TOPOLOGICAL TENSOR PRODUCTS OF TOPOLOGICAL SEMIGROUPS AND THEIR COMPACTIFICATIONS

A.R. MEDGHALCHI - H.R. RAHIMI

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Abstract. In this paper we first develop the notion of tensor products for two topological semigroups and then study this new structure, and get a number of interesting results in semigroup compactifications. We show that this structure is very different from other products such as semidirect products, or Sherier products.

1. Introduction

Our main references in this paper are the books [1], [3]. A semigroup S is called a right [left] topological semigroup if there is a topology on S with $s \longrightarrow st$ [$s \longrightarrow ts$] being continuous. S is called a semitopological [topological] semigroup if $(s,t) \longrightarrow st$ is separately [jointly] continuous. A topological semigroup S is called a topological group if it is a group and the inverse mapping $s \longrightarrow s^{-1}$ is continuous.

Let S be a topological semigroup. We recall that the pair (ψ, X) is called a semigroup compactification of S if X is a compact, Hausdorff right topological semigroup and $\psi : S \longrightarrow X$ is a continuous homomorphism such that $\psi(S)^- = X$, $\psi(S) \subseteq \Lambda(X)$ where $\Lambda(X) = \{t \in X : s \longrightarrow ts \in X \text{ is continuous}\}$. We say that the compactification (ψ, X) of S has the left [right] joint continuity property if the mapping $(s,x) \longrightarrow \psi(s)x$ [$(x,s) \longrightarrow x\psi(s)$] is continuous.

Following Howie [3], for a relation l on a set X, we write l^∞ for $l^\infty = \{l^n | n \geq 1\}$, where $l^n = l \circ l \circ \ldots \circ l$. Let l be an equivalence relation on a set X. Then the intersecton of all equivalence relations containing l, is said to be the equivalence generated by l. Following [3, Lemma 1.4.8], if l is a reflexive relation on X, then l^∞ is the smallest transitive relation on X containing l. We denote $[l \cup l^{-1} \cup 1_X]^\infty$ by l^e where $l^{-1} = \{(y,x) | (x,y) \in l\}$ and $1_X = \{(x,x) | x \in X\}$, and by [3, Proposition 1.4.9], we have that l^e is an equivalence generated by l. So, if l^∞ is an equivalence generated by l, then $(x,y) \in l^e$ if and only if, either $x = y$ or, for some $n \in \mathbb{N}$, there is a sequence of translations $x = z_1 \longrightarrow z_2 \longrightarrow z_3 \longrightarrow \ldots \longrightarrow z_n = y$ such that, for each $1 \leq i \leq n-1$, either $(z_i, z_{i+1}) \in l$ or, $(z_{i+1}, z_i) \in l$ [3, Proposition 1.4.10].

An equivalence τ on a semigroup S is called a left [right] S-congruence if $(x,y) \in \tau$ and $s \in S$, implies $(sx, sy) \in \tau$ [$(xs, ys) \in \tau$], and is called an S-congruence if it is both a right and a left S-congruence.

2. Topological S-systems

Let S, T be two topological semigroups with identities and X be a non-empty topological space. Then X is called a topological left S-system if there is an action $(s,x) \longrightarrow sx$ of

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$S \times X$ into X which is jointly continuous and $s_1(s_2x) = (s_1s_2)x, 1_Sx = x$ $(s_1, s_2 \in S, x \in X)$. A topological right S-system is defined similarly. A topological left S-system which is also a topological right T-system is called a topological $(S-T)$-bisystem if $(sx)t = s(xt)$ $(s \in S, t \in T, x \in X)$.

Let X, Y be two topological left S-systems and $\phi : X \longrightarrow Y$ be a continuous map. We say that ϕ is a topological left S-map if $\phi(sx) = s\phi(x)(x \in X, s \in S)$. Similarly, we can define a topological right T-map.

