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A CONNENTED TYCHONOFF IRRESOLVABLE SPACE AND RESOLVABILITY CONDITIONS

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Abstract. We show that the Niemytzki plane, which is connected Tychonoff, is not resolvable.This addresses a question raised by Comfort as to whether a concrete example of an irresolvable connected Tychonoff without isolated points exists. To demonstrate the case we present the following results: (a) A resolvable Cech-complete contains a dense Cech-complete subspace whose complement is dense.
(b) A resolvable space contains some infinite subsets whose derived sets are not empty.
(c) Necessary and sufficient conditions of resolvability and irresolvability are also given.

1 INTRODUCTION Hewitt[7] called a topological space X = (X, T) resolvable if there is a subset A of X such that both A and its complement (X-A) are dense in X. Clearly every resolvable space X is dense-in-itself,i.e, no point of X is isolated in X. Resolvable spaces have been studied extensively in the literature beginning with Hewitt[7]. Related results were proved by Padmavally[8], Comfort and Li Feng[2], Comfort etal[4] and Anderson[1]. Pavlov proved in[9] that a resolvable Baire space contains a dense Baire subspace whose complement is dense. Comfort showed in[3] that regular countably compact spaces are resolvable. In this paper, we prove that a resolvable Cech-complete space contains a dense Cech-complete subspace whose complement is dense. We display an example of a connected Tychonoff space which is not resolvable answering the question raised by Comfort and Garcia[3],whether every connected Tychonoff space without isolated points is resolvable, in the negative. Although the direct reverse of Comfort result (Every regular countably compact is resolvable)in[3] is not true; We prove necessary and sufficient conditions of resolvability and irresolvability in terms of filterbases.

2 Resolvable, Cech-complete and countably compact properties .

Lemma 1: A Resolvable Cech-complete space contains a dense Cech-complete subspace whose complement is dense.

Proof: Let (X,T) be a Cech-complete space and resolvable,i.e.,there exists a subset A of X such that clA = X and cl(X-A)= X. Assume that for every closed subset F of X,$F \cap A$ and $F \cap (X-A)$ are not Cech-complete. Thus there are closed subsets F_1 and F_2 such that $F_1 \cap A$ and $F_2 \cap (X-A)$ each contains a union of countably many nowhere-dense sets whose complements are not dense. Then there is a closed subset F^1 of X ($F^1 = (X-G_1) \cap (X-G_2)$, $F_1 = F \cap (X-G_1)$ and $F_2 = F \cap (X-G_2)$ where G_1 and G_2 are open subsets) in which there is a union of countably many nowhere-dense sets whose complement is not dense. This is is a cotradiction. Since X is Cech-complete and Cech-completeness is hereditary with respect to closednesss, there is a $\pi - base\Im$ of closed subsets of X such that for every $F \in \Im$ either $F \cap A$ or $F \cap (X-A)$ is Cech-complete. Let ν be disjoint subfamily of $\Im$ whose union is dense in X. For every $F \in \nu$, pick the Cech-complete one from $\{F \cap A, F \cap (X-A)\}$, say

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$F \cap A$ and denote it by X(F). Then the set $\cup\{X(F) : F \in \nu\}$ is dense Cech-complete whose complement is dense. □

Note: $Int(A) = \emptyset \Rightarrow \emptyset = int(F) \cap int(A) \Rightarrow$
$\Rightarrow (X - cl(X - F \cap A) = \emptyset \Rightarrow cl(X - F \cap A) = X$.

Example: The example which will be displayed is the known Niemytzki plane as in Steen[10] and Engelking[5].
Niemytzki plane is the space $X = P \cup L$, where $P = \{(x, y) : x, y \in R, y \gg 0\}$ is the open upper half-plane with the Euclidean topology τ , and L is the real axis. The topology generated on $X = P \cup L$ is by adding to τ all sets of the form $\{x\} \cup D$, where $x \in L$ and D is an open disk in P which is tangant to L at the point x.

