HEREDITY OF τ-PSEUDOCOMPACTNESS

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Received May 1, 2005; revised May 12, 2005

Abstract. S. García-Ferreira and H. Ohta gave a construction that was intended to produce a τ-pseudocompact space, which has a regular-closed zero set A and a regular-closed C-embedded set B such that neither A nor B is τ-pseudocompact. We show that although their sets A, B are not regular-closed, there are at least two ways to make their construction work to give the desired example.

1 Introduction All spaces considered in this paper are Tychonoff, i.e., $T_{3\frac{1}{2}}$-spaces. Let $\tau \geq \omega$ denote an infinite cardinal number, and $\mathbb{R}^\tau$ the product of τ copies of the real line with the product topology. J. F. Kennison defined a space X to be *τ-pseudocompact* provided for every continuous $f : X \to \mathbb{R}^\tau$, $f(X)$ is a closed subset of $\mathbb{R}^\tau$ [7]. He proved that a space X is τ-pseudocompact if and only if whenever $\mathcal{F}$ is a family of zero sets of X with the finite intersection property (FIP) and $|\mathcal{F}| \leq \tau$, then $\cap\mathcal{F} \neq \emptyset$ [7, Theorem 2.2]. It is known and easy to prove that ω-pseudocompactness is equivalent to the well-known notion of pseudocompactness (e.g., see [7, Theorem 2.1]).

Recall that a subset H of a topological space is called *regular-closed* if H is the closure of an open set. H is called a *zero set* provided there exists a continuous $f : X \to [0, 1]$ such that $H = f^{-1}(0)$, and H is called *C-embedded in X* if for every continuous $f : H \to \mathbb{R}$, there is a continuous $g : X \to \mathbb{R}$ such that g extends f. A set $Y \subset X$ is said to be *countably compact in X* if every infinite subset of Y has a limit point in X [4]

There are several known examples that show τ-pseudocompactness is not hereditary to various kinds of closed sets. Kennison showed that τ-pseudocompactness is not hereditary to closed C-embedded sets [7, p.440]. T. Retta showed (for $\tau \geq \mathfrak{c}$) that τ-pseudocompactness is not hereditary to regular-closed subsets [8], and a different construction to show the same thing was given by S. García-Ferreira, M. Sanchis, and S. Watson [5, Corollary 1.4], assuming $cf(\tau) > 2^{\mathfrak{c}}$. These examples demonstrate a difference between the countable and uncountable cases: pseudocompactness (i.e., ω-pseudocompactness) is hereditary to C-embedded subsets and to regular-closed sets (e.g., see [6, 9.13]), but for $\tau \geq \mathfrak{c}$, τ-pseudocompactness is not necessarily hereditary to either kind of closed set. Concerning cardinals not covered by the previous examples, H. Ohta (see [5]) constructed an example to show that ω_1-pseudocompactness is not hereditary to regular-closed sets. García-Ferreira and Ohta [4, Example 2.4] generalized this construction to all uncountable cardinals. They stated the following

Example 1.1 (García-Ferreira and Ohta) *For all $\tau \geq \omega_1$, there exists a τ-pseudocompact space X with two regular-closed sets A, B such that A is a zero set, B is C-embedded, and neither is τ-pseudocompact.*

2000 *Mathematics Subject Classification.* Primary 54B05, 54D80, 54G20, 54F05.

Key words and phrases. τ-pseudocompact, zero set, C-embedded set, lexicographic product, long line, Alexandroff duplicate.

There is, however, a small gap in the constructions of Ohta in [5] and of García-Ferreira and Ohta in [4]. The purpose of this paper is to show in §2 that the sets A and B that they claim in [4] to be regular-closed are not, and to show in §3 that a simple modification of their construction suffices to prove Example 1.1. The modification is to replace the cardinals τ^+ and ω_1 in the García-Ferreira and Ohta construction with their long line counterparts. Possibly the previous sentence is sufficient for our main goal of establishing Example 1.1, but we elaborate a bit more on this in §2. In §3 we present another way to modify their construction and give a different, possibly simpler, proof of Example 1.1.

