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SET-REPRESENTATION OF A QUIVER AND RELATIONSHIP WITH LINEAR-REPRESENTATION*

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Abstract. It is well-known that, for a finite quiver Γ and its path algebra $k\Gamma$ over a field k, the linear-representation category **Lin-Rep**Γ is equivalent to the $k\Gamma$-module category $k\Gamma$-Mod . The purpose of this paper is to generalize the conclusion to the so-called set-representation category **Set-Rep**Γ and its equivalent category $P(\Gamma)$-$\mathcal{SET}^{\wr}$.

The authors firstly introduce the definition of the set-representation category **Set-Rep**Γ and find out its equivalent category $P(\Gamma)$-$\mathcal{SET}^{\wr}$. Secondly, through a finite connected quiver Γ on which all objects of $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are (positively) graded, they find some interesting relations between the two categories $k\Gamma$-Mod and $P(\Gamma)$-$\mathcal{SET}^{\wr}$ (see Corollary 3.8 and Corollary 3.9), although one of them is abelian while the other is not. Under the equivalence of categories, such relations also exist between **Lin-Rep**Γ and **Set-Rep**Γ.

2000 Mathematics Subject Classifications: 16G20; 20M30

1 Preliminaries Firstly, we explain some concepts and notations used in this paper, where those on quivers and the representation theory of algebras can be found in [1][2], and those on S-Systems and the theory of semigroups are from [3].

(1) Quiver

A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph, where Γ_0 is the set of the vertices and Γ_1 is the set of arrows between vertices. A *sub-quiver* of Γ is just its oriented sub-graph.

We say a quiver Γ is *a finite quiver* if Γ_0 and Γ_1 are both finite sets. We denote by $s : \Gamma_1 \to \Gamma_0$ and $t : \Gamma_1 \to \Gamma_0$ the maps, where $s(\alpha) = i$ and $t(\alpha) = j$ when $\alpha : i \to j$ is an arrow from the vertex i to the vertex j.

A *path* p in the quiver Γ is either an ordered sequence of arrows $p = \alpha_n \cdots \alpha_2\alpha_1$ with $t(\alpha_l) = s(\alpha_{l+1})$ for $1 \le l \le n$, or the symbol e_i for $i \in \Gamma_0$. We call the path e_i *trivial path* and we define $s(e_i) = t(e_i) = i$. For a non-trivial path $p = \alpha_n \cdots \alpha_2\alpha_1$, we define $s(p) = s(\alpha_1)$, and $t(p) = t(\alpha_n)$.

A vertex i in Γ_0 is called a *sink* if there is no arrow α with $s(\alpha) = i$ and a *source* if there is no arrow α with $t(\alpha) = i$.

(2) S-System

Let S be a semigroup and M a non-empty set. If the map $\varphi : S \times M \longrightarrow M$ satisfies $\varphi(s_2, \varphi(s_1, m)) = \varphi(s_2s_1, m)$, $\forall s_1, s_2 \in S$, $\forall m \in M$, then (M, φ) is called a *left S-System*, or says, S acts on the left of M.

For short, denote $\varphi(s, m)$ by sm, left S-System (M, φ) just as M. Similarly, we can define *right S-Systems*.

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Let M, N are two S-Systems, $f : M \longrightarrow N$ is called a *S-morphism* from M to N, if $f(sm) = sf(m)$, $\forall s \in S$ and $\forall m \in M$. All left S-Systems and all S-morphisms between them constitute a category, denoted by S-$\mathcal{SET}$.

Clearly, if the semigroup S contains zero element, then any S-System M must have an element θ, such that $s\theta = \theta$, $\forall s \in S$. If moreover, M contains a unique element θ_M satisfying $s\theta_M = \theta_M$, $0m = \theta_M$, $\forall s \in S$ and $\forall m \in M$, we call such S-System M *central.* All central S-Systems and S-morphisms between them also constitute a category. Clearly it is a full sub-category of S-$\mathcal{SET}$.

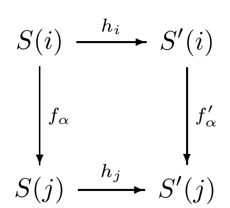
(3) Notations

In this paper, $\#A$ or $\mid A \mid$ stands for the cardinal number of a set A. $\dot{\cup}_{i\in I} A_i$ denotes the disjoint union of a family of sets $\{A_i\}_{i\in I}$. And, $\mathbf{Z}$ denotes the set of all integers.

2 Set-Representations of A Quiver

Definition 2.1 *Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with Γ_0 the set of vertices and Γ_1 the set of arrows between vertices. A* set-representation *(S, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is a set of sets $\{S(i) : i \in \Gamma_0\}$ together with maps f_α: $S(i) \to S(j)$ for each arrow α: $i \to j$.*

A *morphism* h: $(S, f) \to (S', f')$ between two set-representations of Γ is a collection $\{h_i : S(i) \to S'(i)\}_{i\in\Gamma_0}$ of maps such that for each arrow α: $i \to j$ in Γ_1 the diagram:



Figure(I)

commutes. If h: $(S, f) \to (S', f')$ and g: $(S', f') \to (S'', f'')$ are two morphisms between set-representations, then the composition gh is defined to be the collection of maps $\{g_ih_i$: $S(i) \to S''(i)\}_{i\in\Gamma_0}$. In this way, we get *the category of set-representations* of Γ, which we denote by **Set-Rep**Γ.

If we think from any set X, there is a unique map $\bar{0} : X \to \emptyset$ and to any set Y, there is a unique map $\underline{0} : \emptyset \to Y$, then we can define the zero object in **Set-Rep**Γ as follows: (S, f) is called the *zero object*, which we denote by $(\emptyset, 0)$, if $S(i) = \emptyset$ for all $i \in \Gamma_0$ and $f_\alpha = 1_\emptyset$ for each arrow α in Γ_1.

