INTUITIONISTIC FUZZY IDEALS OF INCLINE ALGEBRAS

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ABSTRACT. We consider the intuitionistic fuzzication of the concept of ideals in incline algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionstic fuzzy of an incline algebra and investigate some related properties.

1.Introduction and Preliminaries After the introduction of the concept of fuzzy sets by Zadeh [12],several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of " intuitionistic fuzzy sets" was first published by Atanassov[1,2], as a generation of the notion of fuzzy sets. Cao et al[6] introduced the notion of incline algebras in their book. Kim and Roush [9] studied algebraic structures of inclines, and they with Markowsky [11] discussed the reprentation of inclines, and Ahn [3] investigated permant over inclines. Ahn and Kim [4] introduced the notion of positive implicative incline and studied some relations between R(L)-maps and positive implicative. In [5], Ahn and Jun constructed the quotient incline algebras. In [8], Jun introduced the concept of fuzzy subinclines(ideals) and give some results. In this paper, we consider the intuitionistic fuzzication of the concept of ideals in incline algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionstic fuzzy of an incline algebra and investigate some related properties.

Inclines are a generalization of Boolean and fuzzy algebras, and a special type of a semiring, and give a way to combine algebras and ordered structures to express the degree of intensity of binary relations.

An incline algebra is a set $\mathcal{H}$ with two binary operations denoted by " $+$ " and " $*$ " satisfies the following axioms for all $x, y, z \in \mathcal{H}$:

(i) $x + y = y + x$
(ii) $x + (y + z) = (x + y) + z$
(iii) $x * (y * z) = (x * y) * z$
(iv) $x * (y + z) = x * y + x * z$
(v) $(y + z) * x = y * x + z * x$
(vi) $x + x = x$
(vii) $x + (x * y) = x$
(viii) $y + (x * y) = y$

For convenience, we pronounce " $+$ "(resp. " $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice(as simiring) if and only if $x * x = x$ for all $x \in \mathcal{H}$. Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in \mathcal{H}$. A subincline of an incline $\mathcal{H}$ is a subset $\mathcal{M}$ of $\mathcal{H}$ closed under addition and multiplication . An ideal in an incline $\mathcal{H}$ is a subincline $\mathcal{M} \subset \mathcal{H}$ such that if $x \in \mathcal{M}$ and $y \leq x$ then $y \in \mathcal{M}$. By a homomorphism of incline $\mathcal{H}$ into an incline $\mathcal{I}$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{H}$.

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Definition 1.1([8]). $A \in F(\mathcal{H})$ is called a fuzzy subincline of $\mathcal{H}$ if $A(x+y) \wedge A(x*y) \geq A(x) \wedge A(y)$ for all $x, y \in \mathcal{H}$. A fuzzy subset $A \in F(\mathcal{H})$ is said to be order reversing if $A(x) \geq A(y)$ whenever $x \leq y$.

Definition 1.2([8]). A fuzzy subincline A is called a fuzzy ideal of $\mathcal{H}$ if it is order reversing.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \rightarrow [0,1]$ and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x) = 1 - \mu(x)$. For $t \in [0,1]$, the set $U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}$ is called an upper t-level cut of and the set $L(\mu; t) = \{x \in X \mid \mu(x) \leq t\}$ is called a lower t-level cut of μ. We shall write $a \wedge b$ for $min\{a, b\}$ for $max\{a, b\}$, where a and b are any real numbers.

An intuitionistic fuzzy set (briefly, IFS) A in a nonempty set X is an object having the form

$$A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$$

where the functions $\alpha_A : X \rightarrow [0,1]$ and $\beta_A : X \rightarrow [0,1]$ denote the degree of membership and the degree of non membership respectively, and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1, \quad forall x \in X$.

An intuitionistic fuzzy set $A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (α_A, β_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\alpha_A, \beta_A)$ for the $IFSA = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}$.

2.Intuitionistic Fuzzy ideals In what follows, $F(\mathcal{H})$ denotes the set of all fuzzy subsets in $\mathcal{H}$, i.e., maps from $\mathcal{H}$ into $([0,1], \vee, \wedge)$,where $[0,1]$ is the set of reals between 0 and 1.