García-Ferreira and Ohta also proved that τ-pseudocompactness is hereditary to any subset that is both a zero set and a C-embedded set (regular-closed or not) [4, Theorem. 1.4]. Thus Example 1.1 seems to be about as strong as possible, and is therefore an important example in the theory of τ-pseudocompactness.

2 The Construction of García-Ferreira and Ohta First we recall the Alexandroff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times 2$, where $2 = \{0,1\}$. In the topology of $A(X)$, each point of $X \times \{1\}$ is isolated, and each point $(x,0) \in X \times \{0\}$ has basic open neighborhoods of the form $U \times 2 \setminus \{(x,1)\}$, where U is an open neighborhood of x in X (see [3]). Let $Y \subset X$. The space $A(X,Y)$ is defined to be the set $(X \times \{0\}) \cup (Y \times \{1\})$ with the subspace topology from $A(X)$ [4, §2].

Now we recall the construction of García-Ferreira and Ohta [4, Example 2.4]. Let $\tau \geq \omega_1$ be an infinite cardinal, and τ^+ the first cardinal larger than τ. As is well known, the spaces τ^+ and ω_1 with the order topology satisfy the following properties:

(1) τ^+ is initially τ-compact (i.e., every open cover of cardinality at most τ has a finite subcover [1]) and ω_1 is initially ω-compact (i.e., countably compact).

(2) every real-valued continuous function defined on τ^+ or ω_1 is eventually constant.

Let $S_1 = (\tau^+ + 1) \times (\omega + 1)$, and $S_2 = (\omega_1 + 1) \times (\omega + 1)$. Next consider the quotient of the disjoint union $S_1 \oplus S_2$ obtained by identifying (τ^+, n) and (ω_1, n) for every $n \in \omega$. Let φ denote the quotient map from $S_1 \oplus S_2$ onto the quotient space. Then let X denote the quotient space minus the point $\varphi((\tau^+, \omega)) = \varphi((\omega_1, \omega))$. Let $Y_1 = \tau^+ \times \{\omega\} \subset S_1$, $Y_2 = \omega_1 \times \{\omega\} \subset S_2$, and $Y = \varphi(Y_1 \cup Y_2)$, and $Z = \varphi(Y_2)$.

The space for Example 1.1 given by García-Ferreira and Ohta is $A(X,Y)$ where X, Y were defined in the previous paragraph, and the two subsets are $A = Y \times 2$ and $B = Z \times 2$.

A gap in the proof by García-Ferreira and Ohta occurs because neither of $A = Y \times 2$ or $B = Z \times 2$ is a regular-closed set. To see this let $int_X(H)$ denote the interior of H in X, and note the following fact: For any space X, if H is closed in X and there is a point $p \in H$ such that $p \notin int_X(H)$ and p is relatively isolated in H, then H is not regular-closed. For the sets $A = Y \times 2$ and $B = Z \times 2$, take any isolated ordinal $\alpha < \omega_1$ and put $p = (\varphi(\alpha), 0)$. Then p is relatively isolated in A and B, hence A, B are not regular-closed.

The following Lemma indicates a way to repair this gap.

Lemma 2.1 *If $Y \subset X$ is dense-in-itself, then $Y \times 2$ is regular-closed in $A(X,Y)$.*

Proof. We claim that $Y \times 2 = cl_{A(X,Y)}(Y \times \{1\})$. Since $Y \times 2$ is closed in $A(X,Y)$, we need only show that $Y \times \{0\} \subset cl_{A(X,Y)}(Y \times \{1\})$. For any $y \in Y$, and any neighborhood U of y in X, $(U \times 2) \setminus \{(y,1)\}$ is a basic neighborhood of the point $(y,0)$ in $A(X)$. Since Y is dense-in-itself there is $z \in U \cap Y$ such that $z \neq y$. Then $(z,1) \in (U \times 2) \setminus \{(y,1)\}$ which shows that $(y,0) \in cl_{A(X,Y)}(Y \times \{1\})$.