An object (S, f) is called a *sub-object* of an object (S', f') in **Set-Rep**Γ, if $S(i) \subseteq S'(i)$ for all $i \in \Gamma_0$ and $f_\alpha = f'_\alpha|_{S(i)}$ for each arrow α starting from i.

A *sum* of two objects (S, f) and (S', f') in **Set-Rep**Γ is the object (W, g), where $W(i) = S(i) \coprod S'(i)$ for each $i \in \Gamma_0$ and $g_\alpha = f_\alpha \coprod f'_\alpha$ for all $\alpha \in \Gamma_1$. An object (S, f) is said to be *indecomposable* if it can not be written as the sum of any two nonzero set-representations. An object (S, f) is *simple* if it has no proper nonzero sub-objects. Clearly, a simple object is indecomposable.

Next, we illustrate with some examples.

Example 2.1 *Let (S, f) be an object in* **Set-Rep**Γ *and $V(i) = \{(a, a) \mid a \in S(i)\}$, $g_\alpha = (f_\alpha \coprod f_\alpha) \mid_{V(i)}$ for each $i \in \Gamma_0$ and an arrow α starting from i, then (V, g) is a sub-object of $(S, f) \coprod (S, f)$, which is denoted as $1_{(S,f)}$.*

Example 2.2 *Let Γ be the quiver $1\cdot \to \cdot 2$, (S, f) and (S', f') be two set-representations, where $S(1) = \{x_1, y_1\}$, $S(2) = \{x_2, y_2\}$, $S'(1) = \{x'_1, y'_1\}$, $S'(2) = \{x'_2, y'_2\}$, $f_\alpha(x_1) = x_2$,*

$f_\alpha(y_1) = y_2$, $f'_\alpha(x'_1) = f'_\alpha(y'_1) = x'_2$. Let $h_1 : S(1) \to S'(1)$ with $h_1(x_1) = y'_1$ and $h_1(y_1) = x'_1$, $h_2 : S(2) \to S'(2)$ with $h_2(x_2) = h_2(y_2) = x'_2$, then $h = \{h_1, h_2\}$ is a morphism from (S, f) to (S', f').

In the category **Set-Rep**Γ, for a morphism $h = \{h_i\}_{i\in\Gamma_0} : (S, f) \to (S', f')$, we define the image Imh to be the suboject (U, g) of (S', f'), where $U(i) = Imh_i$ and $g_\alpha = f'_\alpha|_{Imh_i}$ for each arrow $\alpha : i \to j$. We define the kernel $Kerh$ to be the sub-object (V, f'') of $(S, f)\coprod(S, f)$, where $V(i) = \{(a, b)|a, b \in S(i) \,\text{with}\, h_i(a) = h_i(b)\}$ and $f''_\alpha = (f_\alpha \coprod f_\alpha)|_{V(i)}$ for each arrow $\alpha : i \to j$.

If each h_i is injective (respectively surjective), we call h a *monomorphism* (respectively an *epimorphism*), and h is an *isomorphism* if and only if h is both monomorphic and epimorphic. The morphism h given in Example 2.2 is neither monomorphic nor epimorphic. We call the sequence $(S, f) \xrightarrow{h} (S', f') \xrightarrow{h'} (S'', f'')$ a *related exact sequence* if $(Imh \coprod Imh) \bigcup 1_{(S',f')} = Kerh'$. Then, we have

Proposition 2.1 *(i) The sequence $(\emptyset, 0) \to (S, f) \xrightarrow{h} (S', f')$ is related exact if and only if h is a monomorphism.*

(ii) Suppose $| S'(i) | \geq 2$ for all $i \in \Gamma_0$, then the sequence $(S, f) \xrightarrow{h} (S', f') \to (\emptyset, 0)$ is related exact if and only if h is an epimorphism.

Proof: (i) $(\emptyset, 0) \to (S, f) \xrightarrow{h} (S', f')$ related exact
$\Longleftrightarrow Kerh = 1_{(S,f)}$
$\Longleftrightarrow (Kerh)(i) = \{(a, a)|a \in S(i)\}, \forall i \in \Gamma_0$
$\Longleftrightarrow \{(a, b)|a, b \in S(i), h_i(a) = h_i(b)\} = \{(a, a)|a \in S(i)\}, \forall i \in \Gamma_0$
$\Longleftrightarrow h_i(a) = h_i(b)$ implies $a = b$, $\forall i \in \Gamma_0$ and $a, b \in S(i)$
$\Longleftrightarrow h_i$ is injective, $\forall i \in \Gamma_0$
$\Longleftrightarrow h$ is monomorphic.
(ii) $(S, f) \xrightarrow{h} (S', f') \to (\emptyset, 0)$ related exact
$\Longleftrightarrow (Imh \coprod Imh) \cup 1_{(S',f')} = (S', f')\coprod(S', f')$
$\Longleftrightarrow (Imh_i \coprod Imh_i) \cup \{(a', a')|a' \in S'(i)\} = S'(i) \coprod S'(i), \forall i \in \Gamma_0$
$\Longleftrightarrow$ if $a', b' \in S'(i)$ and $a' \neq b'$, then $(a', b') \in Imh_i \coprod Imh_i, \forall i \in \Gamma_0$
$\Longleftrightarrow h_i$ is surjective, $\forall i \in \Gamma_0$
$\Longleftrightarrow h$ is epimorphic.
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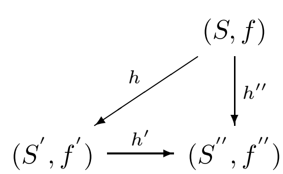
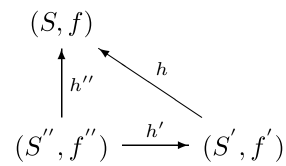
An object (S, f) is said to be *projective* if for an arbitrary epimorphism $h' : (S', f') \to (S'', f'')$, and an arbitrary morphism $h'' : (S, f) \to (S'', f'')$, there exists a morphism $h : (S, f) \to (S', f')$ such that $h'' = h'h$, i.e. we have the commutative diagram



