H-FILTERS OF HILBERT ALGEBRAS

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Abstract. We introduce the concept of Hilbert filter (H-filter, in abbreviation) in Hilbert algebras, and study how to generate an H-filter by a set.

1. Introduction

Following the introduction of Hilbert algebras by A. Diego [5], the algebra and related concepts were developed by D. Busneag [2 - 4]. The present author [7, 8] gave a characterization of a deductive system in a Hilbert algebra, and introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra. In this paper, we introduce the concept of a Hilbert filter (H-filter, in abbreviation) in Hilbert algebras, and study how to generate an H-filter by a set. We also discuss how to generate an H-filter by an H-filter and an element.

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to [2 - 4] and [5].

A *Hilbert algebra* is a triple $(H, \to, 1)$, where H is a nonempty set, " $\to$ " is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:

(i) $x \to (y \to x) = 1$,
ii) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(iii) if $x \to y = y \to x = 1$ then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H, which will be called the *natural ordering* on H. With respect to this ordering 1 is the largest element of H. A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0 relative to the natural ordering. In a bounded Hilbert algebra H we define a unary operation " C " on H by $C(x) := x \to 0$ for all $x \in H$.

In a Hilbert algebra H, the following hold:

(1) $x \leq y \to x$,
(2) $x \to 1 = 1$,
(3) $x \to (y \to z) = (x \to y) \to (x \to z)$,
(4) $1 \to x = x$,
(5) $x \to (y \to z) = y \to (x \to z)$,
(6) $x \to x = 1$.
(7) $x \leq y$ implies $z \to x \leq z \to y$ and $y \to z \leq x \to z$,
(8) $C(0) = 1$ and $C(1) = 0$,
(9) $x \leq y$ implies $C(y) \leq C(x)$,
(10) $x \leq C(C(x))$,
(11) $x \to y \leq (y \to z) \to (x \to z)$,

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(12) $x \to y \le C(y) \to C(x)$,
(13) $x \le (x \to y) \to y$ and $((x \to y) \to y) \to y = x \to y$.

In the sequel, the binary operation " $\to$ " will be denoted by juxtaposition.

For any x, y in a Hilbert algebra H, we define $x \vee y$ by $(yx)x$. Note that $x \vee y$ is an upper bound of x and y.

A Hilbert algebra H is said to be *commutative* ([8, Definition 2.1]) if for all $x, y \in H$,

$$(yx)x = (xy)y, \text{ i.e., } x \vee y = y \vee x.$$

A subset D of a Hilbert algebra H is called a *deductive system* of H if it satisfies:

(i) $1 \in D$,
(ii) $x \in D$ and $xy \in D$ imply $y \in D$.

2. H-filters

In the sequel H will denote a bounded Hilbert algebra unless otherwise specified. We begin with the following definition.

Definition 2.1. A non-empty subset F of H is called a *Hilbert filter* (H*-filter,* in abbreviation) if

(F1) $0 \in F$,
(F2) $C(C(y)C(x)) \in F$ and $y \in F$ imply $x \in F$,
for all $x, y \in H$.

Under this definition $\{0\}$ and H are the simplest examples of H-filters.

Example 2.2. Consider a bounded Hilbert algebra $H := \{1, x, y, z, 0\}$ with Cayley table as follows:

	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

It is easily verified that $F := \{0, y\}$ and $G := \{0, z\}$ are H-filters of H.

Theorem 2.3. Let F be an H-filter of H and $y \in F$. If $C(y) \le C(x)$, then $x \in F$ for all $x \in H$.

Proof. If $C(y) \le C(x)$, then $C(y)C(x) = 1$ and so $C(C(y)C(x)) = C(1) = 0 \in F$. It follows from (F2) that $x \in F$, ending the proof. □

Since $x \le y$ implies $C(y) \le C(x)$ by (9), we have the following corollary.

Corollary 2.4. Let F be an H-filter of H and $y \in F$. If $x \le y$, then $x \in F$ for all $x \in H$.

