Theorem 3.10 need not to be true in general.

Example 3.11. Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Then it is a BCI-algebra ([7]), but not a 0-commutative B-algebra, since $3*(0*2)=0\neq 2*(0*3)$.

Theorem 3.12. *Every* 0*-commutative* B*-algebra is a* p*-semisimple* BCI*-algebra.*

Proof. It follows from Theorems 2.6 and 3.10. □

Theorem 3.13. *Every* p*-semisimple* BCI*-algebra is a* 0*-commutative* B*-algebra.*

Proof. It is enough to show (A3). For any $x,y,z\in X$, we have

$$\begin{aligned} x*(z*(0*y)) &= (x*0)*(z*(0*y)) && [\text{(A2)}] \\ &= (x*z)*(0*(0*y)) && [\text{Theorem 2.6-(5)}] \\ &= (x*z)*y, && [\text{Theorem 2.6-(3)}] \end{aligned}$$

proving the theorem. □

By Theorems 3.12 and 3.13, we conclude that *the class of* p*-semisimple* BCI*-algebras is equivalent to the class of* 0*-commutative* B*-algebras* in some sense (refer [15, p. 27-28]).

Since it is well known that every abelian group is equivalent to a p-semisimple BCI-algebra, we conclude that:

$$\begin{aligned} \text{abelian groups} &\Longleftrightarrow p-\text{semisimple } BCI-\text{algebras} \\ &\Longleftrightarrow 0-\text{commutative } B-\text{algebras} \end{aligned}$$

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