Figure(II)

Dually, an object (S, f) is said to be *injective*, if for an arbitrary monomorphism $h' : (S'', f'') \to (S', f')$ and an arbitrary morphism $h'' : (S'', f'') \to (S, f)$, there exists a morphism $h : (S', f') \to (S, f)$, such that $h'' = hh'$, i.e. we have the commutative diagram



$Figure(III)$

Let $P(\Gamma)$ be the set consisting of 0 and all paths in the quiver Γ . Define a multiplication $\cdot$ on $P(\Gamma)$ as follows: $0 \cdot \rho = \rho \cdot 0 = 0$ for all $\rho \in P(\Gamma)$, for any two paths ρ_{ji} from i to j and ρ_{tk} from k to t, $\rho_{ji} \cdot \rho_{tk} = \begin{cases} 0, & \text{if } i \neq t \\ \rho_{ji}\rho_{tk}, & \text{if } i = t \end{cases}$ where $\rho_{ji}\rho_{tk}$ means the connection of ρ_{ji} and ρ_{tk} for $i = t$.

Then $P(\Gamma)$ becomes a semigroup with zero 0 under the multiplication $\cdot$. Omitting $\cdot$, we usually write $\rho_1\rho_2$ instead of $\rho_1 \cdot \rho_2$.

Now, we define a subcategory $P(\Gamma)$-$\mathcal{SET}^{\wr}$ of the category $P(\Gamma)$-$\mathcal{SET}$: the objects M are $P(\Gamma)$-Systems satisfying (i) $P(\Gamma)M = M$; (ii) there is a unique element $\theta_M \in M$ such that $0m = \theta_M$, for all $m \in M$ (Here θ_M acts as the "*zero element*" of M); (iii) if $e_i m \neq \theta_M$, then $\alpha m \neq \theta_M$, for all arrows α starting from i.

Note that for any $\rho \in P(\Gamma)$, we always have $\rho\theta_M = \rho(0m) = (\rho \cdot 0)m = 0m = \theta_M$. And clearly $P(\Gamma)$ is an object of $P(\Gamma)$-$\mathcal{SET}^{\wr}$, called the *regular object*, and $\{\theta\}$ is the *zero object* of $P(\Gamma)$-$\mathcal{SET}^{\wr}$, if we define the action as $\rho\theta = \theta$ for all $\rho \in P(\Gamma)$.

For two objects M and N in $P(\Gamma)$-$\mathcal{SET}^{\wr}$, a *morphism* φ: $M \to N$ is defined as a map satisfying (i) $\varphi(\rho m) = \rho\varphi(m)$ for any $m \in M$ and $\rho \in P(\Gamma)$; (ii) $\varphi(m) \neq \theta_N$, if $m \neq \theta_M$.

Note that (ii) is equivalent to say $\varphi(M \setminus \{theta_M\}) \subseteq N \setminus \{theta_N\}$, and when $\rho = 0$, from (i), it must hold that $\varphi(\theta_M) = \theta_N$.

Then, $P(\Gamma)$-$\mathcal{SET}^{\wr}$ is exactly a subcategory of the category $P(\Gamma)$-$\mathcal{SET}$.

We have known from [1][2] that, for a field k and a finite quiver Γ, there exists an equivalence between the two categories **Lin-Rep**Γ and $k\Gamma$-Mod, where **Lin-Rep**Γ is the category of k-linear representations of Γ and $k\Gamma$-Mod the $k\Gamma$-module category. It is interesting for us to find that the similar result also holds between the two weaker categories **Set-Rep**Γ and $P(\Gamma)$-$\mathcal{SET}^{\wr}$, that is, we have :

Theorem 2.2 *The two categories* **Set-Rep**Γ *and* $P(\Gamma)$-$\mathcal{SET}^{\wr}$ *are equivalent.*

Proof: We start by defining two functors F: **Set-Rep**$\Gamma \to P(\Gamma)$-$\mathcal{SET}^{\wr}$ and H: $P(\Gamma)$-$\mathcal{SET}^{\wr}$ $\to$ **Set-Rep**Γ.

For an object (S, f) in **Set-Rep**Γ, set $M = \dot{\cup}_{i \in \Gamma_0} S(i) \cup \{\theta_M\}$, where θ_M is an element which is not in $S(i)$ for all $i \in \Gamma_0$. Define the action of $P(\Gamma)$ on the set M as follows: for any $m \in M$, $\rho \in P(\Gamma)$,

(i) $\rho m = \theta_M$, if $\rho = 0$;

(ii) $\rho m = m$, if $m \in S(i)$ and $\rho = e_i$;

(iii) $\rho m = f_\alpha(m)$, if $m \in S(i)$ and ρ is an arrow $\alpha : i \to j$;

(iv) $\rho m = f_{\alpha_s} \cdots f_{\alpha_1}(m)$, if $m \in S(i)$, $\rho = \alpha_s \cdots \alpha_1$ where $\alpha_s, \cdots, \alpha_1$ are arrows and α_1 starts from i.

From this definition, it is easy to see that $P(\Gamma)\{\theta_M\} = \{\theta_M\}$, and that if $m \in S(i)$ but ρ does not start from i, then $\rho m = \rho(e_i m) = (\rho \cdot e_i)m = 0m = \theta_M$.

