THE TOPOLOGICAL CENTER OF $L^1(K)^{**}$

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Abstract. Let K be a (commutative) locally compact hypergroup with a left Haar measure and $L^1(K)$ be the hypergroup algebra of K. In this paper we show that the topological (algebraic) center of the Banach algebra $L^1(K)^{**}$ is $L^1(K)$.

1 Introduction The theory of hypergroups was initiated by Dunkl [4], Jewett [7] and Spector [17] in the early 1970's and has received a good deal of attention from harmonic analysts (note that Jewett calls hypergroups "convos" in his paper [7]). In [12], Pym also considers convolution structures which are close to hypergroups. A fairly complete history is given in Ross's survey article [13] (see also [14]). Hypergroups arise in a natural way as a double coset space, and the space of conjugacy classes of a compact group ([13] and [1]). In particular, locally compact groups are hypergroups. Here we follow Jewett [7]. It is still unknown if an arbitrary hypergroup admits a left Haar measure but all the known examples do [7, §5].
Throughout, K will denote a hypergroup with a left Haar measure λ.
Let $L^1(K)$ denotes the hypergroup algebra of K i.e. all Borel measurable functions ϕ on K with $\|\phi\|_1 = \int_K |\phi(x)|\, d\lambda(x) < \infty$ (with functions equal almost everywhere identified), and the multiplication defined by

$$\phi * \psi(x) = \int_K \phi(x * y)\psi(\check{y})\, d\lambda(y) \qquad \text{(see [7, §5.5])}.$$

Let the second dual $L^1(K)^{**}$ $(= L^\infty(K)^*)$ of $L^1(K)$ be equipped with the first Arens product [3]. Then $L^1(K)^{**}$ is a Banach algebra with this product. The *topological center* of $L^1(K)^{**}$ is defined by

$$Z(L^1(K)^{**}) = \{m \in L^1(K)^{**} : \text{the mapping } n \longmapsto mn \text{ is } w^* - \text{continuous on } L^1(K)^{**}\}.$$

In [9], Lau and Losert have shown that the topological center of $L^1(G)^{**}$ is $L^1(G)$ where G is a locally compact group (see also [2], [10]).

Note that when K is commutative, then $Z(L^1(K)^{**})$ is precisely the algebraic center of $L^1(K)^{**}$.

This paper is organized as follows:
Section 2 consists of some notation and preliminary results that we need in the sequel. The technical Lemma 2.7 in this section plays a key role in proving our main results (Theorem 3.5).

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In section 3 we show that $Z(L^1(K)^{**}) = L^1(K)$ (Theorem 3.5).
This paper will form part of the author's Ph.D thesis under the guideness of Professor Anthony T. Lau, to whom he wishes to express his deepest gratitude for the valuable suggestions and constant encouragement.

2 Preliminaries and some technical lemmas The notation used in this paper is that of [7] with the following exceptions:
$x \to \check{x}$ denotes the involoution on the hypergroup K, δ_x the Dirac measure concentrated at x ($x \in K$), and 1_X the characteristic function of the non-empty set $X \subseteq K$. For $C \subseteq K$ and $y \in K$, let $C * y$ denote the subset $C * \{y\}$ in K.

Lemma 2.1 *Let K be a locally compact non-compact hypergroup. Then there exists a family $\{C_i : i \in I\}$ of compact subsets of K, indexed by I, $y_i, z_i \in K$, $i \in I$ such that ${C_i}^\circ$ (the interior of C_i) is non-empty, $\cup_{i\in I}{C_i}^\circ = K$, $\{C_i : i \in I\}$ is closed under finite unions, and*

(a) the families $\{C_i * y_i : i \in I\}$ and $\{C_i * z_i : i \in I\}$ are pairwise disjoint.

(b) $C_i * y_i * \check{y}_j \cap C_p * z_p * \check{z}_q = \emptyset$, $i \neq j$ and $p \neq q$, $i, j, p, q \in I$.

