Scientiae Mathematicae Japonicae Online, **e-2005**, 281–289

PROPERTY $B(P,\omega)$ AND WEAK $\overline{\theta}$-REFINABILITY OF PRODUCT SPACES

KEIKO CHIBA

Received January 7, 2005

Abstract. In this paper we shall show: (1) Let X be a zero-dimensional metric space and Y be a P-space. If Y has *property* $B(D,\omega)$, then $X \times Y$ has *property* $B(D,\omega)$.

(2) Let X be a regular σ-space and Y be a P-space. If Y has *property* $B(LF,\omega)$, then $X \times Y$ has *property* $B(LF,\omega)$.

(3) Let X be a normal strong Σ-space and Y be a P-space. If Y has *property* $B(LF,\omega)$, then $X \times Y$ has *property* $B(LF,\omega)$.

(4) Let X be a strong Σ-space and Y be a P-space. If Y is weak $\overline{\theta}$-refinable (resp. weak $\overline{\delta\theta}$-refinable), then $X \times Y$ is weak $\overline{\theta}$-refinable. (resp. weak $\overline{\delta\theta}$-refinable).

1. Introduction

Throughout this paper we assume that each space is a Hausdorff space. Each map is assumed to be continuous.

Smith [13] introduced the notion of weak $\overline{\theta}$-refinability and has shown that

θ-refinable $\Rightarrow$ weakly $\overline{\theta}$-refinable $\Rightarrow$ weakly θ-refinable

and the implications are not reversible. And he [15] introduced the notions of *property* $B(D,\omega)$ and *property* $B(LF,\omega)$ and he proved the following:

property $B(D,\omega) \Rightarrow$ weakly $\overline{\theta}$-refinable;

θ-refinable $\Rightarrow$ *property* $B(LF,\omega)$.

It is obvious that

paracompact $\Rightarrow$ *property*$B(D,\omega)$ $\Rightarrow$ *property* $B(LF,\omega)$.

In this paper we shall investigate the conditions for the product space $X \times Y$ has *property* $B(D,\omega)$, *property* $B(LF,\omega)$ and weak $\overline{\theta}$-refinability.

The following results are known.

Theorem A. Suppose X is a Σ-space and Y is a P-space. If X and Y are both paracompact (regular Lindelöf, regular subparacompact, submetacompact (θ-refinable), weakly θ-refinable, weak $\delta\theta$-refinable), then so is $X \times Y$. (The Lindelöf case and paracompact case are proved in Nagami [8], the subparacompact case in Lutzer [5], the submetacompact case in Burke [2, pp.400-401] and the weak θ-refinability case in Yajima [16]. For the weak $\delta\theta$-refinability case, the proof is similar to the case of weak θ-refinability.)

Yajima's theorem is a more general form, i.e., the following.

Theorem B. ([16]). Suppose X is a strong Σ-space and Y is a P-space. If Y is a weakly θ-refinable, then so is $X \times Y$.

Same result for the weak $\delta\theta$-refinability case can be shown similarly.

2000 *Mathematics Subject Classification.* Primary 54B10, 54D20. Secondary 54G20.

Key words and phrases. Property $B(D,\omega)$, Property $B(LF,\omega)$, Property $B(CP,\omega)$, weak $\overline{\theta}$-refinable, P-space, σ-space, Σ-space.

In this paper we shall show that the similar results of Theorem B hold for the case of weak $\overline{\theta}$-refinability and weak $\overline{\delta\theta}$-refinability.

It is known that

X is a metric space $\Rightarrow$ X is a σ-space $\Rightarrow$ X is a Σ-space.

Let Y be a P-space. We consider *property* $B(D,\omega)$ of $X \times Y$ when X is a zero-dimensional metric space and *property* $B(LF,\omega)$ of $X \times Y$ when X is a regular σ-space or a normal Σ-space.

