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# A NOTE ON NORMAL SUBALGEBRAS IN $B$-ALGEBRAS

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ABSTRACT. The concept of normal subalgebras in $B$- algebras is due to J. Neggers and H. S. Kim. We give an equivalent definition and show that the center $\mathrm{Z}(\mathbf{A})$ of a $B$-algebra $\mathbf{A}$ is a normal subalgebra of $\mathbf{A}$. Moreover, we prove that the notion of a normal subalgebra is equivalent to the normal subgroup of the derived group. Hence the lattices of normal subalgebras (and also the congruence lattices) of $B$-algebras are modular.

**1 Preliminaries** $B$-algebras have been introduced by J. Neggers and H. S. Kim in [4]. They defined a $B$-algebra as an algebra $(A; *, 0)$ of type $(2, 0)$ (i.e., a nonempty set $A$ with a binary operation $*$ and a constant 0) satisfying the following axioms:

(A1) $x * x = 0$,
(A2) $x * 0 = x$,
(A3) $(x * y) * z = x * [z * (0 * y)]$.

In [3], Y. B. Jun, E. H. Roh, and H. S. Kim introduced $BH$-algebras, which are a generalization of $BCH/BCK/B$-algebras. For another useful generalization of $B$-algebras see [6]. J. R. Cho and H. S. Kim ([2]) proved that every $B$-algebra is a quasigroup. The following results show that every group determines a $B$-algebra and every $B$-algebra is group-derived.

**Proposition 1.1.** ([1], Proposition 3.1) *Let* $\mathbf{G} = (G; \cdot, ^{-1}, e)$ *be a group. We put* $0 = e$ *and define the binary operation* $*$ *on* $G$ *by setting*

$$x * y = x \cdot y^{-1}.$$

*Then* $(G; *, 0)$ *is a* $B$*-algebra, which is called the group-derived* $B$*-algebra; it will be denoted by* $A(\mathbf{G})$.

**Proposition 1.2.** ([1], Theorem 3.4) *Let* $\mathbf{A} = (A; *, 0)$ *be a* $B$*-algebra. For* $x, y \in A$*, define*

$$x + y = x * (0 * y) \ \text{ and } \ -x = 0 * x.$$

*Then* $(A; +, -, 0)$ *is a group, which we denote by* $G(\mathbf{A})$.

We will need the following two lemmas.

**Lemma 1.3.** ([5], Proposition 2.8) *If* $(A; *, 0)$ *is a* $B$*-algebra, then*

$$x * (y * z) = (x * (0 * z)) * y$$

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*for any* $x, y, z \in A$.

**Lemma 1.4.** ([2], Proposition 2.2) *A B-algebra* $(A; *, 0)$ *obeys the equation*

$$(x * y) * (0 * y) = x.$$

**Proposition 1.5.** *If* $(A; *, 0)$ *is a B-algebra, then*
(a) $(x * z) * (y * z) = x * y$,
(b) $0 * (x * y) = y * x$

*for all* $x, y, z \in A$.

*Proof.* (a): By (A3) we obtain

$$(x * z) * (y * z) = x * [(y * z) * (0 * z)] = x * [y * ((0 * z) * (0 * z))].$$

Hence, applying (A1) and (A2), we get (a).
(b): Using Lemma 1.4 and (A3) we have $y * x = y * [(x * y) * (0 * y)] = (y * y) * (x * y)$. Finally, by (A1) we obtain (b). □

Following J. Neggers and H. S. Kim ([4]) we give

**Definition 1.6.** A $B$-algebra $(A; *, 0)$ is said to be 0-*commutative* if $a * (0 * b) = b * (0 * a)$ for all $a, b \in A$.

In [2], J. R. Cho and H. S. Kim showed the following result.

**Proposition 1.7.** *A B-algebra* $\mathbf{A} = (A; *, 0)$ *is* 0*-commutative if and only if the equation* $x * (x * y) = y$ *holds in* $\mathbf{A}$.

