SUPER COMMUTATIVE D-ALGEBRAS AND BCK-ALGEBRAS IN THE SMARANDACHE SETTING

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Received March 8, 2005

Abstract. In this paper we introduce the notion of a super commutative d-algebra and we show that if $(X;*)$ is a commutative binary system, then by adjoining an element 0 and adjusting the multiplication to $x*x=0$, we obtain a super commutative d-algebra, thereby demonstrating that the class of such algebras is very large. We also note that the class of super commutative d-algebras is Smarandache disjoint from the class of BCK-algebras, once more indicating that the class of d-algebras is quite a bit larger than the class of BCK-algeras and leaving the problem of finding further classes of d-algebras of special types which are Smarandache disjoint from the classes of BCK-algebras and super commutative d-algebras as an open question. Lastly the idea of a super Smarandache class of algebras is also defined and investigated.

1 Introduction. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([7]) introduced a new notion, called a BH-algebra, i.e., (I), (II) and (V) $x*y=0$ and $y*x=0$ imply $x=y$, which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim ([10]) introduced and investigated a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, they demonstrated a rather interesting connection between B-algebras and groups. P. J. Allen et al. ([1]) included several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. In this paper we introduce the notion of a super commutative d-algebra and we show that if $(X;*)$ is a commutative binary system, then by adjoining an element 0 and adjusting the multiplication to $x*x=0$, we obtain a super commutative d-algebra, thereby demonstrating that the class of such algebras is very large. We also note that the class of super commutative d-algebras is Smarandache disjoint from the class of BCK-algebras, once more indicating that the class of d-algebras is quite a bit larger than the class of BCK-algeras and leaving the problem of finding further

2000 *Mathematics Subject Classification.* 06F35, 08C99.
Key words and phrases. BCK/d-algebras, Smarandache-type, Smarandache disjoint, super commutative.

classes of d-algebras of special types which are Smarandache disjoint from the classes of BCK-algebras and super commutative d-algebras as an open question. Lastly the idea of a super Smarandache class of algebras is also defined and investigated.

2 Main results. A *d-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$.

(III) $x * y = y * x = 0$ implies $x = y$,

for any $x, y \in X$.

A d-algebra X is said to be *super commutative* if for any non-zero $x, y \in X$, $x \neq y$, $x * y = y * x \neq 0$. Notice that for $BCK/BCI/d$-algebras, the commutativity means $x * (x * y) = y * (y * x)$ for any $x, y \in X$.

Example 2.1. Let $X := \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	0	0
1	1	0	2
2	1	2	0

Then X is a d-algebra, but not a BCK-algebra, since $(2 * (2 * 0)) * 0 = (2 * 1) * 0 = 2 * 0 = 1 \neq 0$. It is easy to check that X is super commutative, but not commutative, since $1 * (1 * 2) = 2 \neq 0 = 2 * (2 * 1)$.

Construction of super commutative d-algebras. Let X be a non-empty set and $0 \notin X$. Let $S = X \cup \{0\}$. Define a binary operation "$*$" on S as follows:

(i) $x * x = 0 = 0 * x, \forall x \in S$;

(ii) $x * 0 \in X, \forall x \in X$;

(iii) $x * y = y * x \in X$, for any $x \neq y \in X$.

Then $(S; *, 0)$ is a super commutative d-algebra.

Note that every d-algebra X, $|X| \leq 2$, is obviously super commutative. A super commutative d-algebra $(X; *, 0)$ is said to be *non-trivial* if $|X| \geq 3$, since in that case, there is an $x, y \neq 0, x \neq y$ pair so that $x * y = y * x \neq 0$.

Theorem 2.2. *If $(X; *, 0)$ is a BCK-algebra, then it can not contain a non-trivial super commutative d-algebra $(A; *, 0)$.*

Proof. Assume that X contains a non-trivial super commutative d-algebra A. Then there exist $x, y \in X - \{0\}$ such that $x \neq y$, $x * y = y * x \neq 0$. We claim that $x \neq x * y$. If $x = x * y$, then $x = x * y = y * x \neq 0$. Since X is a BCK-algebra, $0 = (y * x) * y = x * y = x$, a contradiction. Hence $x \neq x * y$. Since A is super commutative, we obtain $(x * y) * x = x * (x * y) \neq 0$, which is a contradiction to the fact that $(x * y) * x = 0$ in the BCK-algebra X. □

Theorem 2.3. *If $(X;*,0)$ is a non-trivial super commutative d-algebra, then it cannot contain a BCK-algebra $(A;*,0)$ with $|A| \geq 3$.*

Proof. Note that if $x \in X$, then $x*x = 0*x = 0$ and if $x*0 = x$, then $\{x,0\}$ is a two-element BCK-algebra. Thus, arbitrary d-algebras may easily contain two element subalgebras which are BCK-algebras. Assume that $(A;*,0)$ is a BCK-algebra with $|A| \geq 3$. Let $x, y \in A - \{0\}$ with $x \neq y$. Then $x*y = y*x \neq 0$, since X is super commutative. We claim that $x = x*y, y = y*x$. Suppose that $x \neq x*y$. Then $x*(x*y) = (x*y)*x \neq 0$, since X is super commutative. Since A is a BCK-algebra, we have $(x*y)*x = 0$. Thus $0 = (x*y)*x \neq 0$, a contradiction. Similarly, $y = y*x$. Since X is super commutative, $x = x*y = y*x = y$, a contradiction. Thus X cannot contain a BCK-algebra $(A;*,0)$ with $|A| \geq 3$. □

Let $(X,*)$ be a binary system/algebra. Then $(X,*)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y,*)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y,*)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [2]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X,*)$ (type-P_1) and $(X,\circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

(A) If $(X,*)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X,\circ)$;

(B) If $(X,\circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X,*)$.