Now, let X be a topological $(S-U)$-bisystem, Y be a topological $(U-T)$-bisystem and Z be a topological $(S-T)$-bisystem. Then $X \times Y$ has the structure of a topological $(S-T)$-bisystem (i.e., $s_1s_2(x, y) = s_1(s_2x, y)$, $1_S(x, y) = (x, y)$, $(x, y)t_1t_2 = (x, yt_1)t_2, (x, y)1_T = (x, y)$, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$). Let $X \times Y$ be equipped with the product topology and $\beta : X \times Y \longrightarrow Z$ be a topological $(S-T)$-map (i.e., β is a topological left S-map and a topological right T-map). We say that β is a topological bimap if further $\beta(xu, y) = \beta(x, uy)$ $(u \in U)$.

Let (ψ, X) be a compactification of S with the left [right] joint continuity property. In this case we can regard X as a topological left [right] S-system where $sx = \psi(s)x$ $[xs = x\psi(s)]$ $(s \in S, x \in X)$.

3. The structure of topological tensor products and compactifications

Let S and T be two topological semigroups with identities $1_S, 1_T$, respectively. Let $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then T can obviously be regarded as a topological $(S-T)$-bisystem by $s * t = \sigma(s)t$ $(s \in S, t \in T)$, and S can be regarded as a topological $(S-S)$-bisystem where the action of S on S is just its multiplication.

Definition 3.1. *Consider S, T and σ as above. Let C be a topological $(S-T)$-bisystem and $\beta : S \times T \longrightarrow C$ be a topological $(S-T)$-map. We say that β is a topological σ-bimap if $\beta(ss', t) = \beta(s, \sigma(s')t)$ $(s, s' \in S, t \in T)$.*

Definition 3.2. *In the situation of Definition 3.1, by a topological tensor product we mean a pair (P, ψ) where P is a topological $(S-T)$-bisystem and $\psi : S \times T \longrightarrow P$ is a topological σ-bimap such that for every topological $(S-T)$-bisystem C and every topological σ-bimap $\beta : S \times T \longrightarrow C$, there exists a unique topological $(S-T)$- map $\bar{\beta} : P \longrightarrow C$ such that the diagram*

$$\begin{array}{ccc} S \times T & \xrightarrow{\psi} & P \\ \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\bar{\beta}} & \\ C & & \end{array}$$

commutes.

In the following theorem we prove the existence of the topological tensor product of S and T with respect to σ, which is denoted by $S \otimes_\sigma T$.

Theorem 3.3. *Let S and T be two topological semigroups with identities and $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then there is a unique (up to isomorphism) topological tensor product of S and T.*

Proof We regard $S \times T$ with the product topology as a topological $(S-T)$-bisystem.

Let τ_1 be the equivalence relation on $S \times T$ generated by

$$\{((ss', t), (s, \sigma(s')t)) : s, s' \in S, t \in T\}.$$

Let $\tau = \{(a, b) \in (S \times T) \times (S \times T) : u, v \in S \times T, (uav, ubv) \in \tau_1\}$. By [3, Proposition 1.5.10], τ is the largest congruence on $S \times T$ contained in τ_1. Now, we denote $\frac{S \times T}{\tau}$ by $S \otimes_\sigma T$ and the elements of $\frac{S \times T}{\tau}$ by $s \otimes_\sigma t$. We use the techniques of [3, Proposition 8.1.8] to show that if $s_1 \otimes_\sigma t_1 = s_2 \otimes_\sigma t_2$ then $s_1 = s_2$ and $t_1 = t_2$, or there exist $a_1, \ldots, a_{n-1} \in S$, $b_1, \ldots, b_{n-1} \in T$, $u_1, \ldots, u_n, v_1, \ldots, v_n \in S$ (see the introduction) such that

$$(*) \qquad \begin{aligned} s_1 &= a_1 u_1, & \sigma(u_1)t_1 &= \sigma(v_1)b_1, \\ a_1 v_1 &= a_2 u_2, & \sigma(u_2)b_1 &= \sigma(v_2)b_2, \\ &\vdots \\ a_i v_i &= a_{i+1} u_{i+1}, & \sigma(u_{i+1})b_i &= \sigma(v_{i+1})b_{i+1} \quad (i = 2, \ldots, n-2), \\ &\vdots \\ a_{n-1} v_{n-1} &= s_2 u_n & \sigma(u_n)b_{n-1} &= t_2. \end{aligned}$$

Let $\psi : S \times T \longrightarrow S \otimes_\sigma T$ be defined by $\psi(x, t) = s \otimes_\sigma t$. We have $s_1 \psi(s_2, t_1) = \psi(s_1 s_2, t_1)$, $\psi(s_1, t_1 t_2) = \psi(s_1, t_1)t_2$ and $\psi(s_1 s_2, t_1) = \psi(s_1, \sigma(s_2)t_1)$ for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$. So ψ is a topological σ-bimap.