3 The First Modification To repair Example 1.1 we start over the construction of García-Ferreira and Ohta, but this time we use the long line counterparts of the cardinals τ^+, and ω_1. The following lemmas indicate that the counterparts have the key properties needed in the construction, and since each of these counterparts is dense-in-itself (in fact, connected), Lemma 2.1 fixes the gap and Example 1.1 follows.

Notation: Fix an uncountable cardinal τ. Let $T = \tau^+ \times_{lex} [0,1)$ and $W = \omega_1 \times_{lex} [0,1)$ where the products are given the lexicographic order and the order topology.

Lemma 3.1 *W is countably compact, and T is initially τ-compact.*

Lemma 3.2 *(cf. [6, 16H]) Every real-valued continuous function defined on W or T is eventually constant.*

To get counterparts to $\tau^+ + 1$ and $\omega_1 + 1$, let $W + 1 = W \cup \{w\}$ and $T + 1 = T \cup \{t\}$, where w, t are points not in $T \cup W$. Extend the order of W and T so that w acts as the last element of W and t acts as the last element of T. Let $S_1 = (T + 1) \times (\omega + 1)$, and $S_2 = (W + 1) \times (\omega + 1)$. Next consider the quotient of the disjoint union $S_1 \oplus S_2$ obtained by identifying (t, n) and (w, n) for every $n \in \omega$. Let φ denote the quotient map from $S_1 \oplus S_2$ onto the quotient space. Then let X denote the quotient space minus the point $\varphi((t,\omega)) = \varphi((w,\omega))$. Let $Y_1 = T \times \{\omega\} \subset S_1$, $Y_2 = W \times \{\omega\} \subset S_2$, and $Y = \varphi(Y_1 \cup Y_2)$, and $Z = \varphi(Y_2)$.

The space for Example 1.1 is $A(X, Y)$ where X, Y were defined in the previous paragraph, and the two subsets are $A = Y \times 2$ and $B = Z \times 2$. Since T and W are connected (e.g., see [6, 16H]), each of the sets Y and Z is dense-in-itself, so by Lemma 2.1 they are regular-closed in $A(X, Y)$. The other properties required in Example 1.1 follow as in [4].

4 Another Modification In this section we present another modification of the construction of García-Ferreira and Ohta, suggested to us by Alan Dow, which gives a second proof of Example 1.1. This modification does not use lexicographic products. First we formalize a variation of the Alexandroff duplicate construction, which is probably not new.

Let $C = \{\frac{1}{n} : n \geq 1\} \cup \{0\}$ denote the usual convergent sequence. Let X be a space and put $M(X) = X \times C$. Define a topology on $M(X)$ as follows. All points of the form $(x, \frac{1}{n})$ for $n \geq 1$ are isolated, and basic neighborhoods for a point $(x, 0)$ are defined to be sets of the form $(U \times C) \setminus F$ where U is an open neighborhood of x in X and F is a finite set. It is routine to check that this topology on $M(X)$ is $T_{3\frac{1}{2}}$.

For $Y \subset X$, we define $M(X, Y) = (X \times \{0\}) \cup (Y \times \{\frac{1}{n} : n \geq 1\})$ with the subspace topology from $M(X)$. Note that $M(X) = M(X, X)$. Let π denote the projection map $\pi : M(X) \to X$ defined by $\pi(x, e) = x$ for all $e \in C$. By abuse of notation we also let π denote the restriction of this projection map to $M(X, Y)$.

Lemma 4.1 *If Y is a zero set of X, then $Y \times C$ is a zero set of $M(X, Y)$.*

Proof. This follows because the projection map π is continuous.

Lemma 4.2 *If $Z \subset Y$ is C-embedded in X then $Z \times C$ is C-embedded in $M(X, Y)$.*

Proof. Given a continuous function $f : (Z \times C) \to \mathbb{R}$, we may continuously extend $f \restriction (Z \times \{0\})$ to all of $X \times \{0\}$ because Z is C-embedded in X; so we may assume f is defined on $X \times \{0\} \cup Z \times C$. Then define $g : M(X, Y) \to \mathbb{R}$ by

$$g(p) = \begin{cases} f(p) & \text{if } p \in X \times \{0\} \cup Y \times C \\ f((y, 0)) & \text{if } p = (y, e) \text{ and } y \in Y \setminus Z \end{cases}$$

The function g is continuous by a standard gluing lemma, and extends f to $M(X, Y)$.