Proof: Let $\{C_i : i \in I\}$ be a family of compact subsets of K with ${C_i}^\circ$ nonempty, $K = \cup_{i\in I}{C_i}^\circ$ and that the index set I has minimal cardinality among all such families. By taking finite union of such sets, we may assume that $\{C_i : i \in I\}$ is closed under finite unions. We may also assume that I is well ordered in such a way that each nontrivial order segment $\{i \in I : i \leq j\}$, $j \in I$, of I has smaller cardinality than I. We now proceed with the selection of y_i, z_i, $i \in I$ by transfinite induction. Assume that y_j, z_j have been selected for $j < i$. Then y_i has to meet the following requirements:
from (a) for $p < i$

$$C_i * y_i \cap C_p * y_p = \emptyset,$$

from (b) for any $p \neq q$, $j < i$

$$((C_i * y_i) * \check{y}_j) \cap ((C_p * z_p) * \check{z}_q) = \emptyset,$$

and (by changing i with j in (b))

$$((C_j * y_j) * \check{y}_i) \cap ((C_p * z_p) * \check{z}_q) = \emptyset.$$

Now by using 4.1B in [7] they are equivalent to

$$\begin{aligned} y_i &\notin \check{C}_i * (C_p * y_p) && (1)\\ y_i &\notin (\check{C}_i * (((C_p * z_p) * \check{z}_q) * y_j)) && (2)\\ y_i &\notin ((C_p * z_p) * \check{z}_q)\check{} * (C_j * y_j) \text{ for } j, p, q < i \text{ and } p \neq q && (3) \end{aligned}$$

respectively.
Such choice of y_i is possible, since the collection of compact sets (see [7, 3.2B]) on the right hand side of (1), (2) and (3) do not cover K, by minimality of I. Similarly, we can find z_i such that

$$\begin{aligned} z_i &\notin \check{C}_i * (C_p * z_p) && p < i,\\ z_i &\notin \check{C}_i * (((C_p * y_p) * \check{y}_q) * z_j) &&\\ \text{and} \quad z_i &\notin ((C_p * y_p) * \check{y}_q)\check{} * (C_j * z_j) && \text{for} \quad j, p, q < i \text{ and } p \neq q. \end{aligned}$$

Now by transfinite induction the proof is complete. □

For a Borel function f on K and $x \in K$, ${}_xf$ denotes the left translation

$$ {}_xf(y) = f(x * y) = \int_K f(t)\ d\delta_x * \delta_y(t), $$

and f_x the right translation

$$ f_x(y) = f(y * x) = \int_K f(t)\ d\delta_y * \delta_x(t), $$

if the integrals exist. We write ${}_{x*y}f$ and f_{x*y} for ${}_y({}_xf)$ and $(f_y)_x$ respectively.
The function $\check{f}$ is given by $\check{f}(x) = f(\check{x})$. The integral $\int \ldots d\lambda(x)$ is often denoted by $\int \ldots dx$. Let $(L^P(K), \|.\|_p), 1 \leq p \leq \infty$, denote the usual Banach spaces of Borel functions on K [7, §6.2]. Then $L^\infty(K)$ is a commutative Banach algebra with pointwise multiplication and the essential supremum norm $\| \ . \ \|_\infty$, $L^\infty(K) = L^1(K)^*$ [7, §6.2.]. We say that $X \subseteq L^\infty(K)$ is *translation invariant* if ${}_xf \in X$ and $f_x \in X$ for $f \in X$, $x \in K$; also X is *topologically translation invariant* if $\phi * f \in X$ and $f * \check{\phi} \in X$ for $f \in X, \phi \in P^1(K) = \{\phi \in L^1(K) : \phi \geq 0, \| \ \phi \ \|_1 = 1\}$.
In addition, we make use of the following abbreviations:

$C_{00}(K)$: the set of continuous functions with compact support on K.
$C(K)$: the set of bounded continuous functions on K.
$UC_r(K) = \{f \in C(K) : x \longmapsto {}_xf$ is continuous from K into $(C(K), \| \ . \ \|_\infty)\}$.
$UC_l(K) = \{f \in C(K) : x \longmapsto f_x$ is continuous from K into $(C(K), \| \ . \ \|_\infty)\}$.