Definition 2.1. An $IFSA$=(α_A, β_A) in an incline algebra $\mathcal{H}$ is called an intuitionistic fuzzy ideal of $\mathcal{H}$ if

(I) $\alpha_A(x+y) \wedge \alpha_A(x*y) \geq \alpha_A(x) \wedge \alpha_A(y)$,
(II) $\beta_A(x+y) \vee \beta_A(x*y) \leq \beta_A(x) \vee \beta_A(y)$,
(III) $\alpha_A(x) \geq \alpha_A(y)$ whenever $x \leq y$
(IV) $\beta_A(x) \leq \beta_A(y)$ whenever $x \leq y$

for all $x, y \in \mathcal{H}$.

Example 2.2. Note that for any $x \in \mathcal{H}$, the set $\mathcal{M} = \{a | a \leq x\}$ is an ideal of $\mathcal{H}$ (see [4,Example 1.1.5]). Define $A \in F(\mathcal{H})$ by

$$\alpha_A(x) = \begin{cases} 0.7 & \text{if} \quad x \in \mathcal{M} \\ 0.2 & \text{otherwise} \end{cases} \qquad \beta_A(x) = \begin{cases} 0.2 & \text{if} \quad x \in \mathcal{M} \\ 0.6 & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{H}$. .

It's easy to check that $IFSA$=(α_A, β_A) is an intuitionistic fuzzy ideal of $\mathcal{H}$.

Lemma 2.3. An $IFSA$=(α_A, β_A) is an intuitionistic fuzzy ideal of an incline algebra $\mathcal{H}$ if and only if the fuzzy sets α_A and $\overline{\beta}_A$ are ideals of $\mathcal{H}$.

Proof. Let $IFSA = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $\mathcal{H}$, Clearly α_A is an ideal of $\mathcal{H}$. For any $x, y \in \mathcal{H}$, we have $\overline{\beta}_A(x+y) \wedge \overline{\beta}_A(x*y) = (1 - \beta_A(x+y)) \wedge (1 - \beta_A(x*y)) = 1 - \beta_A(x+y) \vee \beta_A(x+y) \geq 1 - \beta_A(x) \vee \beta_A(y) = (1 - \beta_A(x)) \wedge (1 - \beta_A(y)) = \overline{\beta}_A(x) \wedge \overline{\beta}_A(y)$. Now, let $x \leq y$, then $\overline{\beta}_A(x) = 1 - \beta_A(x) \geq 1 - \beta_A(y) = \overline{\beta}_A(y)$ by Definition 2.1. Hence $\overline{\beta}_A$ is a fuzzy ideal of $\mathcal{H}$.

Conversely, assume that α_A and $\overline{\beta}_A$ are ideals of $\mathcal{H}$, then (I) and (II) are true. For any $x, y \in \mathcal{H}$, we get $1 - \beta_A(x+y) \vee \beta_A(x*y) = (1 - \beta_A(x+y)) \wedge (1 - \beta_A(x*y)) = \overline{\beta}_A(x+y) \wedge \overline{\beta}_A(x*y) \geq \overline{\beta}_A(x) \wedge \overline{\beta}_A(y) = (1 - \beta_A(x)) \wedge (1 - \beta(y)) = 1 - \beta_A(x) \vee \beta_A(y)$, that is, $\beta_A(x+y) \vee \beta_A(x*y) \leq \beta_A(x) \vee \beta_A(y)$. Hence $IFSA$=(α_A, β_A) is an intuitionistic fuzzy ideal of $\mathcal{H}$.

Theorem 2.4. $IFSA = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of an incline algebra $\mathcal{H}$ if and only if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are ideals of $\mathcal{H}$.

Proof. If $IFSA = (\alpha_A, \beta_A)$ is an an intuitionistic fuzzy ideal of $\mathcal{H}$, then $\alpha_A = \overline{\overline{\alpha}}_A$ and β_A are ideals of $\mathcal{H}$ from Lemma 2.3, hence $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $\mathcal{H}$. Conversely, if $\Box A = (\alpha_A, \overline{\alpha}_A)$ and $\diamond A = (\overline{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $\mathcal{H}$, then α_A and $\overline{\beta}_A$ are ideals of $\mathcal{H}$, hence $IFSA$=(α_A, β_A) is an intuitionistic fuzzy ideal of $\mathcal{H}$.

Theorem 2.5. An $IFSA$=(α_A, β_A) is an intuitionistic fuzzy ideal of an incline algebra $\mathcal{H}$ if and only if for all $s, t \in [0,1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $\mathcal{H}$.