Facts about Niemytzki plane X, see[5]:
1- X is Cech-complete.
2- P is dense Cech-complete whose complement is not dense.
3- every closed subset of X is G_δ.
4- X is a connected Tychonoff space without isolated points.
Corollary 2: The Niemytzki plane X is irresolvable.
Proof: If the Niemytzki plane X is resolvable, then there is a dense Cech- -complete subspace whose complement is dense. From the above facts about the Niemytzki plane, P is dense Cech-complete whose complement is not dense in X. Assume there is a subset A of X other than the P which is dense Cech- complete and whose complement is dense.i.e.,clA = X and cl(X-A)= X. Let $B = \cup A_i$, where A_i are nowhere dense in A, then cl(X-A) ⊂ cl(X-B)=clA=X. This is a contradiction. □

In the following result we assume the space X is Hausdorf and without isolated points.
Lemma 3: If X is resolvable, then for some infinite subsets B ⊂ X, $accB \neq \emptyset$.

Proof: Assume X is resolvable,i.e., there is A ⊂ X such that $clA = X$, $cl(X - A) = X$ and $X = A \cup (X - A)$. Suppose for every infinite B⊂ X, $accB = \emptyset$, then such a B is closed.

(X is countably compact if every infinite A⊂ X satisfies $accA \neq \emptyset$).
We study the following cases : (i) B ⊂ A, we use the fact that if every subset of A is closed then $accA = \emptyset \Rightarrow clA = A \subset$ X.
(ii) B⊂ (X-A), by the above argument, then cl(X-A) ⊂ X.
(iii) $B = B_1 \cup B_2$, where $B_1 \subset$ A, and $B_2 \subset$ (X-A), if $accB = \emptyset$, then clA⊂ X, and cl(X-A)⊂ X. This is a contradiction. □

The above result cannot be generalized to every infinite subset of X, since there are many examples, in the literature, of resolvable spaces which are not countably compact,and the reverse of Comfort[3] is not true.
Filterbases have been used by Ganster[6] for results equivalent to irresolvability in the context of dense subsets. We generalize the result of Comfort[3] using filterbases, as shown in the following theorem:
Theorem 4:(a) A space X is irresolvable iff $\{(X - clA) : intA = \emptyset\}$ is a filterbase on X.
(b) A space X is resolvable iff $\{accA : intA = \emptyset\}$ is a filterbase on X.

Proof of (a):(i)⇒ Let X be irresolvable, then for no A ⊂ X, $(X - clA) \neq \emptyset)$ with $intA = \emptyset$. Suppose $\{(X - clA) : intA = \emptyset\} = \zeta$ is not a filterbase, then for no $C_3 \subset$

$C_1 \cap C_2$, $C_3 \in \zeta$. Let C_1, $C_2 \in \zeta$ and $C_3 \subset C_1 \cap C_2$, then $\emptyset \subset C_3 \subset C_1 \cap C_2 \Rightarrow intC_3 \subset$ int($C_1 \cap C_2) = int(C_1 \cap C_2)$=$\emptyset$. That implies $C_3 \in \zeta$ and this is a contradiction.
(ii)$\Leftarrow$ Let $\{(X - clA) : intA = \emptyset\}$=$\zeta$ be a filterbase on X and X is resolvable, then $(X - clA) = \emptyset$, $\emptyset \in \zeta$. This is a contradiction. $\square$

Proof of (b): (i)$\Rightarrow$ Let X be resolvable, then by lemma 3 $accA \neq \emptyset$, and $intA = \emptyset$. Suppose $\{accA : intA = \emptyset\}$=$\zeta$ is not a filterbase, then for any $C_3 \subset C_1 \cap C_2 \Rightarrow intC_3 = \emptyset$. This is a contradiction.
(ii)$\Leftarrow$ $\{accA : intA = \emptyset\}$=$\zeta$ is a filterbase on X and X is irresolvable, then $accA = \emptyset$. $\square$

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