It is known that

$UC_r(K) = \{f \in C(K) : x \longmapsto {}_xf$ is continuous from K into $(C(K))$ with the weak-topology$\}$

[16, Theorem 4.2.2, p 88].
Each of the spaces $UC_r(K)$ and $UC_l(K)$ is a normed closed, conjugate closed, translation invariant and topologically translation invariant subspace of $C(K)$ containing the constant functions and $C_0(K)$ [15, Lemma 2.2.]. Furthermore

(i) $UC_r(K) = L^1(K) * UC_r(K) = L^1(K) * L^\infty(K)$

(ii) $UC_l(K) = UC_l(K) * L^1(K)\check{} = L^\infty(K) * L^1(K)\check{}$ [15, Lemma 2.2.].

Note that $UC_r(K)$ is in general not an algebra [15, Remark 2.3(b)].
For $\phi \in L^1(K)$, we write $\tilde{\phi}(x) = \Delta(\check{x})\phi(\check{x})$ where Δ is the modular function on K; then $\|\tilde{\phi}\| = \|\phi\|_1$. If $f \in L^P(K), 1 \leq p \leq \infty, x \in K$, then $\| {}_xf\|_p \leq \|f\|$, and this is in general not an isometry [7, §3.3]. The mapping $x \longmapsto {}_xf$ is continuous from K to $(L^P(K), \|.\|_p), 1 \leq p < \infty$, [7, 2.2B and 5.4H].
It is easy to show that $L^1(K)$ has a bounded approximate identity(B.A.I) $\{e_i : i \in I\} \subseteq C_{00}^+(K)$ such that $\|e_i\| = 1$ (see [15, Lemma 2.1]).
For any Banach space X, we denote by X^* and X^{**} its first and second dual. Let A be a Banach algebra. For any $f \in A^*$ and $a \in A$, we may define a linear functional fa on A by $\langle fa, b\rangle = \langle f, ab\rangle, (b \in A)$. One can check that $fa \in A^*$ and $\|fa\| \leq \|f\|\|a\|$. Now for $n \in A^{**}$, we may define $nf \in A^*$ by $\langle nf, a\rangle = \langle n, fa\rangle$; clearly we have $\|nf\| \leq \|n\|\|f\|$. Next for $m \in A^{**}$, define $mn \in A^{**}$ by $\langle mn, f\rangle = \langle m, nf\rangle$. We have $\|mn\| \leq \|m\|\|n\|$, and

A^{**} becomes a Banach algebra with the multiplication mn, just defined, referred to as the first *Arens product*, versus another multiplication on A^{**} called the second Arens product, which is denoted by $m \circ n$ and defined successively as follows:

$\langle m \circ n, f\rangle = \langle n, fm\rangle$, in which $\langle fm, a\rangle = \langle m, af\rangle$, where $\langle af, b\rangle = \langle f, ba\rangle$, herein m, n, f, a, and b are taken as above.

¿From now on A^{**} will always be regarded as a Banach algebra with the first Arens product. Let $Z(A^{**})$ denote all $m \in A^{**}$ such that

$$mn = m \circ n$$

for all $n \in A^{**}$. We call $Z(A^{**})$ the *topological center* of A^{**}.

Lemma 2.2 *$Z(A^{**})$ is a closed subalgebra of A^{**} containing A.*

For a proof see [3, p.310] or [9, Lemma 1].

Lemma 2.3 *For any $m \in A^{**}$, the following are equivalent:*

(a) $m \in Z(A^{**})$;

(b) the map $n \to mn$ from A^{**} into A^{**} is w^* -w^* continuous;

(c) the map $n \to mn$ from A^{**} into A^{**} is w^* -w^* continuous on norm bounded subsets of A^{**}.

For a proof see [3, p.313].
Note that for n fixed in A^{**}, the mapping $m \longmapsto mn$ is always w^* -w^* continuous.
We collect here some facts about the Arens product on $L^1(K)^{**}$ that we shall need.

Lemma 2.4 *Let $\phi, \psi \in L^1(K)$, $f \in L^\infty(K)$. Then*

(i) $\langle \psi f, \phi\rangle = \langle f\phi, \psi\rangle$.