Let Ω be a set. Denote $\Omega^n = \{(\alpha_0,\alpha_1,...,\alpha_{n-1})|\alpha_i \in \Omega, i = 0,...,n-1\}$ for each $n \in \omega, \Omega^{<\omega} = \bigcup_{n\in\omega} \Omega^n$ and $\Omega^\omega = \{(\alpha_0,\alpha_1,...,\alpha_n,...,)|\alpha_n \in \Omega$ for each $n \in \omega\}$. For each $\sigma = (\alpha_0,\alpha_1,...,\alpha_{n-1}) \in \Omega^n$ and $\alpha \in \Omega$, we denote $\sigma \vee \alpha = (\alpha_0,\alpha_1,...,\alpha_{n-1},\alpha)$. For each $\sigma = (\alpha_0,\alpha_1,...,\alpha_n,...) \in \Omega^\omega$, we denote $\sigma \upharpoonright n = (\alpha_0,\alpha_1,...,\alpha_{n-1})$. It is obvious that $\sigma \upharpoonright n \in \Omega^n$.

A space Y is said to be a P-space ([7]) if for any open cover $\{U(\sigma)|\sigma \in \Omega^{<\omega}\}$ of Y where $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$ and $\alpha \in \Omega$, then there is a closed cover $\{K(\sigma)|\sigma \in \Omega^{<\omega}\}$ of X such that

(i) $K(\sigma) \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,

(ii) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} U(\sigma \upharpoonright n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \upharpoonright n) = Y$.

For a space X, dim$X$ denotes the covering dimension of X and X is a zero-dimensional space means dim$X = 0$.

A subset A of X is called a "*clopen*" set if A is both an open set and a closed set of X.

The following lemmas $1 \sim 3$ are well known.

Lemma 1. *If X is a zero-dimensional metric space, then X has a base $\mathcal{B}$ satisfying the following conditions:*

(i) $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, $\mathcal{B}_n$ is a discrete cover of X by clopen sets,

(ii) $\mathcal{B}_n = \{B(\sigma)|\sigma \in \Omega^n\}, B(\sigma) = \bigcup_{\alpha\in\Omega} B(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{B(\sigma \upharpoonright n)|n \in \omega\}$ is a local base of x in X.

Lemma 1 follows from the following.

Theorem C (Katětov [4], Morita [6], or cf. 12.2 Theorem in [10]). A space X is a metric space with dim $X \leq 0$ if and only if X is a subset of a Baire 0-dimensional space.

A collection $\mathcal{F}$ of subsets of X is called a *net* of X if for each $x \in X$ and each open set U, there is an $F \in \mathcal{F}$ such that $x \in F \subset U$.

A space X is called a σ-space if σ-locally finite net ([9], [11]).

Lemma 2. *([9, Theorem 1]). If X is a σ-space, then X has a net $\mathcal{F}$ satisfying the following conditions:*

(i) $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{F}_n$ is a locally finite closed cover of X,

(ii) $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}, F(\sigma) = \bigcup_{\alpha\in\Omega} F(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{F(\sigma \upharpoonright n)|n \in \omega\}$ is a net of x.

A space X is called a Σ-space if X has a Σ-net ([8]).

Lemma 3. *([8, 1.4. Lemma]). If X is a Σ-space, then X has a spectral Σ-net $\mathcal{F}$, i. e., satisfying the following conditions:*

(i) $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{F}_n$ is a locally finite closed cover of X,

(ii) $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}, F(\sigma) = \bigcup_{\alpha\in\Omega} F(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{F(\sigma \upharpoonright n)|n \in \omega\}$ is a K-net of $C(x)$, i. e., if U is an open set in X such that $C(x) \subset U$, then $F(\sigma \upharpoonright n) \subset U$ for some n. Here $C(x) = \bigcap_{n\in\omega} F(\sigma \upharpoonright n)$.

A space X is called a strong Σ-space if X has a Σ-net such that $C(x)$ is compact for each $x \in X$.

The following lemma is obvious.

Lemma 4. *Let us define* $\phi : \omega \times \omega \to \omega$ *by* $\phi(n,m) = n + \frac{(n+m)(n+m+1)}{2}$.

Then ϕ *is a bijection satisfying the conditions:*
if $m', m \in \omega$ *and* $m' < m$, *then* $\phi(n,m') < \phi(n,m)$ *for each* $n \in \omega$.