Proof. Let $IFSA = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $\mathcal{H}$. First let $x, y \in U(\alpha_A; t)$ for all $t \in [0,1]$. Then $\alpha_A(x+y) \wedge \alpha_A(x*y) \geq \alpha_A(x) \wedge \alpha(y) \geq t$, which implies that $\alpha_A(x+y) \geq t$ and $\alpha_A(x*y) \geq t$, i.e., $x+y \in U(\alpha_A; t)$ and $x*y \in U(\alpha_A; t)$. Let $x \in U(\alpha_A; t)$ and $y \leq x$. Then $\alpha_A(y) \geq \alpha(x) \geq t$, and so $y \in U(\alpha_A; t)$. Hence U$(\alpha_A; t)$ is an ideal of $\mathcal{H}$. Now let $x, y \in L(\beta_A; s)$ for all $s \in [0,1]$. Then $\beta_A(x+y) \vee \beta_A(x*y) \leq \beta_A(x) \vee \beta_A(y) \leq s$, which implies that $\beta_A(x+y) \leq s$ and $\beta_A(x*y) \leq s$, i.e., $x+y \in L(\beta_A; s)$ and $x*y \in L(\beta_A; s)$. Now let $x \in L(\beta_A; s)$ and $y \leq x$, then $\beta_A(y) \leq \beta_A(x) \leq s$, and so $y \in L(\beta_A : s)$. Hence $L(\beta_A; s)$ is an ideal of $\mathcal{H}$.

Conversely, assume that for each $s, t \in [0,1]$, the nonempty sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $\mathcal{H}$. If there exist $x_0, y_0 \in \mathcal{H}$ such that $\alpha_A(x_0+y_0) \wedge \alpha_A(x_0*y_0) < \alpha_A(x_0) \wedge \alpha_A(y_0)$, then taking $t_0 = (\alpha_A(x_0+y_0) \wedge \alpha_A(x_0*y_0) + \alpha_A(x_0) \wedge \alpha_A(y_0))/2$, we have $\alpha_A(x_0+y_0) \wedge \alpha(x_0*y_0) < t_0 < \alpha_A(x_0) \wedge \alpha_A(y_0)$. It follows that $x_0+y_0 \notin U(\alpha_A; t_0)$, $x_0*y_0 \notin U(\alpha_A; t_0)$ and $x_0, y_0 \in U(\alpha_A; t_0)$, that is , $U(\alpha_A; t_0)$ ia not an ideal of $\mathcal{H}$. This is a contradiction. If there exist $x_0, y_0 \in \mathcal{H}$ such that $\beta_A(x_0+y_0) \vee \beta_A(x_0*y_0) > \beta_A(x_0) \vee \beta_A(y_0)$, then taking $s_0 = (\beta_A(x_0+y_0) \vee \beta_A(x_0*y_0) + \beta_A(x_0) \vee \beta_A(y_0))/2$, we have $\beta_A(x_0) \vee \beta_A(y_0) < s_0 < \beta_A(x_0+y_0) \vee \beta_A(x_0*y_0)$, it follows that $x_0+y_0 \notin L(\beta_A; s_0)$, $x_0*y_0 \notin L(\beta_A; s_0)$ and $x_0, y_0 \in L(\beta_A; s_0)$, and that $L(\beta_A; s_0)$ is not an ideal of $\mathcal{H}$. This is a contradiction. Suppose that $x_0, y_0 \in \mathcal{H}$ such that $x_0 \leq y_0$ and $\alpha_A(x_0) < \alpha_A(y_0)$. If we take $m_0 = (\alpha_A(x_0) + \alpha_A(y_0))/2$, then $\alpha_A(x_0) < m_0 < \alpha_A(y_0)$, and so $y_0 \in U(\alpha_A; m_0)$. It follows from the hypothesis that $x_0 \in U(\alpha_A; m_0)$, which is contradiction. Finally, if there exist $x_0 \leq y_0$ and $\beta_A(x_0) > \beta_A(y_0)$. If we take $m_0 = (\beta_A(x_0) + \beta_A(y_0))/2$, then $\beta_A(y_0) < m_0 < \beta_A(x_0)$, and so $y_0 \in L(\beta_A; m_0)$. It follows from the hypothesis that $x_0 \in L(\beta_A; m_0)$, which is contradiction. Therefore $IFSA = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $\mathcal{H}$.