(ii) $\psi f = f * \breve{\psi} \in UC_l(K)$, $\quad f\phi = \tilde{\phi} * f \in UC_r(K)$.

(iii) ${}_a(\psi f) = \psi({}_a f)$, $(f\phi)_a = (f_a)\phi$ *for* $a \in K$.

Proof: immediate.

Lemma 2.5 *Let $0 \neq m \in L^\infty(K)^*$. Then there is a net $\{u_\alpha\}$ in $L^1(K)$ such that $\|u_\alpha\| \leq \|m\|$, all u_α have compact support and $u_\alpha \to m$ in the w^*-topology of $L^\infty(K)^*$.*

Proof: This follows from Goldstine's theorem and the density of $C_{00}(K)$ in $L^1(K)$. □

Lemma 2.6 *If $m \in Z(L^1(K)^{**})$ and $f \in L^\infty(K)$, then $fm \in UC_r(K)$ and $(fm)(x * y) = \langle m, f_{x*y}\rangle$.*

Proof: We may assume that $m \neq 0$. Let $\{u_\alpha\}$ be the net in the Lemma 2.5. Then $\langle n, fm\rangle = \langle m \circ n, f\rangle = \langle mn, f\rangle = \langle m, nf\rangle = \lim_\alpha\langle u_\alpha, nf\rangle = \lim_\alpha\langle u_\alpha n, f\rangle = \lim_\alpha\langle n, fu_\alpha\rangle$, for all $n \in L^\infty(K)^*$. That is, $fu_\alpha \to fm$ weakly for all $f \in L^\infty(K)$. Replacing $\{u_\alpha\}$ by a suitable convex combinations, we may assume $fu_\alpha \to fm$ in norm. Note that $fu_\alpha = \tilde{u_\alpha} * f \in UC_r(K)$ (Lemma 2.4(ii)). It follows that $fm \in UC_r(K)$.
Furthermore, if $f \in L^\infty(K)$, $y \in K$ then

$$fm(y) = \lim_\alpha \tilde{u_\alpha} * f(y) = \lim_\alpha \int_K \tilde{u_\alpha}(x) f(\breve{x} * y)dx =$$

$$\lim_\alpha \int_K u_\alpha(x)f(x*y)dx = \lim_\alpha \langle u_\alpha, f_y\rangle = \langle m, f_y\rangle$$

(using [7, 5.5A]).
Now if $\phi \in L^1(K)$, $a \in K$, then by what we have seen above

$$\langle \phi, (fm)_a\rangle = (fm)\phi(a) = f(m\circ\phi)(a) = f(m\phi)(a) = \langle m\phi, f_a\rangle = \langle m\circ\phi, f_a\rangle = \langle \phi, (f_a)m\rangle$$

since $m \in Z(L^1(K)^{**})$; i.e. $(fm)_a = (f_a)m$.
Now for $x, y \in K$,

$$\begin{aligned} fm(x*y) &= (fm)_y(x) &= (f_y)m(x) \\ &= \langle m, (f_y)_x\rangle &= \langle m, f_{x*y}\rangle \end{aligned}$$

Hence, $(fm)(x*y) = \langle m, f_{x*y}\rangle$. □

Lemma 2.7 *If $n \in Z(L^1(K)^{**})$ and $u \in L^1(K)$ be such that $(n-u)(f) = 0$ for all $f \in C_0(K)$, then $n = u$.*

Proof: By Lemma 2.2, it is enough to show that any element of $Z(L^1(K)^{**})$ vanishing on $C_0(K)$ is zero. So let $n \in Z(L^1(K)^{**})$ such that $n(f) = 0$ for all $f \in C_0(K)$. First we show that $n(f) = 0$ for all $f \in L^\infty(K)$ vanishing outside a compact subset of K. Let $\varepsilon > 0$ be given. Since fn is continuous (Lemma 2.6), we can find $V \subseteq \{x : |fn(x) - fn(e)| < \varepsilon\}$ such that V is open with compact closure. Put $\nu = \frac{1_V}{\lambda(V)}$; then $f * \check{\nu} \in C_0(K)$. Hence by using 2.4(ii),

$$fn * \check{\nu}(e) = \nu(fn)(e) = (\nu f)n(e) = (f*\check{\nu})n(e) = \langle n, f*\check{\nu}\rangle = 0.$$