Clearly, M is a $P(\Gamma)$-System under the action defined above. Moreover, we can say M is an object of $P(\Gamma)$-$\mathcal{SET}^{\wr}$. Firstly, the element θ_M satisfies $0m = \theta_M$ for all $m \in M$. And, obviously, $P(\Gamma)M \subseteq M$. Conversely, for all $m \in M$, when $m \neq \theta_M$, suppose $m \in S(i)$ for some i, then $m = e_i m$; when $m = \theta_M$, we have $P(\Gamma)\{\theta_M\} = \{\theta_M\}$. Hence $M \subseteq P(\Gamma)M$.

It follows that $P(\Gamma)M = M$. If $e_im \neq \theta_M$, which implies $m \in S(i)$, then for all arrows as $\alpha : i \to j$, $\alpha m = f_\alpha(m) \in S(j)$, so $\alpha m \neq \theta_M$. Then M is an object of $P(\Gamma)$-$\mathcal{SET}^\wr$.

Now, we can start to define the functors F: **Set-Rep**$\Gamma \to P(\Gamma)$-$\mathcal{SET}^\wr$ by $F(S, f) = M$.

Let h be a morphism from (S, f) to (S', f') in the category **Set-Rep**Γ. Then, for each $i \in \Gamma_0$, we have a map $h_i : S(i) \to S'(i)$ satisfying the Figure (I), i.e. $h_jf_a = f'_ah_i$ for each arrow α from i to j. It has been known that $M = F(S, f) = \dot\cup_{i\in\Gamma_0}S(i) \cup \{\theta_M\}$ and $M' = F(S', f') = \dot\cup_{i\in\Gamma_0}S'(i) \cup \{\theta_{M'}\}$, Introducing a map $\tilde h : M \to M'$ satisfying that $\tilde h|_{S(i)} = h_i$ for all i and $\tilde h(\theta_M) = \theta_{M'}$. Thus, we can get $\tilde h(\alpha m) = \alpha\tilde h(m)$ for each $m \in M$. Moreover, $\tilde h(\rho m) = \rho\tilde h(m)$ for each $m \in M$, $\rho \in P(\Gamma)$. And, when $m \neq \theta_M$, $\tilde h(m) \neq \theta_{M'}$ since $\tilde h(S(i)) = h_i(S(i)) \subseteq S'(i)$. Therefore $\tilde h$ is a morphism from M to M'. This means one can set $F(h) = \tilde h$.

We next want to define a functor H: $P(\Gamma)$-$\mathcal{SET}^\wr \to$ **Set-Rep**Γ. For an object M in category of $P(\Gamma)$-$\mathcal{SET}^\wr$, let $S(i) = e_iM \setminus \{\theta_M\}$. For all arrows $\alpha : i \to j$, define $f_\alpha : S(i) \to S(j)$ as follows: for all $m \in S(i)$, suppose $m = e_im^{'}$, let $f_\alpha(m) = \alpha m$, it is well-defined since $\alpha m = \alpha(e_im^{'}) = \alpha m^{'} \neq \theta_M$ and $\alpha m = (e_j\alpha)m = e_j(\alpha m) \in S(j)$. Therefore let $H(M) = (S, f)$, where $S = \{S(i) : i \in \Gamma_0\}$, and $f = \{f_\alpha :$ there is an arrow α from i to $j\}$. Then $H(M)$ is an object of category **Set-Rep**Γ.

If $\varphi : M \to M'$ is a morphism in $P(\Gamma)$-$\mathcal{SET}^\wr$, we have $H(M) = (S, f)$, $H(M') = (S', f')$, where $S(i) = e_iM \setminus \{\theta_M\}$ and $S'(i) = e_iM' \setminus \{\theta_{M'}\}$. Since $\varphi(e_iM) = e_i\varphi(M) \subseteq e_iM'$ and $\varphi(m) \neq \theta_{M'}$ for all $m \in M$ and $m \neq \theta_M$, then we get $\varphi_i : e_iM \setminus \{\theta_M\} \to e_iM' \setminus \{\theta_{M'}\}$ by restriction, i.e. $\varphi_i = \varphi|_{S(i)} : S(i) \to S'(i)$. For each arrow $\alpha : i \to j$, we have $\alpha\varphi(m) = \varphi(\alpha m)$, for all $m \in M$. So $\alpha\varphi_i(m) = \varphi_j(\alpha m)$, for all $m \in S(i)$. Hence $f'_\alpha\varphi_i(m) = \varphi_jf_\alpha(m)$, for all $m \in S(i)$. Then, $f'_\alpha\varphi_i = \varphi_jf_\alpha$ for any arrow $\alpha : i \to j$. Therefore we can set $H(\varphi) = \{\varphi_i\}_{i\in\Gamma_0}$, which is a morphism in **Set-Rep**Γ.

Next, we will prove F and H are mutual-inverse equivalent functors. Let (S, f) be an object in **Set-Rep**Γ, then $M = F(S, f) = \dot\cup_{j\in\Gamma_0}S(j) \cup \{\theta_M\}$ and $e_iM \setminus \{\theta_M\} = e_i(\dot\cup_{j\in\Gamma_0}S(j)) \setminus \{\theta_M\} = e_iS(i) \setminus \{\theta_M\} = S(i)$. For an arrow $\alpha : i \to j$ in Γ_1, the map $f_\alpha : S(i) \to S(j)$ induces the map $\tilde f_\alpha : F(S, f) \to F(S, f)$ satisfying $\tilde f_\alpha(m) = \alpha m$ for all $m \in F(S, f)$. The restriction of $\tilde f_\alpha$ on $e_iF(S, f) \setminus \{\theta_M\} = S(i)$ is just f_α. So $HF(S, f) = (S, f)$.