A BCK-algebra $(X;*,0)$ is said to be *strict* if $|X| \geq 3$. Putting Theorem 2.2 and 2.3 together we obtain the following conclusion:

Theorem 2.4. *The class of strict BCK-algebras and the class of non-trivial super commutative d-algebras are Smarandache disjoint.*

Another construction of super commutative d-algebras. Consider the set $\underline{n} := \{0,1,2,\cdots,n\}$ and define a product $*$ by $0*0 = 0*k = 0$; $k*0 = k$; $i*j = j*i = |i-j|$. Then $(\underline{n} : *, 0)$ is a super commutative d-algebra. For example, if $n = 2$, then we have a table:

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	1	0

which is another non-trivial super commutative d-algebra of order 3, not isomorphic to Example 2.1.

A class $\{(X_i,*)\}$ of algebras is said to be *super Smarandache* if it contains subclasses $\{(A_i,*)\}$ and $\{(B_i,*)\}$ such that these algebras are Smarandache disjoint. As an example, the class of groups and the class of left semigroups are both classes of semigroups which we have shown to be Smarandache disjoint.If we take the class of cyclic groups of prime power order then it cannot have Smarandache disjoint subclasses,essentially because it has an index function of the right kind.

Theorem 2.5. *The class* $\mathbf{N} := \{\underline{n} \,|\, n = 2, 3, \cdots\}$ *of non-trivial super commutative d-algebras is not a super Smarandache.*

Proof. If $\mathbf{N}$ is a super Smarandache class of algebras and if $\{(A_i, *)\}$ and $\{(B_j, *)\}$ are Smarandache disjoint subclasses of $\mathbf{N}$, then $(A_{i_0}, *) = \underline{a}$ and $(B_{j_0}, *) = \underline{b}$ means that $\underline{a}$ is isomorphic to a subalgebra of $\underline{b}$ if $\underline{a} \leq \underline{b}$ or conversely, if $\underline{b} \leq \underline{a}$. Since $n \geq 2$ for all $\underline{n} \in \mathbf{N}$, we obtain a contradiction and the theorem follows. □

Remark. The proof of Theorem 2.5 admits of considerable generalization. Thus, let $L = \{(S_i, *)\}_{i \in I}$ be a class of algebras with the property that if $(S_{i_1}, *)$ and $(S_{i_2}, *)$ are elements of L, then there is an $(S_{i_3}, *)$ in L such that $| S_{I_3} | \geq 2$ and $(S_{i_3}, *) \cong (A, *) \cong (B, *)$ with $(A, *)$ a subalgebra of $(S_{i_1}, *)$ and $(B, *)$ a subalgebra of $(S_{i_2}, *)$. It is sufficient to show that the proof of Theorem 2.5 continues to hold and thus that the class L is once more not a super Smarandache class. By this observation the class of finite fields of characteristic p (a fixed prime) is not a super Smarandache class for example.

Now, consider $\widehat{n} := \{0, 1, 2, \cdots, n\}$ with the product $*$ defined by $0 * 0 = 0 * k = 0$, and $i * j := \max\{0, i - j\}$ for any $i, j \in \widehat{n}$. Then $(\widehat{n}; *, 0)$ is a BCK-algebra generated by the order $i * j = 0$ iff $i \leq j$, with a counting of the length of intervals added. We have for $\widehat{3}$ a table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

If $m \leq n$, then $\widehat{m}$ is a subalgebra of $\widehat{n}$.

Theorem 2.6. *The class of BCK-algebras* $\{\widehat{n} \,|\, n = 2, 3, \cdots\}$ *is not a super Smarandache.*

Proof. The proof is the same as Theorem 2.5, we omit the proof. □

Remark. The two families, viz., $\{\underline{n} \,|\, n = 2, 3, \cdots\}$ and $\{\widehat{n} \,|\, n = 2, 3, \cdots\}$ give two different ways to describe the natural numbers ≥ 2, one of them the super commutative d-algebra way (cardinal d-algebras) and one of them the BCK-algebra way (ordinal BCK/d-algebras. From Theorem 2.5 we have already seen that these classes are Smarandache disjoint. They therefore represent very different ways to look at the natural numbers.

Proposition 2.7. *If we define* $(a, b) \oplus (c, d) := (a*c, b*d)$ *on* $X := \underline{m} \times \widehat{n}$*, then* $(X; \oplus, (0, 0))$ *is a d-algebra.*

Proof. Straightforward. □

References

[1] P. J. Allen, H. S. Kim and J. Neggers, *Smarandache disjoint in BCK/d-algebras,* Sci. Math. Japo. Online **e-2004** (2004), 485-487.

[2] Jung R. Cho and H. S. Kim, *On B-algebras and quasigroups*, Quasigroups and Related systems **8** (2001), 1–6.

[3] Qing Ping Hu and Xin Li, *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313-320.

[4] Qing Ping Hu and Xin Li, *On proper BCH-algebras* , Math. Japonica **30** (1985), 659–661.

[5] K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras* , Math. Japnica **23** (1978), 1–26.

[6] K.Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125–130.

[7] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematicae **1** (1998), 347–354.

[8] J. Meng and Y. B. Jun, BCK-algebras, Kyung Moon Sa Co., Seoul, 1994.

[9] J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca **49** (1999), 19–26.

[10] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54** (2002), 21–29.

[11] W. B. V. Kandasamy, *Smarandache semirings, semifield, and semivector spaces* American Research Press, Rehoboth, 2002.

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