It follows from Lemma 2.6 that

$$|n(f)| = |fn(e) - (fn*\check{\nu}(e))| = \frac{1}{\lambda(V)}\left|\int_V (fn(e) - fn(x))dx\right| \le \frac{\varepsilon}{\lambda(V)}\int_V dx = \varepsilon.$$

Hence, we may assume that K is non-compact. Now let $\{u_\alpha\}$ be the net in Lemma 2.5. Replacing $\{u_\alpha\}$ by a convex combination of $\{u_\alpha\}$ if necessary, we may assume that for any $f \in L^\infty(K)$:

$$\| fu_\alpha - fn \| \to 0. \tag{1}$$

(see the proof of Lemma 2.6)
If $n \neq 0$, we may assume that n is positive and $\| n \| = 1$. Now for any probability measure $\mu \in M(K)$, $\| n\mu \| = 1$.
Let $0 < \varepsilon < 1/6$, then (using separation theorem for locally convex spaces) there exists $f \in L^\infty(K)$ such that

$$|\langle n\mu, f\rangle| > 1 - \varepsilon, \tag{2}$$

for all probablity measure $\mu \in M(K)$ with compact support. Let $\{C_i : i \in I\}$ be a family of compact subsets of K and let $y_i, z_i \in K$, and $i \in I$ satisfying the conditions of Lemma 2.1. For each i, define

$$f_i'(x) = \begin{cases} f(x*\check{y}_i) & \text{if } x \in C_i * y_i, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$f_i^{''}(x) = \begin{cases} f(x * \check{z}_i) & \text{if } x \in C_i * z_i, \\ 0 & \text{otherwise.} \end{cases}$$

For any finite subset σ of I, let

$$f'_\sigma = \Sigma_{i\in\sigma} f'_i \quad \text{and} \quad f^{''}_\sigma = \Sigma_{i\in\sigma} f^{''}_i.$$

Since the positive and negative parts of $Re(f'_\sigma)$, $Im(f'_\sigma)$, $Re(f^{''}_\sigma)$ and $Im(f^{''}_\sigma)$ are monotonically increasing bounded nets of positive functions in $L^\infty(K)$, there exists f', $f^{''}$ in $L^\infty(K)$ such that

$$f'_\sigma \to f' \qquad \text{and} \qquad f^{''}_\sigma \to f^{''}$$

in the w^*-topology of $L^\infty(K)$.
By (1) we may choose α such that

$$\| fu_\alpha - fn \| < \varepsilon, \qquad \| f'u_\alpha - f'n \| < \varepsilon \quad \text{and} \quad \| f^{''}u_\alpha - f^{''}n \| < \varepsilon.$$

Since each measure u_α has compact support, the family $\{C_i : i \in I\}$ is closed under finite unions and $K = \cup\{C_i^\circ : i \in I\}$, there exists $i_0 \in I$ such that *supp* $u_\alpha \subseteq C_{i_0}$. Let $g' = f'_{y_{i_0}}$ and $g^{''} = f^{''}_{z_{i_0}}$, then

$$|n(g')| > 1 - 3\varepsilon \qquad \text{and} \qquad |n(g^{''})| > 1 - 3\varepsilon \qquad (3).$$

Indeed, since $1_{C_{i_0}}(f'_\sigma)_{y_{i_0}} \to 1_{C_{i_0}}(f'_{y_{i_0}})$ in the w^*-topology of $L^\infty(K)$ and $1_{C_{i_0}}(f'_\sigma)_{y_{i_0}} = 1_{C i_0} f_{y_{i_0} * \check{y}_{i_0}}$ for σ containing i_0, it follows that $1_{C_{i_0}} g' = 1_{C_{i_0}} f_{y_{i_0} * \check{y}_{i_0}}$, namely $g' = f_{y_{i_0} * \check{y}_{i_0}}$ on C_{i_0}. Hence by Lemma 2.6,

$$\begin{aligned} \varepsilon &> |f'u_\alpha(y_{i_0}) - f'n(y_{i_0})| &=& |u_\alpha(f'_{y_{i_0}}) - n(f'_{y_{i_0}})| \\ &= |u_\alpha(f_{y_{i_0} * \check{y}_{i_0}}) - n(g')| &=& |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - n(g')|. \end{aligned}$$