Lemma 2.5. If H is commutative, then

(i) $C(C(x)C(y)) = C(yx)$,
(ii) $C(C(x)) = x$, for all $x, y \in H$.

Proof. (i) From (9) and (12) we know that $C(C(x)C(y)) \leq C(yx)$. Now

$$\begin{aligned} C(yx)C(C(x)C(y)) &= ((yx)0)(((x0)(y0))0) \\ &= ((x0)(y0))(((yx)0)0) && \text{[by (5)]} \\ &= (y((x0)0))(((yx)0)0) && \text{[by (5)]} \\ &= (y((0x)x))((0(yx))(yx)) && \text{[by commutativity]} \\ &= (yx)(yx) = 1, && \text{[by (4) and (6)]} \end{aligned}$$

which implies that $C(yx) \leq C(C(x)C(y))$. Hence $C(C(x)C(y)) = C(yx)$.

(ii) Note that $x \vee 0 = (0x)x = 1x = x$ and $0 \vee x = (x0)0 = C(C(x))$. By the commutativity, we have $C(C(x)) = x$. □

If H is commutative, then we have a characterization of an H-filter by using Lemma 2.5(i).

Theorem 2.6. Assume that H is commutative and let F be a non-empty subset of H. Then F is an H-filter if and only if it satisfies (F1) and

(F3) $C(xy) \in F$ and $y \in F$ imply $x \in F$ for all $x, y \in H$.

For a non-empty subset F of H, we define

$$C(F) := \{C(x) | x \in F\}.$$

In general, $C(F)$ may not be a deductive system even if F is an H-filter of H. In fact, in Example 2.2, $C(F) = \{1, z\}$ is not a deductive system of H since $zx = 1 \in C(F)$ and $x \notin C(F)$.

Theorem 2.7. Assume that H is commutative. If F is an H-filter, then $C(F)$ is a deductive system of H.

Proof. If F is an H-filter, then $0 \in F$ and so $C(0) = 1 \in C(F)$. Let $x \in C(F)$ and $xy \in C(F)$ for all $x, y \in H$. Then $x = C(u)$ and $xy = C(v)$ for some $u, v \in F$. By using Lemma 2.5(ii), we have

$$C(C(u)C(C(y))) = C(C(u)y) = C(xy) = C(C(v)) = v \in F.$$

It follows from (F2) that $C(y) \in F$ so that $y = C(C(y)) \in C(F)$. This completes the proof. □

Observation 2.8 Suppose $\mathcal{F}$ is a non-empty family of H-filters of H. Then $F = \cap\mathcal{F}$ is also an H-filter of H.

Let A be a subset of H. The least H-filter containing A is called the *H-filter generated* by A, written $\langle A \rangle$.

Since H is clearly an H-filter containing A, in view of Observation 2.8 we know that the definition is well-defined. We start with the following.

Observation 2.9. Let A and B be subsets of H. Then the following hold:

(i) $\langle\{0\}\rangle = \{0\}$, $\langle\emptyset\rangle = \{0\}$.
(ii) $\langle H \rangle = H$, $\langle\{1\}\rangle = H$.
(iii) $A \subseteq B$ implies $\langle A \rangle \subseteq \langle B \rangle$.
(iv) $x \leq y$ implies $\langle\{x\}\rangle \subseteq \langle\{y\}\rangle$.
(v) if A is an H-filter of H, then $\langle A \rangle = A$.

The next statement gives a description of elements of $\langle A \rangle$.

Theorem 2.8. If A is a non-empty subset of H, then

$$\langle A\rangle = \{x \in H | C(a_n)(...(C(a_1)C(x))...) = 1 \text{ for some } a_1, ..., a_n \in A\}.$$

In order to prove Theorem 2.10 we need the following facts: For any natural number n we define $x^n y$ recursively as follows: $x^1 y = xy$ and $x^{n+1}y = x(x^n y)$. By (5) and induction we know that

(14) $z(x_n(...(x_1 y)...)) = x_n(...(x_1(zy))...)$.