For a morphism $h = \{h_i\}_{i\in\Gamma_o}$: $(S, f) \to (S', f')$, we have $F(h) = \tilde h$ where $\tilde h\ |_{S(i)}= h_i$, $\tilde h(\theta_M) = \theta_{M'}$. Due to the definition of H, it follows $HF(h) = \{h_i\}_{i\in\Gamma_0}$. Thus, $HF = \mathbf{id}$ the identity functor in **Set-Rep**Γ.

Let M be an object in $P(\Gamma)$-$\mathcal{SET}^\wr$, then $H(M) = (S, f)$, where $S(i) = e_iM \setminus \{\theta_M\}$ and

$$f = \{f_\alpha : S(i) \to S(j) \mid \quad f_\alpha(m_i) = \alpha m_i \text{ for an arrow } \alpha : i \to j \text{ and } m_i \in S(i)\}.$$

When $i \neq j$, if there exists two elements $m, m^{'} \in M$, such that $e_im = e_jm^{'} \neq \theta_M$, then for an arrow $\alpha : i \to k$, $\alpha(e_im) = \alpha m \neq \theta_M$, but $\alpha(e_jm^{'}) = (\alpha e_j)m^{'} = 0m^{'} = \theta_M$, this is a contradiction. Hence $S(i) \cap S(j) = \emptyset$ when $i \neq j$. So if we can prove $M = \cup_{i\in\Gamma_o}S(i) \cup \{\theta_M\}$, then $FH(M) = M$. In fact, $\cup_{i\in\Gamma_o}S(i) \cup \{\theta_M\} \subseteq P(\Gamma)M = M$. Conversely, for all $m \in M$, if $m = \theta_M$, it is clearly that $m \in \cup_{i\in\Gamma_o}S(i) \cup \{\theta_M\}$, when $m \neq \theta_M$, since $m \in M = P(\Gamma)M$, there is $\rho_{ji} \in P(\Gamma), m' \in M$, such that $m = \rho_{ji}m'$. Clearly $m' \neq \theta_M$, so $m = \rho_{ji}m' = e_j(\rho_{ji}m') \in e_jM \setminus \{\theta_M\} = S(j)$. Therefore, $M \subseteq \cup_{i\in\Gamma_o}S(i) \cup \{\theta_M\}$.

For a morphism $\varphi : M \to M'$, we have $H(\varphi) = \{\varphi_i : S(i) \to S'(i) \mid \quad \varphi_i = \varphi\ |_{S(i)}\}_{i\in\Gamma_0}$. Moreover, due to the definition of F, it follows $FH(\varphi) = \varphi$. Therefore, $FH = \mathbf{id}$ the identity functor in $P(\Gamma)$-$\mathcal{SET}^\wr$.

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As a corollary, the following holds naturally:

Corollary 2.3 *(i) An object* (V, f) *in the category* **Set-Rep**Γ *is projective (respectively injective, simple, indecomposable) if and only if* $F(V, f)$ *is projective (respectively injective, simple, indecomposable) in the category* $P(\Gamma)$-$\mathcal{SET}^{\wr}$*;*

(ii) A sequence $(U, f) \to (V, g) \to (W, h)$ *in the category* **Set-Rep**Γ *is related exact if and only if the induced sequence* $F(U, f) \to F(V, g) \to F(W, h)$ *is related exact in the category* $P(\Gamma)$-$\mathcal{SET}^{\wr}$*.*

A *relation* σ on a quiver Γ is a set of paths which have the same two endpoints. If $\rho = \{\sigma_t\}_{t\in T}$ is a set of relations on Γ, the pair (Γ, ρ) denotes a quiver with relations.

Associating with (Γ, ρ), we define $P(\Gamma, \rho)$ to be $P(\Gamma)/\sim$, where $x \sim y$ in $P(\Gamma)$ if and only if $x = y$ or x and y lie in the same σ_t for a certain $t \in T$. The category **Set-Rep**(Γ, ρ) of representations is the full subcategory of **Set-Rep**Γ, whose objects are (S, f) with $f_{\sigma_{t_1}} = f_{\sigma_{t_2}}$, when σ_{t_1} and σ_{t_2} lie in the same σ_t for some $t \in T$ and here $f_{\sigma_{t_1}}$ stands for $f_{\alpha_s} \cdots f_{\alpha_1}$, when $\sigma_{t_1} = \alpha_s \cdots \alpha_1$ with each α_i an arrow. The subcategory $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$ of category $P(\Gamma)$-$\mathcal{SET}^{\wr}$ can be defined to be with objects M which satisfy $\sigma_{t_1}M = \sigma_{t_2}M$, if σ_{t_1} and σ_{t_2} lie in the same σ_t for some $t \in T$.

Combining this concept with Theorem 2.2, we get :

Proposition 2.4 *Let* (Γ, ρ) *be a quiver with relations, then the functor* F*:* **Set-Rep**$\Gamma \to P(\Gamma)$-$\mathcal{SET}^{\wr}$ *induces an equivalence between* **Set-Rep**(Γ, ρ) *and* $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$*.*

Proof: If (S, f) is an object in **Set-Rep**(Γ, ρ), then by definition, $f_{\sigma_{t_1}} = f_{\sigma_{t_2}}$ if σ_{t_1} and σ_{t_2} lie in the same $\sigma_t \in \rho$. Hence $\sigma_{t_1}F(S, f) = \sigma_{t_2}F(S, f)$, so that $F(S, f)$ is an object of $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$.

Conversely, if $F(S, f)$ is an object of $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$, then $\sigma_{t_1}F(S, f) = \sigma_{t_2}F(S, f)$ when σ_{t_1} and σ_{t_2} lie in the same $\sigma_t \in \rho$, i.e. they have the same two endpoints. So $f_{\sigma_{t_1}} = f_{\sigma_{t_2}}$, and hence (S, f) is an object of **Set-Rep**(Γ, ρ).