Lemma 2.6 ([8]). Let Λ be a totally ordered set and let $\{\mathcal{M}_t | t \in \Lambda\}$ be a family of ideals of an inline algebra $\mathcal{H}$ such that for all $s, t \in \Lambda, s > t$ if and only if $\mathcal{M}_t \subset \mathcal{M}_s$. Then $\bigcup_{t \in \Lambda} \mathcal{M}_t$ and $\bigcap_{t \in \Lambda} \mathcal{M}_t$ are ideals of $\mathcal{H}$.

Let Λ be a nonempty subset of $[0,1]$.

Theorem 2.7. Let $\{\mathcal{M}_t | t \in \Lambda\}$ be a collection of ideals of an incline algebra $\mathcal{H}$ such that $\mathcal{H} = \bigcup_{t \in \Lambda} \mathcal{M}_t$ and for all $s, t \in \Lambda, s > t$ if and only if $\mathcal{M}_s \subset \mathcal{M}_t$. Then an $IFSA = (\alpha_A, \beta_A)$ in $\mathcal{H}$ defined by

$\alpha_A(x) = sup\{t | x \in \mathcal{M}_t\}$, $\beta_A(x) = inf\{t | x \in \mathcal{M}_t\}$

for all $x \in \mathcal{H}$ is an intuitionistic fuzzy ideal of $\mathcal{H}$.

Proof. Following Theorem 2.5, it is sufficient to show that $U(\alpha_A;t)$ and $L(\beta_A;s)$ are ideals of $\mathcal{H}$. In order to prove that $U(\alpha_A;t)$ is an ideal of $\mathcal{H}$. To do this, we divide into the following two cases:

(i)$t = sup\{q \in \Lambda | q < t\}$ and (ii)$s \neq sup\{q \in \Lambda | q < t\}$.

Case (i) implies that

$x \in U(\alpha_A;t) \Leftrightarrow \mathcal{M}_q$ for all $q < t \Leftrightarrow x \in \bigcap_{q<t} \mathcal{M}_q$,

so that $U(\alpha_A;t) = \bigcap_{q<t} \mathcal{M}_q$, which is an ideal of $\mathcal{H}$ by Lemma 2.6. For the case (ii), we claim that $U(\alpha_A;t) = \bigcup_{q\geq t} \mathcal{M}_q$. If $x \in \bigcup_{q\geq t} \mathcal{M}_q$, then $x \in \mathcal{M}_q$ for some $q \geq t$. It follows that $\alpha_A(x) \geq q \geq t$, so that $x \in U(\alpha_A;t)$. This proves that $\bigcup_{q\geq t} \mathcal{M}_q \subseteq U(\alpha_A;t)$. Now assume that $x \notin \bigcup_{q\geq t} \mathcal{M}_q$. Then $x \notin \mathcal{M}_q$ for all $q \geq t$. Since $t \neq sup\{q \in \Lambda | q < t\}$, there exists $\varepsilon > 0$, such that $(t-\varepsilon, t) \cap \Lambda = \varnothing$. Hence $x \notin \mathcal{M}_q$ for all $q > t-\varepsilon$, which means that if $x \in \mathcal{M}_q$, then $t \geq t-\varepsilon$. Thus $\alpha_A(x) \leq t-\varepsilon < t$, and so $x \notin U(\alpha_A;t)$. Therefore $U(\alpha_A;t) \subseteq \bigcup_{q\geq t} \mathcal{M}_q$. Using Lemma 2.6, $U(\alpha_A;t) = \bigcup_{q\geq t} \mathcal{M}_q$ is an ideal of $\mathcal{H}$.

Next we prove that $L(\beta_A;s)$ is an ideal of $\mathcal{H}$. To do this, we divide into the following two cases:

(iii) $s = inf\{r \in \Lambda | s < r\}$ and (iv)$s \neq inf\{r \in \Lambda | s < r\}$.