## 2. Normal subalgebras in $B$-algebras

From now on, $\mathbf{A}$ always denotes a $B$-algebra $(A; *, 0)$. A nonempty subset $N$ of $A$ is called a *subalgebra* of $\mathbf{A}$ if $x * y \in N$ for any $x, y \in N$. It is easy to see that if $N$ is a subalgebra of $\mathbf{A}$, then $0 \in N$.

**Lemma 2.1.** *Let* $N$ *be a subalgebra of* $\mathbf{A}$ *and let* $x, y \in A$. *If* $x * y \in N$, *then* $y * x \in N$.

*Proof.* Let $x * y \in N$. By Proposition 1.5 (b), $y * x = 0 * (x * y)$. Since $0 \in N$ and $x * y \in N$, we see that $0 * (x * y) \in N$. Consequently, $y * x \in N$. □

In [5], J. Neggers and H. S. Kim introduced the notion of a normal subset of a $B$-algebra. A nonempty subset $N$ of $A$ is said to be *normal* (or a *normal subalgebra*) of $\mathbf{A}$ if

$$(x * a) * (y * b) \in N \quad \text{for any } x * y, a * b \in N.$$

In [5], it is proved that any normal subset of a $B$-algebra $\mathbf{A}$ is a subalgebra of $\mathbf{A}$. Obviously, $\{0\}$ and $A$ are normal subalgebras of $\mathbf{A}$.
Let Sub($\mathbf{A}$) and N($\mathbf{A}$) be the sets of all subalgebras and normal subalgebras of $\mathbf{A}$, respectively.

**Theorem 2.2.** *Let $N \in \mathrm{Sub}(\mathbf{A})$. Then the following statements are equivalent:*
(a) *$N$ is a normal subalgebra;*
(b) *If $x \in A$ and $y \in N$, then $x * (x * y) \in N$.*

*Proof.* (a) $\Rightarrow$ (b): Let $x \in A$ and $y \in N$. Then $x * x = 0 \in N$ and $0 * y \in N$. Since $N$ is normal, $(x * 0) * (x * y) \in N$. Thus $x * (x * y) \in N$.

(b) $\Rightarrow$ (a): Let $x * y, a * b \in N$. By Lemma 2.1, $b * a \in N$. Applying Proposition 1.5 (a) we have

$$(0 * a) * (0 * b) = (0 * a) * [(0 * a) * (b * a)]$$

and using (b) we get

$$(0 * a) * (0 * b) \in N. \tag{1}$$

Applying (A3) twice we obtain

$$\begin{aligned} x * (x * [(0 * a) * (0 * b)]) &= x * [(x * b) * (0 * a)] \\ &= (x * a) * (x * b). \end{aligned}$$

From this, combining (b) with (1) we deduce that $(x * a) * (x * b) \in N$. We have

$$[(x * a) * (x * b)] * (y * x) \in N, \tag{2}$$

because $N$ is a subalgebra. Using (A3) and Proposition 1.5 we get

$$\begin{aligned} [(x * a) * (x * b)] * (y * x) &= (x * a) * [(y * x) * (0 * (x * b))] \\ &= (x * a) * [(y * x) * (b * x)] \\ &= (x * a) * (y * b). \end{aligned}$$

Therefore $(x*a)*(y*b) \in N$ by (2), and consequently, $N$ is normal. □

From Proposition 1.7 and Theorem 2.2 we obtain

**Corollary 2.3.** *In 0-commutative $B$-algebras the concepts of subalgebras and normal subalgebras coincide.*

**Proposition 2.4.** *Let $\mathbf{G} = (G; \cdot, ^{-1}, e)$ be a group and let $N \subseteq G$. Then $N$ is a normal subgroup of $\mathbf{G}$ if and only if $N$ is a normal subalgebra of the group-derived $B$-algebra $A(\mathbf{G})$.*