#

Let (Γ, ρ) be a quiver with relations and F : **Set-Rep**$(\Gamma, \rho) \to P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$ be the above equivalence. Then as Corollary 2.3 we have the same conclusion between the two categories **Set-Rep**(Γ, ρ) and $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$ by F.

Corollary 2.5 *(i) An object* (V, f) *in* **Set-Rep**(Γ, ρ) *is projective (respectively injective, simple, indecomposable) if and only if* $F(V, f)$ *is projective (respectively injective, simple, indecomposable) in* $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$ *. (ii) A sequence* $(U, f) \to (V, g) \to (W, h)$ *in* **Set-Rep**(Γ, ρ) *is related exact if and only if the induced sequence* $F(U, f) \to F(V, g) \to F(W, h)$ *is related exact in* $P(\Gamma, \rho)$-$\mathcal{SET}^{\wr}$*.*

3 Relations between Set-Representations and Linear-Representations on A Quiver

This section consists of four parts, every semigroup mentioned contains a zero element, and the quiver Γ is finite. Here we use $\upsilon, \omega, \cdots$ stand for the vertices in the quiver Γ.

PART ONE *(Positively) graded semigroups and (positively) graded* S*-systems*

1) A semigroup S is *graded* if there exists a family of non-empty subsets $\{S_{(i)}\}_{i\in\mathbf{Z}}$, where $S_{(0)}$ is a sub-semigroup, $S = \cup_{i\in\mathbf{Z}}S_{(i)}$, $S_{(i)}S_{(j)} \subseteq S_{(i+j)}$, and $S_{(i)} \cap S_{(j)} = \{0\}$ for $i \neq j$. When $S = \cup_{i\geq 0}S_{(i)}$ is a graded semigroup, S is called *positively graded*. And a positively graded semigroup S is called *strongly graded* if $S_{(i)}S_{(j)} = S_{(i+j)}$ for any $i, j \geq 0$.

Note that, the path semigroup $P(\Gamma)$ which consists of 0 and all paths in Γ has a natural positive gradation: $P(\Gamma) = \cup_{i\geq 0}(P(\Gamma))_{(i)}$, where $(P(\Gamma))_{(i)}$ consists of 0 and all the paths whose length is i. This positive gradation of $P(\Gamma)$ is strongly graded obviously.

2) Let S be a graded semigroup, M be an S-System in S-$\mathcal{SET}^l$, if there exists a family of nonempty subsets $\{M_{(i)}\}_{i\in\mathbf{Z}}$, such that $M=\cup_{i\in\mathbf{Z}}M_{(i)}$, $S_{(i)}M_{(j)}\subseteq M_{(i+j)}$, and $M_{(i)}\cap M_{(j)}=\{\theta_M\}$ for $i\neq j$, then M is said to be *graded*. Similarly, for positively graded semigroup S, we can give the definition of positive gradation for M.

3) Let M be positively graded $P(\Gamma)$-System in $P(\Gamma)$-$\mathcal{SET}^l$, where the gradation of $P(\Gamma)$ is natural, if every homogeneous component is the union of some $M_v=e_vM$, that is, for every vertex $v\in\Gamma_0$, e_vM is contained in some a homogeneous component, then M is said to be *vertex positively graded*. Clearly, if M is positively graded and for any vertex $v\in\Gamma_0$, M_v contains at most one element except for θ_M, then M is vertex positively graded.

PART TWO *Arrow Positive Functions and Symmetric Cycles*

Definition 3.1 *(i) Function $F:\Gamma_0\longrightarrow\boldsymbol{Z}$ is called an* arrow function *on Γ, if $F(t(\alpha))=F(s(\alpha))+1$ for any arrow $\alpha\in\Gamma_1$(see [4]).*

(ii) If function $F:\Gamma_0\longrightarrow\boldsymbol{Z}^+\cup\{0\}$ is an arrow function on Γ, we call F an arrow positive function *on Γ.*

Proposition 3.1 *F is an arrow positive function on a connected quiver Γ, $G:\Gamma_0\longrightarrow\boldsymbol{Z}^+\cup\{0\}$ is another positive function, then G is an arrow positive function on Γ if and only if there exists an integer k, such that $F=G+k$.*

Proof: ($\Leftarrow$) For any arrow $\alpha\in\Gamma_1$, $G(t(\alpha))=F(t(\alpha))-k=F(s(\alpha))+1-k=G(s(\alpha))+1$.

($\Rightarrow$) Consider the function $H:\Gamma_0\longrightarrow\mathbf{Z}$, where $H=F-G$. We have known that F and G are both arrow positive functions, so $H(t(\alpha))=F(t(\alpha))-G(t(\alpha))=(F(s(\alpha))+1)-(G(s(\alpha))+1)=F(s(\alpha))-G(s(\alpha))=H(s(\alpha))$, for any arrow $\alpha\in\Gamma_1$. From the fact that Γ is connected, we have $H(v)=H(\omega)$, for any two vertices $v,\omega\in\Gamma_0$. If we let $H(v)=k$ for any $v\in\Gamma_0$, then $F=G+k$.

#

Definition 3.2 *For a non-trivial path ρ in a quiver Γ, if $s(\rho)=e(\rho)$, we say it is an* oriented cycle*. A sub-quiver Δ of a quiver Γ is said to be a* cycle *, if when omitted the direction of all arrows, the graph, which we call the base graph, is closed . In a cycle , when the number of clockwise arrows equals to the number of anti-clockwise arrows, we say the cycle is* symmetric*.*

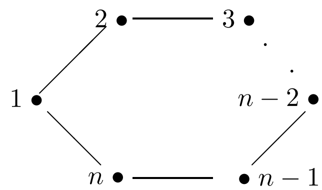
By Definition 3.1 and 3.2, when a quiver has no cycle, we can always define an arrow positive function on it. And it is clearly that, an oriented cycle is not symmetric. Indeed, we have the following conclusion:

Lemma 3.2 *A finite cycle Δ is symmetric if and only if there is an arrow positive function on Δ.*

Proof: ($\Leftarrow$) Suppose $F:\Delta_0\longrightarrow\mathbf{Z}^+\cup\{0\}$ is an arrow positive function, the base graph of Δ is like $Figure(IV)$, we consider the vertex 1 the same as $n+1$, then $F(n+1)=F(1)$. From the definition, $F(n+1)=F(1)+$ #{clockwise arrows in Δ}−#{anti-clockwise arrows in Δ }, that is, #{clockwise arrows in Δ }− #{anti-clockwise arrows in Δ }= $F(n+1)-F(1)=0$. So the cycle Δ is symmetric.