As a special case of (14) we get

(15) $z(x^n y) = x^n(zy)$.

Now let $a, y, x_1, ..., x_n$ be elements of a Hilbert algebra H. Then

$$\begin{aligned}
&(((x_n(...(x_1(ya))...))a)a)(x_n(...(x_1(ya))...)) \\
&= x_n(...(x_1(y((((x_n(...(x_1(ya))...))a)a)a)))...) && \text{[by (5)]} \\
&= x_n(...(x_1(y((x_n(...(x_1(ya))...))a)))...) && \text{[by (13)]} \\
&= (x_n(...(x_1(ya))...))(x_n(...(x_1(ya))...)) && \text{[by (5)]} \\
&= 1, && \text{[by (6)]}
\end{aligned}$$

which implies that

$$((x_n(...(x_1(ya))...))a)a \le x_n(...(x_1(ya))...).$$

The reverse inequality follows from (13). Hence we have

(16) $((x_n(...(x_1(ya))...))a)a = x_n(...(x_1(ya))...)$.

Substituting 0 for a and assuming $x_1 = x_2 = ... = x_n = x$ in (16), we obtain

(17) $C(C(x^n C(y))) = x^n C(y)$.

Proof of Theorem 2.10. Denote

$$U = \{x \in H | C(a_n)(...(C(a_1)C(x))...) = 1 \text{ for some } a_1, ..., a_n \in A\}.$$

We first prove that U is an H-filter. Since A is non-empty, there exists $a \in A$. Then $C(a)C(0) = C(a)1 = 1$, whence $0 \in U$. Let $C(C(y)C(x)) \in U$ and $y \in U$. Then there exist $a_i \in A$ $(i = 1, ..., n)$ and $b_j \in A$ $(j = 1, ...m)$ such that

(18) $C(a_n)(...(C(a_1)C(C(C(y)C(x))))...) = 1$ and $C(b_m)(...(C(b_1)C(y))...) = 1$.

It follows from (17) that (18) implies

$$C(a_n)(...(C(a_1)(C(y)C(x)))...) = 1,$$

and so $C(y) \le C(a_n)(...(C(a_1)C(x))...)$. By using (7) we get

$$\begin{aligned}
1 &= C(b_m)(...(C(b_1)C(y))...) \\
&\le C(b_m)(...(C(b_1)(C(a_n)(...(C(a_1)C(x))...)))...),
\end{aligned}$$

and hence $C(b_m)(...(C(b_1)(C(a_n)(...(C(a_1)(C(x))...)))...) = 1$. This shows that $x \in U$. Therefore U is an H-filter. Now it is clear that $A \subseteq U$. Let V be any H-filter containing A and let $x \in U$. Then $C(a_n)(...(C(a_1)C(x))...) = 1$ for some $a_1, ..., a_n \in A$. Thus

$$\begin{aligned}
1 &= C(a_n)(C(a_{n-1})(...(C(a_1)C(x))...)) \\
&= C(a_n)(C(a_{n-1})(...(C(a_1)(x0))...)) \\
&= C(a_n)(((C(a_{n-1})(...(C(a_1)(x0))...))0)0) && \text{[by (17)]} \\
&= C(a_n)(C(C(C(a_{n-1})(...(C(a_1)C(x))...)))),
\end{aligned}$$

which implies that

$$C(C(a_n)(C(C(C(a_{n-1})(...(C(a_1)C(x))...))))) = C(1) = 0 \in V.$$

Noticing $a_n \in A \subseteq V$ and V to be an H-filter, we have $C(C(a_{n-1})(...(C(a_1)C(x))...)) \in V$. Now

$$
\begin{aligned}
&C(C(a_{n-1})(...(C(a_1)C(x))...)) \\
&= C(C(a_{n-1})(C(a_{n-2})(...(C(a_1)C(x))...))) \\
&= C(C(a_{n-1})(C(C(C(a_{n-2})(...(C(a_1)C(x))...)))))).\text{[by (17)]}
\end{aligned}
$$

Since $a_{n-1} \in A \subseteq V$, it follows from (F2) that $C(C(a_{n-2})(...(C(a_1)C(x))...)) \in V$. Repeating the above argument we conclude that $C(C(x)) \in V$. Since $x \leq C(C(x))$, we have $x \in V$ by Corollary 2.4. This proves that $U \subseteq V$, whence $U = \langle A \rangle$. This completes the proof.