Finally, we show that $(S \otimes_\sigma T, \psi)$ is a topological tensor product of S and T. Let C be a topological $(S - T)$-bisystem and $\beta : S \times T \longrightarrow C$ be a topological σ-bimap. If we define $\bar{\beta} : S \otimes_\sigma T \longrightarrow C$ by $\bar{\beta}(\psi(s, t)) = \beta(s, t)$, then $\bar{\beta}$ is well defined. By equations $(*)$ it is sufficient to find the values of β on generators. So, if $s_1 \otimes_\sigma t_1 = s_2 \otimes_\sigma t_2$, then we have

$$\begin{aligned} \beta(s_1, t_1) &= \beta(a_1 u_1, t_1) = \beta(a_1, \sigma(u_1)t_1) = \cdots = \beta(a_{n-1} v_{n-1}, b_{n-1}) \\ &= \beta(s_2 u_n, b_{n-1}) = \beta(s_2, \sigma(u_n)b_{n-1}) = \beta(s_2, t_2). \end{aligned}$$

Since $\bar{\beta}$ is a topological $(S-T)$-map and $\bar{\beta} \circ \psi = \beta$, it follows that $(S \otimes_\sigma T, \psi)$ is a topological tensor product of S and T.

If (P, ψ) and (P', ψ') are two topological tensor products of S and T, then putting $C = P'$, we can find a unique $(S - T)$-map $\bar{\beta} : P \longrightarrow P'$ such that $\bar{\beta} \circ \psi = \psi'$, i.e.,

$$\begin{array}{ccc} S \times T & \xrightarrow{\psi} & P \\ \downarrow{\scriptstyle \psi'} & \swarrow{\scriptstyle \bar{\beta}} & \\ P' & & \end{array}$$

commutes.

Similarly, we can find a unique $(S - T)$-map $\bar{\alpha} : P' \longrightarrow P$ such that $\bar{\alpha} \circ \psi' = \psi$, i.e.,

$$\begin{array}{ccc} S \times T & \xrightarrow{\psi'} & P' \\ \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \bar{\alpha}} & \\ P & & \end{array}$$

commutes. Thus $\bar{\alpha} \circ \bar{\beta} \circ \psi = \psi$, i.e.,

$$\begin{array}{ccc} S \times T & \xrightarrow{\psi} & P \\ \downarrow \psi & \swarrow \bar{\alpha} \circ \bar{\beta} & \\ P & & \end{array}$$

commutes. Hence by the uniqness property $\bar{\alpha} \circ \bar{\beta} = id_P$, similarly, $\bar{\beta} \circ \bar{\alpha} = id_{P'}$, so $P \simeq P'$ (semigroup isomorphism and onto). □

Proposition 3.4. *Let S be a right topological semigroup, let R be a congruence on S, and let the quotient semigroup S/R have the quotient topology. Then the following assertions hold.*

(i) *S/R is a right topological semigroup.*

(ii) *If S is semitopological, then so is S/R.*

(iii) *If S is a compact right topological (respectively, semitopological, topological) semigroup and if R is closed (in $S \times S$), then S/R is a compact, Hausdorff, right topological (respectively, semitopological, topological) semigroup.*

Proof See [1, Proposition 1.3.8]. □

Theorem 2.5. *Let S and T be two topological semigroups with identities, and $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then the following assertions hold:*

a) *$S \otimes_\sigma T$ is a topological semigroup with an identity.*

b) *If S and T are topological groups, then $S \otimes_\sigma T$ is a topological group.*

c) *If S and T are compact Hausdorff topological semigroups (groups), then so is $S \otimes_\sigma T$.*