Now since

$$|fn(y_{i_0} * \check{y}_{i_0})| = |n(f_{y_{i_0} * \check{y}_{i_0}})| > 1 - \varepsilon \qquad \text{(by (2))}$$

and

$$|fn(y_{i_0} * \check{y}_{i_0}) - fu_\alpha(y_{i_0} * \check{y}_{i_0})| = |n(f_{y_{i_0} * \check{y}_{i_0}}) - u_\alpha(f_{y_{i_0} * \check{y}_{i_0}})| < \varepsilon,$$

we have

$$|fu_\alpha(y_{i_0} * \check{y}_{i_0})| \geq$$
$$|fn(y_{i_0} * \check{y}_{i_0})| - |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - fn(y_{i_0} * \check{y}_{i_0})| > 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon.$$

Consequently,

$$|n(g')| \geq |fu_\alpha(y_{i_0} * \check{y}_{i_0})| - |fu_\alpha(y_{i_0} * \check{y}_{i_0}) - n(g')| > 1 - 3\varepsilon.$$

Similarly, $|n(g^{''})| > 1 - 3\varepsilon$.
By (a) and (b) of Lemma 2.1, the support of $g'g^{''}$ is contained in the compact set $D_0 = (C_{i_0} * y_{i_0} * \check{y}_{i_0}) \cap (C_{i_0} * z_{i_0} * \check{z}_{i_0})$. Let $h \in C_0(K)$ with $h = 1$ on D_0 and $\| 1 - h \| \leq \frac{1}{\|f\|+1}$ (see [**?**, Lemma 4]), we have by (3) and what we have shown first,

$$\langle |g'(1-h)|, n\rangle \geq |n(g' - g'h)| = |n(g')| > 1 - 3\varepsilon$$

and similarly, $\langle |g^{''}(1-h)|, n\rangle > 1-3\varepsilon$.
Hence, by adding these inequalities,we get

$$\langle (|g'| + |g^{''}|)(|1-h|), n\rangle > 2-6\varepsilon > 1.$$

But $\| \, (|g'|+|g^{''}|)(|1-h|) \, \| \leq 1$. This contradicts the assumption $\| \, n \, \| = 1$ and we are done. □

3 Topological center of $\mathbf{L}^1(\mathbf{K})^{**}$ In this section, we shall show that the topological center of $L^1(K)^{**}$ is $L^1(K)$.

Lemma 3.1 *If $\{e_i\}_{i\in I}$ be a bounded approximate identity for $L^1(K)$ and $f \in UC_r(K)$, then $\int_K f(x)\, e_i(x)\, dx \longmapsto f(e)$.*

Proof: By [15, Lemma 2.2(i)] and Lemma 2.4(ii), $f = g\phi$ for some $g \in UC_r(K)$ and $\phi \in L^1(K)$. Then by using Lemma 2.3, Lemma 2.6, and [15, Lemma 2.2(i)], we have

$$\int_K f(x)\, e_i(x)\, dx = \langle e_i, f\rangle = \langle e_i, g\phi\rangle = \langle \phi e_i, g\rangle \longmapsto \langle \phi, g\rangle = g\phi(e) = f(e). \qquad \square$$

Lemma 3.2 *Let H be a compact subhypergroup of K with the normalized Haar measure λ_H and $\{U_n\}$ be a decreasing sequence of relatively compact neighborhoods of H with $H = \cap_{n=1}^{\infty} U_n$. Put $\mu_n = \frac{1_{U_n}}{\lambda(U_n)}$, then $\mu_n \longmapsto \lambda_H$ in the $\sigma(M(K), C(K))$ topology of $M(K)$.*