If $A = \{a_1, ..., a_n\}$, we will denote $\langle \{a_1, ..., a_n\} \rangle = \langle a_1, ..., a_n \rangle$ for the sake of convenience.

The following corollary is immediate from Theorem 2.10.

Corollary 2.9. For any $a \in H$, we have

$$\langle a \rangle = \{x \in H | C(a)^n C(x) = 1 \text{ for some natural number } n\}.$$

The following theorem shows how to generate an H-filter by given an H-filter and an element.

Theorem 2.10. Let F be an H-filter of H and $a \in H$. Then

$$\langle F \cup \{a\} \rangle = \{x \in H | C(C(a)^n C(x)) \in F \text{ for some natural number } n\}.$$

Proof. Denote

$$U = \{x \in H | C(C(a)^n C(x)) \in F \text{ for some natural number } n\}.$$

Since $C(C(a)^n C(a)) = C(1) = 0 \in F$, therefore $a \in U$. Let $x \in F$. Since $C(x) \leq C(a)C(x) = C(C(C(a)C(x)))$, it follows from Theorem 2.3 that $C(C(a)C(x)) \in F$ so that $x \in U$. Hence $F \cup \{a\} \subseteq U$. In order to prove that U is an H-filter, let $C(C(y)C(x)) \in U$ and $y \in U$. Then there are natural numbers n and m such that

(19) $C(C(a)^n C(C(C(y)C(x)))) \in F$ and

(20) $C(C(a)^m C(y)) \in F$, respectively.

From (17) it follows that (19) is precisely the following

(21) $C(C(a)^n (C(y)C(x)) \in F$.

(20) and (21) imply that $C(C(a)^n (C(y)C(x)) = u$ and $C(C(a)^m C(y)) = v$ for some $u, v \in F$. Using (17) we get

(22) $C(a)^n (C(y)C(x)) = C(C(C(a)^n (C(y)C(x))) = C(u)$ and

(23) $C(a)^m C(y) = C(C(C(a)^m C(y))) = C(v)$.

From (22) we know that $C(y) \leq C(u)(C(a)^n C(x))$, which implies from (5), (7) and (23) that

$$C(v) = C(a)^m C(y) \leq C(u)(C(a)^{m+n} C(x)).$$

Hence

$$
\begin{aligned}
&C(v)(C(u)(C(C(C(a)^{m+n} C(x))))) \\
&= C(v)(C(u)(C(a)^{m+n} C(x))) \qquad \text{[by (17)]} \\
&= 1.
\end{aligned}
$$

Since $u, v \in F$, it follows from Observation 2.9(v) and Theorem 2.10 that

$$C(C(a)^{m+n} C(x)) \in F$$

so that $x \in U$. Clearly, $0 \in U$. Therefore U is an H-filter. Finally let V be an H-filter containing F and a. If $x \in U$, then there exists a natural number n such that $C(C(a)^n C(x)) \in F \subseteq V$. Thus, by (17), we have

$$C(C(a)(C(C(C(a)^{n-1} C(x))))) = C(C(a)^n C(x)) \in V.$$

Combining $a \in V$ and using (F2) we get $C(C(a)^{n-1}C(x)) \in V$. Repeating the procedure above, we conclude that $C(C(x)) \in V$. It follows from (10) and Corollary 2.4 that $x \in V$. This proves that $U \subseteq V$. Therefore U is the least H-filter containing F and a, i.e., $\langle F \cup \{a\} \rangle = U$. This completes the proof. □

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