Proof a) Clearly, $S \times T$ is a topological semigroup with identity. Hence by Theorem 3.4 $S \otimes_\sigma T$ is so as well.

b) It is easy to see that $S \otimes_\sigma T$ is a group whenever S and T are. So, if S and T are topological groups, then again by Theorem 3.4, $S \otimes_\sigma T$ is a topological group.

c) First, we show that τ (defined in Theorem 3.3) is a closed congruence on $S \times T$. Let $s_\alpha \longrightarrow s$, $s'_\alpha \longrightarrow s'$, $t_\alpha \longrightarrow t$, $t'_\alpha \longrightarrow t'$, and $s_\alpha \otimes_\sigma t_\alpha = s'_\alpha \otimes_\sigma t'_\alpha$. By an argument similar to the one in the proof of Theorem 3.3, continuity of σ and joint continuity of the multiplications on S and T imply that $s \otimes_\sigma t = s' \otimes_\sigma t'$. So by Theorem 3.4, $S \otimes_\sigma T$ is a compact, Hausdorff topological semigroup (group). □

Theorem 3.6. *Let (ψ_1, X_1) and (ψ_2, X_2) be two topological semigroup compactifications of topological semigroups S and T, respectively. Let $\sigma : S \longrightarrow T$, $\eta : X_1 \longrightarrow X_2$ be two continuous homomorphisms such that $\eta \circ \psi_1 = \psi_2 \circ \sigma$. Then $X_1 \otimes_\eta X_2$ is a topological semigroup compactification of $S \otimes_\sigma T$.*

Proof If we define the action of S on X_1 by $(s, x_1) \longrightarrow \psi_1(s)x_1$, then X_1 is a topological $(S - X_1)$-bisystem. Similarly, X_2 is a topological $(X_2 - T)$-bisystem, where the action of T on X_2 is defined by $(x_2, t) \longrightarrow x_2\psi_2(t)$. Also, the action of X_1 on X_2 is defined by $(x_1, x_2) \longrightarrow$

$\eta(x_1)x_2$. By Theorems 3.3 and 3.5, $X_1 \otimes_\eta X_2$ exists and is a compact Hausdorff topological semigroup and a topological $(S-T)$-bisystem. Now, let $\phi_1 = \psi_1 \times \psi_2 : S \times T \longrightarrow X_1 \times X_2$, and $\phi_2 : S \times T \longrightarrow S \otimes_\sigma T$ be a topological σ-bimap and $\phi_3 : X_1 \times X_2 \longrightarrow X_1 \otimes_\eta X_2$ be a topological η-bimap. We first observe that $\phi_3 \circ \phi_1$ is a topological $(S-T)$-map. Let $s, s' \in S$ and $t, t' \in T$. Indeed

$$\begin{aligned}\phi_3 \circ \phi_1(ss', t) &= \phi_3(\psi_1(ss'), \psi_2(t)) = \phi_3(\psi_1(s)\psi_1(s'), \psi_2(t)) \\ &= \psi_1(s)\phi_3(\psi_1(s'), \psi_2(t)) = \psi_1(s)(\phi_3 \circ \phi_1(s', t)).\end{aligned}$$

Similarly, $\phi_3 \circ \phi_1(s, tt') = (\phi_3 \circ \phi_1(s, t))\psi_2(t')$. Moreover, we have:

$$\begin{aligned}\phi_3 \circ \phi_1(ss', t) &= \phi_3(\psi_1(s)\psi_1(s'), \psi_2(t)) \\ &= \phi_3(\psi_1(s), [\eta \circ \psi_1(s')]\psi_2(t)) \\ &= \phi_3(\psi_1(s), [\psi_2 \circ \sigma(s')]\psi_2(t)) \\ &= \phi_3(\psi_1(s), \psi_2(\sigma(s')t)) \\ &= \phi_3 \circ \phi_1(s, \sigma(s')t).\end{aligned}$$