Case (iii) implies that

$x \in L(\beta_A;s) \Leftrightarrow \mathcal{M}_r$ for all $s < r \Leftrightarrow x \in \bigcap_{s<r} \mathcal{M}_r$,

so that $L(\beta_A;s) = \bigcap_{s<r} \mathcal{M}_r$, which is an ideal of $\mathcal{H}$ by Lemma 2.6. For the case (iv), we claim that $L(\beta_A;s) = \bigcup_{s\geq r} \mathcal{M}_r$. If $x \in \bigcup_{s\leq r} \mathcal{M}_r$, then $x \in \mathcal{M}_r$ for some $r \leq s$. It follows that $\beta_A(x) \leq r \leq s$, so that $x \in L(\beta_A;s)$. This proves that $\bigcup_{s\geq r} \mathcal{M}_r \subseteq L(\beta_A;s)$. Now assume that $x \notin \bigcup_{s\geq r} \mathcal{M}_r$. Then $x \notin \mathcal{M}_r$ for all $r \leq s$. Since $s \neq inf\{r \in \Lambda | s < r\}$, there exists $\varepsilon > 0$, such that $(s+\varepsilon, s) \cap \Lambda = \varnothing$. Hence $x \notin \mathcal{M}_r$ for all $r < s+\varepsilon$, which means that if $x \in \mathcal{M}_r$, then $r \geqslant s+\varepsilon$. Thus $\beta_A(x) \geqslant s+\varepsilon > s$, and so $x \notin L(\beta_A;s)$. Therefore $L(\beta_A;s) \subseteq \bigcup_{r\leq s} \mathcal{M}_r$. Using Lemma 2.6, $L(\beta_A;s) = \bigcup_{r\leq s} \mathcal{M}_r$ is an ideal of $\mathcal{H}$. Therefore $IFSA = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $\mathcal{H}$.

3. Equivalence relations on $IF(\mathcal{H})$ Let $IF(\mathcal{H})$ be the family of all intuitionistic fuzzy ideals of an incline algebra $\mathcal{H}$ and let $t \in [0,1]$. Define binary relation U^t and L^t on $IF(\mathcal{H})$ as follows:

$(A,B) \in U^t \Leftrightarrow U(\alpha_A;t) = U(\alpha_B), (A,B) \in L^t \Leftrightarrow L(\beta_A;t) = L(\beta_B;t)$

respectively, for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in $IF(\mathcal{H})$. Then clearly U^t and L^t are equivalence relations on $IF(\mathcal{H})$. For any $A = (\alpha_A, \beta_A) \in IF(\mathcal{H})$, let $[A]_{U^t}$ (respectively, $[A]_{L^t}$) denote the equivalence class of A modulo U^t (respectively, L^t), and denote by $IF(\mathcal{H})/U^t$(respectively, $IF(\mathcal{H})/L^t$) the system of all equivalence classes modulo U^t(respectively, L^t); so

$IF(\mathcal{H})/U^t = \{[A]_{U^t} | A = (\alpha_A, \beta_A) \in IF(\mathcal{H})\}$,

respectively,

$IF(\mathcal{H})/L^t = \{[A]_{L^t} | A = (\alpha_A, \beta_A) \in IF(\mathcal{H})\}$,

Now let $I(\mathcal{H})$ denote the family of all ideals of $\mathcal{H}$ and let $t \in [0,1]$. Define maps f_t and g_t from $IF(\mathcal{H})$ to $I(\mathcal{H}) \cup \{\varnothing\}$ by $f_t(A) = U(\alpha_A;t)$ and $g_t(A) = L(\beta_A;t)$, respectively, for all $A = (\alpha_A, \beta_A) \in IF(\mathcal{H})$. Then f_t and g_t are clearly well defined.

Theorem 3.1. For any $t \in (0,1)$ the maps f_t and g_t are surjective from $IF(\mathcal{H})$ to $I(\mathcal{H}) \cup \{\varnothing\}$.

Proof. Let $t \in (0,1)$. Note that $0_\sim = (0,1)$ is in $IF(\mathcal{H})$, where 0 and 1 are fuzzy sets in $\mathcal{H}$ defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in \mathcal{H}$. Obviously $f_t(0_\sim) = U(0;t) = \varnothing = L(1;t) = g_t(0_\sim)$. Let $G(\neq \varnothing) \in I(\mathcal{H})$. For $G_\sim = (\mathcal{X}_G, \overline{\mathcal{X}}_G) \in IF(\mathcal{H})$, we have $f_t(G_\sim) = U(\mathcal{X}_G;t) = G$ and $g_t(G_\sim) = L(\overline{\mathcal{X}}_G;t) = G$. Hence f_t and g_t are surjective.

Theorem 3.2. The quotient sets $IF(\mathcal{H})/U^t$ and $IF(\mathcal{H})/L^t$ are equipotent to $I(\mathcal{H}) \cup \{\varnothing\}$ for all every $t \in (0,1)$.