($\Rightarrow$) Suppose the base graph of Δ is like $Figure(IV)$, inductively define $F:\Delta_0\longrightarrow\mathbf{Z}^+\cup\{0\}$ as follows: $F(1)=n$, and when $m\geq 2$,

$$F(m)=F(m-1)+1,\ \text{ if }\ m-1\to m$$



$Figure(IV)$

$$F(m) = F(m-1) - 1, \text{ if } m \to m-1$$

because Δ is symmetric and $F(n+1) = F(1)$, it is easy to check that the definition above is well-defined and it is an arrow positive function on Δ.

#

The lemma below answers to the question that for what quiver, there exists an arrow positive function.

Lemma 3.3 *Suppose Γ is a finite connected quiver, if any cycle in Γ is symmetric, there must exist an arrow positive function on Γ.*

Proof: Inducing on $\mid \Gamma_0 \mid$. Clearly, the conclusion is right when $\mid \Gamma_0 \mid = 1$.

Since Γ does not contain oriented cycles, Γ_0 contains at least one source, denote by s. Let $\{v_1, v_2, \cdots, v_t\}$ denotes the set of all ending points of the arrows starting from s, throwing the source s, we get a full sub-graph of Γ, which we denote by Γ'. Suppose $\Gamma' = \cup_{i=1}^{l} \Gamma(i)$, where $\Gamma(1), \Gamma(2), \cdots, \Gamma(l)$ are all the connected components of Γ'. Since quiver Γ is connect, then $l \leq t$, and we get a partition of $\{v_1, v_2, \cdots, v_t\}$: $\{v_1, v_2, \cdots, v_t\} = S_1 \cup S_2 \cup \cdots \cup S_l$, where the union is disjoint and $S_i \subseteq \Gamma(i)_0, i = 1, \cdots, l$.

For any $\Gamma(i)$, the cycle in it is also symmetric, and by induction, on each $\Gamma(i)$ we can define an arrow positive function $F_i : \Gamma(i)_0 \longrightarrow \mathbf{Z}^+ \cup \{0\}$.

From the fact that each $\Gamma(i)$ is connected, for any two vertices $v_1, v_2 \in S_i$, there exists a path ρ from v_1 to v_2 or from v_2 to v_1, suppose that is $v_1 \cdot \xrightarrow{p} \cdot v_2$, so $s \cdot \to v_1 \cdot \xrightarrow{p} \cdot v_2 \leftarrow \cdot s$ forms a cycle in Γ, so it is symmetric and hence in the path p the number of the clockwise arrows equals to the number of the anti-clockwise arrows. We know that F_i is an arrow positive function on $\Gamma(i)$, so $F_i(v_2) = F_i(v_1) + \#\{\text{clockwise arrows in } p\} - \#\{\text{anti-clockwise arrows in } p\} = F_i(v_1)$. So for all $v \in S_i$, $F_i(v)$ is fixed.

Now we define another positive function $G_i : \Gamma(i)_0 \longrightarrow \mathbf{Z}^+ \cup \{0\}$, $i = 1, 2, \cdots, l$ as follows:

$G_i = F_i - F_i(v) + k + 1$, where $v \in S_i$, and k is a positive integer large enough such that $G_i \geq 1$ for all $i = 1, 2, \cdots, l$. Then by Proposition 3.1 G_i is also an arrow positive function on $\Gamma(i)$, and $G_i(v) = k + 1$ for any $v \in S_i$, $i = 1, 2, \cdots, l$. We know $\Gamma_0 = \{s\} \cup \{\cup_{i=1}^{l} \Gamma(i)_0\}$, define $F : \Gamma_0 \longrightarrow \mathbf{Z}^+ \cup \{0\}$ by $F(s) = k$, and $F(v) = G_i(v)$ if $v \in \Gamma(i)_0$, then $F(v_1) = F(v_2) = \cdots = F(v_t) = k + 1 = F(s) + 1$. So F is an arrow positive function on Γ.

#

By then, we get the following result:

Proposition 3.4 *Γ is a finite connected quiver, then there exists an arrow positive function on Γ if and only if it does not contain any non-symmetric cycle.*

Proof: Since the restriction of an arrow positive function on its sub-graph is also an arrow positive function, we get the theorem easily from the Lemma 3.3 and Lemma 3.2 above.

#

PART THREE *The quiver Γ on which all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are positively graded*

Lemma 3.5 *For a finite quiver Γ, if all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are positively graded, then any cycle in Γ is symmetric.*

Proof: Suppose Γ contains a cycle Δ with the base graph like $Figure(IV)$, consider a special $P(\Gamma)$-System M in $P(\Gamma)$-$\mathcal{SET}^{\wr}$, its set-representation according to the equivalence in Theorem 2.3 is (S, f), where all $S(\upsilon)$ are equal and contain only one element, the maps between them are all identity maps.