Obviously, $\phi_3 \circ \phi_1$ is continuous, thus $\phi_3 \circ \phi_1$ is a topological σ-bimap. Now by the universal property of topological tensor products, there is a topological $(S-T)$-map $\bar{\beta} : S \otimes_\sigma T \longrightarrow X_1 \otimes_\eta X_2$, we have

$$\begin{aligned}[\bar{\beta}(S \otimes_\sigma T)]^- &= [\bar{\beta}(\phi_2(S \times T))]^- = [\phi_3(\phi_1(S \times T))]^- \supseteq \phi_3(\phi_1(S \times T)^-) \\ &= \phi_3(X_1 \times X_2) = X_1 \otimes_\eta X_2.\end{aligned}$$

Also,

$$\begin{aligned}[\bar{\beta}(S \otimes_\sigma T)] &= \bar{\beta}(\phi_2(S \times T)) = \phi_3(\phi_1(S \times T)) \\ &= \phi_3(\psi_1(S) \times \psi_2(T)) \subseteq \phi_3(\Lambda(X_1) \times \Lambda(X_2)) \\ &= \phi_3(\Lambda(X_1 \times X_2)) = \Lambda(\phi_3(X_1 \times X_2)) \\ &= \Lambda(X_1 \otimes_\eta X_2).\end{aligned}$$

Clearly $\bar{\beta}$ is a continuous homomorphism, since ϕ_1, ϕ_2, ϕ_3 are so. Therefore, $X_1 \otimes_\eta X_2$ is a compactification of $S \otimes_\sigma T$. Note that $X \otimes_\eta X_2$ is in fact the topological tensor product of X_1 and X_2 with respect to η. □

Corollary 3.7. *Let $(\epsilon_i, S_i^{\mathcal{F}_i})$ $(i = 1, 2)$ be two canonical compacitifications of topological semigroups S_i such that $S_i^{\mathcal{F}_i}$ is a topological semigroup. Let $\sigma : S \longrightarrow T$ be a cotinuous homomorphism such that $\sigma^*(\mathcal{F}_2) \subseteq \mathcal{F}_1$. Then $S_1^{\mathcal{F}_1} \otimes_\eta S_2^{\mathcal{F}_2}$ exists and is a compactification of $S \otimes_\sigma T$.* □

4. The spaces of functions on topological tensor products

Theorem 4.1. *Let S and T be two topological semigroups with identities, and $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then $(S \otimes_\sigma T)^{ap} \simeq S^{ap} \otimes_\eta T^{ap}$.*

Proof Let $(\epsilon_{S\otimes_\sigma T}, (S\otimes_\sigma T)^{ap})$, (ϵ_S, S^{ap}) and (ϵ_T, T^{ap}) be topological ap-compactifications of $S\otimes_\sigma T$, S and T respectively. By Theorem 3.6, $(\delta_{S\otimes_\sigma T}, S^{ap} \otimes_\eta T^{ap})$ is a topological semigroup compactification of $S \otimes_\sigma T$.

The universal property of the *ap*-compactification $(\epsilon_{S\otimes_\sigma T}, (S\otimes_\sigma T)^{ap})$ of $S\otimes_\sigma T$ [1, Theorem 1.4.10] gives a continuous homomorphism $\phi : (S\otimes_\sigma T)^{ap} \longrightarrow S^{ap}\otimes_\eta T^{ap}$ such that the following diagram

$$\begin{array}{ccc} S\otimes_\sigma T & \overset{\epsilon_{S\otimes_\sigma T}}{\longrightarrow} & (S\otimes_\sigma T)^{ap} \\ \Big\downarrow \delta_{S\otimes_\sigma T} & \swarrow \phi & \\ S^{ap}\otimes_\eta T^{ap} & & \end{array}$$

commutes.

Also, since $(\epsilon_S\times\epsilon_T, (S\times T)^{ap})$ is a topological semigroup compactification of $S\times T$ via the homomorphism $\theta : S\times T \overset{\pi_1}{\longrightarrow} S\otimes_\sigma T \overset{\epsilon_{S\otimes_\sigma T}}{\longrightarrow} (S\otimes_\sigma T)^{ap}$, there is a continuous homomorphism $\phi_1 : (S\times T)^{ap} \longrightarrow (S\otimes_\sigma T)^{ap}$ such that the diagram