*Proof.* Let $N$ be a normal subgroup of $\mathbf{G}$ and let $x, y \in N$. Then $x * y = x \cdot y^{-1} \in N$, and therefore $N$ is a subalgebra of $A(\mathbf{G})$. If $x \in G$ and $y \in N$, then

$$x * (x * y) = x \cdot (x \cdot y^{-1})^{-1} = x \cdot y \cdot x^{-1} \in N.$$

By Theorem 2.2, $N$ is a normal subalgebra of $A(\mathbf{G})$.
Since $x \cdot y^{-1} = x*y$ and $x \cdot y \cdot x^{-1} = x*(x*y)$, the converse is obvious. □

**Definition 2.5.** ([1]) The set

$$\mathrm{Z}(\mathbf{A}) = \{y \in A : y * (0 * x) = x * (0 * y) \quad \text{for all } x \in A\}$$

is called the *center* of a $B$-algebra $\mathbf{A}$.

In [1], P. J. Allen, J. Neggers, and H. S. Kim asked the following questions:

Is the center $\mathrm{Z}(\mathbf{A})$ a normal subalgebra of $\mathbf{A}$?
Is the notion of a normal subalgebra equivalent to the normal subgroup of the derived group?

By Theorem 4.7 of [1], $\mathrm{Z}(\mathbf{A})$ is a subalgebra of $\mathbf{A}$. Observe that $\mathrm{Z}(\mathbf{A})$ is normal. Indeed, let $x \in A$ and $y \in \mathrm{Z}(\mathbf{A})$. It follows that

$$\begin{aligned} x*(x*y) &= [x*(0*y)]*x && \text{[use Lemma 1.3]} \\ &= [y*(0*x)]*x && [\text{since } y \in \mathrm{Z}(\mathbf{A})] \\ &= y*(x*x) && \text{[use Lemma 1.3]} \\ &= y. && \text{[by (A1) and (A2)].} \end{aligned}$$

Thus $x*(x*y) = y \in \mathrm{Z}(\mathbf{A})$. By Theorem 2.2, $\mathrm{Z}(\mathbf{A})$ is a normal subalgebra of $\mathbf{A}$, that is, the first one of the preceding two questions has a positive answer. Moreover, the next theorem shows that the answer to the second one is also positive.

**Theorem 2.6.** *Let $\mathbf{A} = (A; *, 0)$ be a B-algebra and let $N \subseteq A$. Then $N \in \mathrm{N}(\mathbf{A})$ if and only if $N$ is a normal subgroup of the group $G(\mathbf{A})$.*

*Proof.* By Proposition 1.2, $G(\mathbf{A}) = (A; +, -, 0)$ is a group, where $x + y = x*(0*y)$ and $-x = 0*x$ for all $x, y \in A$. Applying Proposition 1.5 (b) we have

$$x - y = x*[0*(0*y)] = x*(y*0) = x*y$$

and

$$x + (y - x) = x*[0*(y*x)] = x*(x*y).$$

Now, the proof is straightforward. □

**Remark 2.7.** It is easy to see that the center $\mathrm{Z}(\mathbf{A})$ of a $B$-algebra $\mathbf{A}$ is the center of the group $G(\mathbf{A})$. Therefore, from Theorem 2.6 it follows that $\mathrm{Z}(\mathbf{A})$ is a normal subalgebra of $\mathbf{A}$.

Let $\mathrm{Con}(\mathbf{A})$ be the set of all congruences on $\mathbf{A}$. With respect to set inclusion, $\mathrm{Con}(\mathbf{A})$ forms a lattice, which we denote by $\mathbf{Con}(\mathbf{A}) = (\mathrm{Con}(\mathbf{A}); \subseteq)$. Similarly, $\mathbf{Sub}(\mathbf{A}) = (\mathrm{Sub}(\mathbf{A}); \subseteq)$ and $\mathbf{N}(\mathbf{A}) = (\mathrm{N}(\mathbf{A}); \subseteq)$ are lattices. The theory of groups and Theorem 2.6 yield

**Theorem 2.8.** *Let $\mathbf{A}$ be a B-algebra. Then $\mathbf{Con}(\mathbf{A})$ and $\mathbf{N}(\mathbf{A})$ are isomorphic modular lattices. Moreover, the lattice $\mathbf{Sub}(\mathbf{A})$ is modular, if $\mathbf{A}$ is 0-commutative.*

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