Proof: Define $\{\mathcal{U'}_n\}_{n=1}^{\infty}$ in $L^1(K/H)$ by $\mathcal{U'}_n(x*H) = \mathcal{U'}_n(\dot{x}) = \int_H \mu_n(x*t)\, d\lambda(t)$, then by [8, Remark 2.5, p 180] we have

$$\int_{K/H} \mathcal{U'}_n(\dot{x})\, d\dot{x} = \int_{K/H}\int_H \mu_n(x*t)\, d\lambda_H(t)\, d\dot{x} = \int_K \mu_n(x)\, dx = 1,$$

so $\mathcal{U'}_n \in L^1(K/H)$. We have also *supp* $\mathcal{U'}_n \subseteq U_n/H$. Indeed if $x*H = \dot{x} \notin U_n/H = \{a*H : a \in U_n\}$, then $x*t \cap U_n = \emptyset$ for all $t \in H$ (using [7, 10.3A]). Thus $\{\mathcal{U'}_n\}$ is a B.A.I in $L^1(K/H)$. Now for $f \in C_{00}(K)$, by using [8, Remark 2.5] and Lemma 3.1

$$\int_K \mu_n(x)\, f(x)\, dx = \int_{K/H} f'(\dot{x})\mathcal{U'}_n(\dot{x})\, d\dot{x} \longmapsto f'(e*H) = \int_H f(t)\, d\lambda_H(t).$$

Hence $\langle \mu_n, f\rangle \longmapsto \langle \lambda_H, f\rangle$ for all $f \in C_{00}(K)$. On the other hand $\{\mu_n\}$ has a w^*-cluster point in $C(K)^*$, say μ (by Alaoglu theorem). Then $\mu = \lambda_H$ on $C_{00}(K)$ and hence on $C_0(K)$. Now since $\|\lambda_H\| = 1$ as an element of $M(K)$ and $\|mu_n\| = 1 \longmapsto \|\lambda_H\| = 1$, so by [11, Theorem 3.9] we have $\mu = \lambda_H$ on $C(K)$. □

Lemma 3.3 *Let H be a compact subhypergroup of K such that K/H is metrizable and $m \in Z(L^\infty(K)^*)$. Then $m\lambda_H \in L^1(K)$, where λ_H is the normalized Haar measure on H.*

Proof: First we show that if, for $\mu \in M(K)$, there exists a sequence $\{u_n\}$ in $L^1(K)$ converging μ in the $\sigma(M(K), C(K))$ topology, then $m\mu \in L^1(K)$ for any $m \in Z(L^\infty(K)^*)$. Let $u \in L^1(K)$ and $m \in Z(L^\infty(K)^*)$ and $\nu = m|_{C_0(K)}$. Then for $f \in C_0(K)$ by using Lemma 2.4(ii), we have

$$\langle mu, f\rangle = \langle m, uf\rangle = \langle \nu, uf\rangle = \langle \nu * u, f\rangle \text{ (see [7, 4.2E])}.$$

Since $mu \in Z(L^\infty(K)^*)$, it follows from Lemma 2.7 that $mu = \nu * u$. Now if $f \in L^\infty(K)$, $fm \in UC_r(K)$ (Lemma 2.6). Hence, $\langle m\mu, f\rangle = \langle \mu, fm\rangle = \lim_n\langle u_n, fm\rangle = \lim_n\langle mu_n, f\rangle$.

We know that $mu_n \in L^1(K)$ for all n and that $L^1(K)$ is weakly sequentially complete; it follows that $m\mu \in L^1(K)$. Now let $\{U_n\}$ be a decreasing sequence of relatively compact neighborhoods of H such that $H = \cap_{n=1}^{\infty} U_n$. Such a sequence exists, since the canonical map $\pi : K \to K/H$ is continuous, open and onto and K is locally compact Hausdorff and K/H is Hausdorff. Put $\mu_n = \frac{1_{U_n}}{\lambda(U_n)}$; then $\mu_n \in L^1(K)$ and $\mu_n \to \lambda_H$ in the $\sigma(M(K), C(K))$ topology of $M(K)$ (Lemma 3.2); so by what we have shown above, $m\lambda_H \in L^1(K)$. □

To show that $Z(L^\infty(K)^*) = L^1(K)$, we need one more lemma.