Since M is positively graded, from its special construction it is also vertex positively graded. Define a function $F : \Gamma_0 \longrightarrow \mathbf{Z}^+ \cup \{0\}$ as follows: $F(\upsilon) = i$, if $e_\upsilon M \subseteq M_{(i)}$. It is easy to know F is an arrow positive function on Γ, and so it is on Δ. Indeed, if for an arrow $\alpha : \upsilon \to \omega$, $F(\upsilon) = i$, then from the construction of M, $e_\omega M = \alpha(e_\upsilon M) \subseteq \alpha M_{(i)} \subseteq P(\Gamma)_{(1)} M_{(i)} \subseteq M_{(i+1)}$, i.e. $F(\omega) = F(\upsilon) + 1$. By Lemma 3.2, Δ is a symmetric cycle.

#

Thus, we get the main result of this section:

Theorem 3.6 *Suppose Γ is a finite connected quiver, $P(\Gamma)$ is the path semigroup consisting of zero and all paths in Γ, then the following properties are equivalent:*

(i) all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are positively graded;

(ii) any cycle in Γ is symmetric;

(iii) there exists an arrow positive function on Γ;

(iv) all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are vertex positively graded.

Proof: (i)$\Rightarrow$(ii): By Lemma 3.5.

(ii)$\Rightarrow$(iii): By Lemma 3.3.

(iii)$\Rightarrow$(iv): Suppose $F : \Gamma_0 \longrightarrow \mathbf{Z}^+ \cup \{0\}$ is an arrow positive function on quiver Γ, since for any $P(\Gamma)$-System M in $P(\Gamma)$-$\mathcal{SET}^{\wr}$, $M = \cup_{\upsilon \in \Gamma_0} e_\upsilon M$, let $M_{(i)} = \cup_{\upsilon \in \Gamma_0, F(\upsilon)=i} e_\upsilon M$, then $M = \cup_{F(\upsilon)=i, \upsilon \in \Gamma_0} M_{(i)}$ is a positive gradation. Actually, for any arrow α in Γ_1, we have $F(t(\alpha)) = F(s(\alpha)) + 1$. Then when $i \neq F(s(\alpha))$, $\alpha M_{(i)} = \{\theta_M\} \subseteq M_{(i+1)}$, since $\alpha M = \alpha e_{s(\alpha)} M$. And from the definition of $M_{(i)}$ and $\alpha M_{(F(s(\alpha)))} \subseteq e_{t(\alpha)} M \subseteq M_{(F(t(\alpha)))} = M_{(F(s(\alpha))+1)}$, we know M is vertex positively graded.

(iv)$\Rightarrow$(i): By the definition of vertex positively graded.

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Next, we give an example.

Example 3.1 *Let Γ be a quiver as $1 \stackrel{\alpha_1}{\to} 2 \stackrel{\alpha_2}{\to} 3 \stackrel{\alpha_3}{\to} \cdots \stackrel{\alpha_{n-1}}{\to} n$, then all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are positively graded.*

Indeed, if M is a $P(\Gamma)$-System of $P(\Gamma)$-$\mathcal{SET}^{\wr}$, let $M_{(i)} = e_i M$, $i = 1, 2, \cdots, n$. From the proof of Theorem 2.3, we know $M = \cup_{i=1}^n (e_i M \backslash \{\theta_M\}) \cup \{\theta_M\}$ and $(e_i M \backslash \{\theta_M\}) \cap (e_j M \backslash \{\theta_M\}) = \emptyset$ when $i \neq j$, so $M = \cup_{i=1}^n M_{(i)}$ and $M_{(i)} \cap M_{(j)} = \{\theta_M\}$ when $i \neq j$. And the inclusion $(P(\Gamma))_{(i)} M_{(j)} \subseteq M_{(i+j)}$is also easy to prove, here $M_{(i)} = \{\theta_M\}$ for any $i > n$.

PART FOUR *Relations between the two categories of set-representations and linear-representations*

At first, we cite the major theorem in [4] below:

Theorem 3.7 ([4]) *Let Γ be a finite connected quiver, $k\Gamma$ is the corresponding path algebra, then the following properties are equivalent:*
(i) all $k\Gamma$-modules are graded;
(ii) any cycle in Γ is symmetric;
(iii) there exists an arrow function on Γ;
(iv) all $k\Gamma$-modules are vertex graded.

From Theorem 3.6 and Theorem 3.7, we know that for a finite connected quiver Γ, there is an arrow function on it if and only if there is an arrow positive function on it. Since all the proofs in [4] about gradation were based on positive gradation, all conclusions about gradation in [4] can be equivalently replaced by the ones about positive gradation. Similarly, our results about positive gradation in this paper can be equivalently replaced by the ones about gradation. Then through the common statement (ii) in Theorem 3.6 and Theorem 3.7, we have a collection of equivalent statements. In particular, we have the following corollaries:

Corollary 3.8 *Let Γ be a finite connected quiver and k a field, then the following two properties are equivalent:*
(i) all $k\Gamma$-modules are (positively) graded;
(ii) all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are (positively) graded.

Corollary 3.9 *Let Γ be a finite connected quiver and k a field, then the following two properties are equivalent:*
(i) all $k\Gamma$-modules are vertex (positively) graded;
(ii) all $P(\Gamma)$-Systems in $P(\Gamma)$-$\mathcal{SET}^{\wr}$ are vertex (positively) graded.

From the two theorems above, we find that on a finite connected quiver Γ, there are some interesting relations between the two categories $P(\Gamma)$-$\mathcal{SET}^{\wr}$ and $k\Gamma$-Mod, one of which is not abelian while the other is. Since the set-representation category **Set-Rep**Γ is equivalent to the category $P(\Gamma)$-$\mathcal{SET}^{\wr}$, and the linear-representation category **Lin-Rep**Γ is equivalent to the category $k\Gamma$-Mod, there are also some similar relations between the two representation categories **Set-Rep**Γ and **Lin-Rep**Γ

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