$$\begin{array}{ccc} S\times T & \overset{\theta}{\longrightarrow} & (S\otimes_\sigma T)^{ap} \\ \Big\downarrow \epsilon_S\times\epsilon_T & \nearrow \phi_1 & \\ (S\times T)^{ap} & & \end{array}$$

commutes. On the other hand $(S\times T)^{ap} \simeq S^{ap}\times T^{ap}$ [2], [4], [1, Theorem 5.2.4]. Thus we can assume (up to isomorphism), $\phi_1 : S^{ap}\times T^{ap} \longrightarrow (S\otimes T)^{ap}$. By equations $(*)$ in the proof of the Theorem 3.3 it is sufficient to apply ϕ_1 to generators. Indeed, if $vv'\otimes_\eta \mu = v\otimes_\eta \eta(v')\mu$ $(v, v' \in S^{ap}, \mu\in T^{ap})$, we can get nets $\{s_\alpha\}, \{s'_\beta\}$ in S and $\{t_\gamma\}$ in T such that $\lim_\alpha \epsilon_S(s_\alpha) = v$, $\lim_\beta \epsilon_S(s_\beta) = v'$, $\lim_\gamma \epsilon_T(t_\gamma) = \mu$. Thus

$$\begin{aligned} \phi_1(vv'\otimes_\eta \mu) &= \phi_1(\lim_{\alpha,\beta,\gamma} \epsilon_S\times\epsilon_T(s_\alpha s_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \phi_1(\epsilon_S\times\epsilon_T(s_\alpha s_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \epsilon_{S\otimes_\sigma T}(\pi_1(s_\alpha s_\beta, t_\gamma)) \\ &= \lim_{\alpha,\beta,\gamma} \epsilon_{S\otimes_\sigma T}(s_\alpha, \sigma(s_\beta)t_\gamma) \\ &= \phi_1(\lim_{\alpha,\beta,\gamma} \epsilon_S\times\epsilon_T(s_\alpha, \sigma(s_\beta)t_\gamma) \\ &= \phi_1(v\otimes_\eta \eta(v')\mu). \end{aligned}$$

Now by an argument similar to equations $(*)$ of Theorem 3.3 one can get that ϕ_1 preservers congruence. So there exists a continuous homomorphism $\phi_2 : S^{ap}\otimes_\eta T^{ap} \longrightarrow (S\otimes_\sigma T)^{ap}$ such that the diagram

$$\begin{array}{ccc} S^{ap}\times T^{ap} & \overset{\phi_1}{\longrightarrow} & (S\otimes_\sigma T)^{ap} \\ \Big\downarrow \pi_2 & \nearrow \phi_2 & \\ S^{ap}\otimes_\eta T^{ap} & & \end{array}$$

commutes. But $\phi\circ\phi_2$ is the identity on $S^{ap}\otimes_\eta T^{ap}$, for, if $u\otimes_\eta v \in S^{ap}\otimes_\eta T^{ap}$, then we can find a net $\{s_\alpha\}$ in S and a net $\{t_\beta\}$ in T such that $\epsilon_S(s_\alpha) \longrightarrow u$, $\epsilon_T(t_\beta) \longrightarrow v$. Now

$$\begin{aligned} \phi\circ\phi_2(u\otimes_\eta v) &= \phi\circ\phi_2(\pi_2(u,v)) = \phi(\phi_1(u,v)) \\ &= \lim_{\alpha,\beta} \phi(\phi_1(\epsilon_S\times\epsilon_T(s_\alpha, t_\beta)) = \lim_{\alpha,\beta} \phi\circ\theta(s_\alpha, t_\beta) \\ &= \lim_{\alpha,\beta} \phi(\epsilon_{S\otimes_\sigma T}(s_\alpha\otimes_\sigma t_\beta)) = \lim_{\alpha,\beta} \delta_{S\otimes_\sigma T}(s_\alpha\otimes_\sigma t_\beta) \\ &= u\otimes_\eta v. \end{aligned}$$

So $(S \otimes_\sigma T)^{ap} \simeq S^{ap} \otimes_\eta T^{ap}$. □

Theorem 3.2. *Let S and T be two topological semigroups with identities, and $\sigma : S \longrightarrow T$ be a continuous homomorphism. Then $(S \otimes_\sigma T)^{sap} \simeq S^{sap} \otimes_\eta T^{sap}$.*