Lemma 3.4 *Let H be a compact subhypergroup of K and $m \in Z(L^\infty(K)^*)$. If $f \in L^\infty(K)$ is right H-periodic (i.e. $f_x = f$ for all $x \in H$) then $\langle m, f\rangle = \langle m\lambda_H, f\rangle$.*

Proof: As the proof of Lemma 2.6 for any $a \in H$, $(fm)_a = (f_a)m$. Consequently, by Lemma 2.6,

$$\begin{aligned}\langle m, f\rangle = fm(e) &= \int_H fm(e)d\lambda_H(x)\\ &= \int_H fm(x)d\lambda_H(x)\\ &= \langle \lambda_H, fm\rangle = \langle m\lambda_H, f\rangle. \quad \square\end{aligned}$$

Now we are ready to prove the main theorem of this section.

Theorem 3.5 *Let K be a locally compact hypergroup with left Haar measure, then $Z(L^\infty(K)^*) = L^1(K)$.*

Proof: We follow ideas in the proof of Theorem 1 in [9]. By Lemma 2.2, it suffices to show that $Z(L^\infty(K)^*) \subseteq L^1(K)$. Let $m \in Z(L^\infty(K)^*)$ and $\mu = m|_{C_0(K)}$. Also by Lemma 2.7, it is enough to show that $\mu \in L^1(K)$. Let B be a compact subset of K with $\lambda(B) = 0$. We may assume that B contains the identity of K. Then there exists a decreasing sequence of open relatively compact sets $U_n \supseteq B$ such that $(\lambda + |\mu|)(U_n \setminus B) \to 0$ (by regularity).
By induction, we construct a sequence ϕ_n in $C_0(K)$ such that $0 \le \phi_n \le 1$, $\phi_n = 1$ on B and that $\phi_n = 0$ on $U_n \cap V_{n-1}$, where $V_0 = K$ and $V_n = \{y \in K : \phi_n(y) \ne 0\}$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, $d_n(x, y) = \| (\phi_n)_x - (\phi_n)_y \|_\infty$ defines a continuous psedometric on K and $C_n = \{x \in K : d_n(x, e) = 0\}$ is a compact subhypergroup of K.
Indeed C_n is closed; it contains the identity and $C_n * C_n \subseteq C_n$. We note that C_n is compact, since $C_n \subseteq \bar{U}_n$. Moreover, C_n is a compact subhypergroup of K by [7, 10.2F].
But if $C = \cap_{n=1}^{\infty} C_n$, then K/C is metrizable and hence by Lemma 3.3, $m\lambda_C \in L^1(K)$. Consequently, Lemma 3.4 implies $\langle \mu, f\rangle = \langle m\lambda_C, f\rangle$ for all right C-periodic functions f. Also, since $\{V_n\}$ is decreasing $\lambda(V_n) \to \lambda(B) = 0$, hence, $m\lambda_C(V_n) \to 0$ (since $m\lambda_C \in L^1(K)$). Since $B \subseteq V_n \subseteq U_n$, $\mu(V_n) \to \mu(B)$ consequently, $\mu(B) = 0$.
Now by regularity of μ, we have $\mu \ll \lambda$ i.e. $\mu \in L^1(K)$. □

Definition 3.6 *$L^1(K)$ is called Arens regular if $mn = m \circ n$ for all $m, n \in L^1(K)^{**}$.*

The following corollary was proved by Young in [18] (see also [9]) for locally compact groups. For hypergroups, it was shown by Skantharajah [16, Theorem 5.2.3].

Corollary 3.7 *([16]) $L^1(K)$ is Arens regular if and only if K is finite.*

Proof: If K is finite, then $L^1(K)$ is reflexive and hence Arens regular.
If $L^1(K)$ is Arens regular, then $L^1(K)^{**} = L^1(K)$ by Theorem 3.5. Hence, $L^\infty(K)$ is reflexive and consequently of finite dimension. Therefore, K is finite. □

Corollary 3.8 *If K is commutative then $L^1(K)$ is the algebraic center of the algebra $L^1(K)^{**}$.*

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