Proof. For $t \in (0,1)$, let f_t^*(respectively, g_t^*) be a map from $IF()/U^t$(respectively, $IF(\mathcal{H})/L^t$) to $I(\mathcal{H}) \cup \{\varnothing\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (respectively, $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\alpha_A, \beta_A) \in IF(\mathcal{H})$. If $U(\alpha_A; t) = U(\alpha_B; t)$ and $L(\beta_A; t) = L(\beta_B; t)$ for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B) \in IF(\mathcal{H})$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps f_t^* and g_t^* are injective . Now let $G(\neq \varnothing) \in I(\mathcal{H})$. For $G_\sim = (\mathcal{X}_G, \overline{\mathcal{X}}_G) \in IF(\mathcal{H})$, we have

$f_t^*([G_\sim]_{U^t}) = f_t(G_\sim) = U(\mathcal{X}_G; t) = G,$

$g_t^*([G_\sim]_{L^t}) = g_t(G_\sim) = L(\mathcal{X}_G; t) = G.$

Finally, for $0_\sim = (0,1) \in IF(\mathcal{H})$, we get

$f_t^*([0_\sim]_{U^t}) = f_t(0_\sim) = U(0; t) = G,$

$g_t^*([0_\sim]_{L^t}) = g_t(0_\sim) = L(0; t) = G.$

This shows that f_t^* and g_t^* are surjective. This completes the proof.

For any $t \in [0,1]$, we define another relation R^t on $IF(\mathcal{H})$ as follows:

$(A, B) \in R^t \Leftrightarrow U(\alpha_A; t) \cap L(\alpha_A; t) = U(\alpha_B; t) \cap L(\alpha_B; t)$

for any $A = (\alpha, \beta_A), B = (\alpha_B, \beta_B) \in IF(\mathcal{H})$. Then the relation R^t is also an equivalence relation on $IF(\mathcal{H})$.

Theorem 3.3. For any $t \in (0,1)$, the maps $\phi_t : IF(\mathcal{H}) \to I(\mathcal{H}) \cup \{\varnothing\}$ defined by $\phi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\alpha_A, \beta_A) \in IF(\mathcal{H})$ is surjective.

Proof. Let $t \in (0,1)$. For $0_\sim = (0,1) \in IF(\mathcal{H})$,

$\phi_t(0_\sim) = f_t(0_\sim) \cap g_t(0_\sim) = U(0; t) \cap L(1; t) = \varnothing.$

For any $H \in IF(\mathcal{H})$, there exists $H_\sim = (\mathcal{X}_H, \overline{\mathcal{X}}_H) \in IF(\mathcal{H})$ such that

$\phi_t(H_\sim) = f_t(\mathcal{H}_\sim) \cap g_t(H_\sim) = U(\mathcal{X}_H; t) \cap L(\overline{\mathcal{X}}_H; t) = H.$

This proves the proof.

Theorem 3.4. For any $t \in (0,1)$, the quotient set $IF(\mathcal{H})/R^t$ is equipotent to $I(\mathcal{H}) \cup \{\varnothing\}$.

Proof. Let $t \in (0,1)$ and $\phi_t^* : IF(\mathcal{H})/R^t \to I(\mathcal{H}) \cup \{\varnothing\}$ be a map defined by $\phi_t^*([A]_{R^t} = \phi_t(A)$ for all $[A]_{R^t} \in IF(\mathcal{H})/R^t$. If $\phi_t^*([A]_{R^t} = \phi_t^*([B]_{R^t}$ for any $[A]_{R^t}, [B]_{R^t} \in IF(\mathcal{H})/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$, so that ϕ_t^* is injective. For $0_\sim = (0,1) \in IF(\mathcal{H})$,

$\phi_t^*([0_\sim]_{R^t}) = \phi_t(0_\sim) = f_t(0_\sim) \cap g_t(0_\sim) = U(0; t) \cap L(1; t) = \varnothing.$

If $H \in IF(\mathcal{H})$, then for $H_\sim = (\mathcal{X}_H, \overline{\mathcal{X}}_H) \in IF(\mathcal{H})$, we have

$\phi_t^*([H_\sim]_{R^t}) = \phi(H_\sim) = f_t(\mathcal{H}_\sim) \cap g_t(H_\sim) = U(\mathcal{X}_H; t) \cap L(\overline{\mathcal{X}}_H; t) = H.$

Hence ϕ_t^* is surjective, this completes the proof.

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