Proof Since $(\epsilon_{S\otimes_\sigma T}, (S \otimes_\sigma T)^{sap})$ is a universal topological group compactification of $S \otimes_\sigma T$ [1, Theorem 4.3.7], an argument similar to that for Theorem 4.1 (using the universal property of the topological group compactification $(S \otimes_\sigma T)^{sap}$, S^{sap}, T^{sap}, $(S \times T)^{sap}$ and the universal property of the topological tensor product) shows that $(S \otimes_\sigma T)^{sap} \simeq S^{sap} \otimes_\eta T^{sap}$. □

Example 1 (Absorption property). *Let T be a topological commutative semigroup with identity and let S be a topological subsemigroup of T containing the identity, and $\sigma : S \longrightarrow T$ be a continuous homomorphism. We consider the left [right] action of S on T by $(s, t) \longrightarrow \sigma(s)t$ $[(t, s) \longrightarrow t\sigma(s)]$. Clearly, S and T are topological $(S - S)$-bisystems. Then $\tau = \{((s_1 s_2, t), (s_1, \sigma(s_2)t)) : s_1, s_2 \in S, t \in T\}$ is a congruence on $S \times T$. We define $\psi : S \otimes_\sigma T \longrightarrow T$ by $\psi(s \otimes_\sigma t) = \sigma(s)t$. Then ψ is a surjective continuous homomorphism. Also, ψ is one-to-one. For, if $\psi(s_1 \otimes_\sigma t_1) = \psi(s_2 \otimes_\sigma t_2)$, then $\sigma(s_1)t_1 = \sigma(s_2)t_2$. Now $s_1 \otimes_\sigma t_1 = 1_S \otimes_\sigma \sigma(s_1)t_1 = 1_S \otimes_\sigma \sigma(s_2)t_2 = s_2 \otimes_\sigma t_2$. Thus $S \otimes_\sigma T \simeq T$.*

Example 2. *Let S be a topological semigroup with identity such that every member of S is uniquely expressible. Let $\sigma = id_S : S \longrightarrow S$. Now, by the equations $(*)$ in the proof of Theorem 3.3, if $s_1 \otimes_\sigma s_2 = s_3 \otimes_\sigma s_4$, then $s_1 = s_2$ and $s_3 = s_4$ $(s_1, s_2, s_3, s_4 \in S)$. Thus $S \otimes_\sigma S = \{(s_1, s_2) : s_1, s_2 \in S\}$, i.e., $S \otimes_\sigma S = S \times S$.*

Example 3. *Let $S = (\mathbf{R}, +)$ and $\sigma = id_{\mathbf{R}} : \mathbf{R} \longrightarrow \mathbf{R}$. Then we can find $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$, $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ in $\mathbf{R}$ such that for every $(s_1, t_1) \in \mathbf{R} \times \mathbf{R}$ and $(s_2, t_2) \in \mathbf{R} \times \mathbf{R}$ we have $s_1 \otimes_\sigma t_1 = s_2 \otimes_\sigma t_2$. Thus $\mathbf{R} \otimes_\sigma \mathbf{R} = \mathbf{s} \otimes_\sigma \mathbf{t}$ $(s, t \in \mathbf{R})$.*

Note. *The above example shows that our tensor product is very different from other products. In fact $\mathbf{R}Ⓢ\mathbf{R} \cong \mathbf{R} \times \mathbf{R}$, where $\mathbf{R}Ⓢ\mathbf{R}$ is the semidirect product of $\mathbf{R}$ and $\mathbf{R}$. While $\mathbf{R} \otimes_\sigma \mathbf{R} = \mathbf{s} \otimes_\sigma \mathbf{t}$ is just one equivalence class. So it is different from Sherier product [5].*

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Faculty of Mathematical Science and Computer Engineering,
Teacher Training University, 599 Taleghani Ave., Tehran 15614, Iran.
medghalchi@saba.tmu.ac.ir
Department of Mathematics, Faculty of Science,
Islamic Azad University (Central Tehran Branch), Tehran-Iran
h_rahimi2004@yahoo.com