The next result is an analog of [4, Lemma 2.2].

Lemma 4.3 *M(X,Y) is τ-pseudocompact if and only if X is τ-pseudocompact and Y is countably compact in X.*

Proof. Assume that X is τ-pseudocompact and Y is countably compact in X. Let $\{f_\beta^{-1}(0) : \beta < \tau\}$ be a family of zero sets of $M(X, Y)$ with the FIP. If this family traces on $X \times \{0\}$, then the intersection is non-empty because X is τ-pseudocompact. Thus we assume there is $\alpha < \tau$ such that $f_\alpha^{-1}(0) \subset Y \times \{\frac{1}{n} : n \geq 1\}$. Note that if $H \subset Y \times \{\frac{1}{n} : n \geq 1\}$ and H is closed in $M(X, Y)$ then H is finite. This is because either $\{y \in Y : (\exists n \geq 1)((y, \frac{1}{n}) \in H)\}$ is infinite, hence has a limit point $x \in X$ which implies $(x, 0) \in cl_{M(X,Y)}(H) = H$, or there is a $y \in Y$ such that $(y, \frac{1}{n}) \in H$ for infinitely many n, hence $(y, 0) \in cl_{M(X,Y)}(H) = H$, which is again a contradiction. Thus $f_\alpha^{-1}(0)$ is finite, hence compact; so one of the points in $f_\alpha^{-1}(0)$ is in $\cap\{f_\beta^{-1}(0) : \beta < \tau\}$. Thus $M(X, Y)$ is τ-pseudocompact.

Conversely, suppose $M(X, Y)$ is τ-pseudocompact. Let $\mathcal{F} = \{f_\beta^{-1}(0) : \beta < \tau\}$ be a family of zero sets of X with the FIP. Since the projection map π is continuous, the maps $g_\alpha = f_\alpha \circ \pi$ are continuous on $M(X)$ for all $\alpha < \tau$. Thus $\{g_\beta^{-1}(0) : \beta < \tau\}$ is a family of zero sets on $M(X)$. Since

$$f_\alpha^{-1}(0) \times \{0\} \subset g_\alpha^{-1}(0) \cap (X \times \{0\}) \subset M(X, Y),$$

$\mathcal{G} = \{g^{-1}(0) \cap M(X, Y) : \alpha < \tau\}$ has the FIP. By assumption, there exists $p \in \cap\mathcal{G}$; so $\pi(p) \in \cap\mathcal{F}$. Thus X is τ-pseudocompact. To see that Y is countably compact in X, suppose otherwise, i.e., suppose there is an infinite subset of Y that has no limit points in X. Then there is an infinite subset of $Y \times \{1\}$ that forms a closed discrete set of isolated points, but this is impossible because $M(X, Y)$ is pseudocompact.

To complete the construction, let X, Y, Z be the space and subsets defined in §2 and put $A = Y \times C$, and $B = Z \times C$. Clearly B is clopen in A. Since Y is a zero set in X, $A = Y \times C$ is a zero set in $M(X, Y)$. Further $A = cl_{M(X,Y)}(Y \times \{\frac{1}{n} : n \geq 1\})$; so A is a regular-closed set (although Y is not dense-in-itself). Similarly, B is regular-closed.

To complete our second proof of Example 1.1, we use the next lemma which follows the method of García-Ferreira and Ohta.

Lemma 4.4 *A and B are not ω_1-pseudocompact.*

Proof. Since B is a clopen subset of A, it suffices to show that B is not ω_1-pseudocompact. Now $B = Z \times C = M(Z) = M(Z, Z)$. Since Z is a copy of ω_1, Z contains a decreasing family of ω_1 many clopen sets with empty intersection; so Z is not ω_1-pseudocompact. By Lemma 4.3, $M(Z) = B$ is not ω_1-pseudocompact, hence A is not.

We thank Alan Dow for suggesting the space discussed in §4 and for helpful remarks concerning it.

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