2. Property $B(D,\omega)$

Definition 1. *(Smith [15]). A space* X *is called to have property* $B(D,\omega)$ *if for every open cover* $\mathcal{G}$ *of* X, *there is a cover* $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$ *of* X *such that* $\mathcal{H}$ *is a refinement of* $\mathcal{G}$ *and satisfies the conditions: for each* $n < \omega$,
$(i)_n$ $\mathcal{H}_n$ *is a discrete collection of closed subsets of* $X \setminus \bigcup_{m<n}(\bigcup \mathcal{H}_m)$,
$(ii)_n$ $\bigcup_{m<n}(\bigcup \mathcal{H}_m)$ *is closed in* X. *Here* $\bigcup_{m<0}(\bigcup \mathcal{H}_m) = \emptyset$.

It is obvious that *property* $B(D,\omega)$ is closed hereditary.

Notation. Let $\mathcal{K}$ be a collection of subsets in X and A a subset of X. Then we denote $\mathcal{K}|A = \{K \cap A | K \in \mathcal{K}\}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets in X. We denote $\mathcal{V} \prec \mathcal{U}$ if $\mathcal{V}$ is a refinement or a partial refinement of $\mathcal{U}$.

Lemma 5. *In Definition 1, the condition (ii) follows from (i).*

Proof. (By induction) . The condition $(ii)_1$ follows from $(i)_0$. Put $A_n = \bigcup_{m<n}(\bigcup \mathcal{H}_m)$. Assume that $(ii)_n$ hold. By $(i)_n$, $\overline{\bigcup \mathcal{H}_n} \cap (X \setminus A_n) = \bigcup \mathcal{H}_n$ and by $(ii)_n$, $\overline{A_n} = A_n$. Therefore $\overline{A_{n+1}} = \overline{(\bigcup \mathcal{H}_n) \cup A_n} = ((\overline{\bigcup \mathcal{H}_n}) \cap (X \setminus A_n)) \cup ((\overline{\bigcup \mathcal{H}_n}) \cap A_n) \cup \overline{A_n} = (\bigcup \mathcal{H}_n) \cup A_n = A_{n+1}$. □

Theorem 1. *Let* X *be a zero-dimensional metric space and* Y *be a* P*-space. If* Y *has property* $B(D,\omega)$, *then* $X \times Y$ *has property* $B(D,\omega)$.

Proof. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ be a base of X satisfying the conditions in Lemma 1. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ is an open cover of $X \times Y$. For each $\sigma \in \Omega^{<\omega}$ and each $\xi \in \Xi$, let $U(\sigma;\xi) = \bigcup\{U | U$ is an open set in $Y, B(\sigma) \times U \subset G_\xi\}$. Then $U(\sigma;\xi)$ is an open set in Y and $B(\sigma) \times U(\sigma;\xi) \subset G_\xi$. Put $U(\sigma) = \bigcup_{\xi\in\Xi} U(\sigma;\xi)$. Then

(1) $\{U(\sigma) | \sigma \in \Omega^{<\omega}\}$ is an open cover of Y.

Proof. Let $y \in Y$. Let us choose a point $x \in X$ and a $\sigma \in \Omega^\omega$ such that $\{\sigma \restriction n; n \in \omega\}$ be a local base of x in X. Since $\mathcal{G}$ is a cover of $X \times Y$, there is a $\xi \in \Xi$ such that $(x,y) \in G_\xi$. Then there are an n and an open set U with $(x,y) \in B(\sigma \restriction n) \times U \subset G_\xi$. By the definition of $U(\sigma \restriction n), U \subset U(\sigma \restriction n)$. Thus $y \in U(\sigma \restriction n)$.

(2) $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$ and $\alpha \in \Omega$.

Proof. This follows from $B(\sigma \vee \alpha) \subset B(\sigma)$.

Since Y is a P-space, there is a closed cover $\{K(\sigma) | \sigma \in \Omega^{<\omega}\}$ of Y such that
(3) $K(\sigma) \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,
(4) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} U(\sigma \restriction n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$.

The following holds.

(5) Let x be an arbitrary element of X and let $\sigma \in \Omega^\omega$ such that $\{B(\sigma \restriction n) : n < \omega\}$ is a local base of x. Then $\bigcup_{n<\omega} U(\sigma \restriction n) = Y$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{U}_\sigma = \{U(\sigma;\xi) | \xi \in \Xi\}$ is an open cover of $U(\sigma)$. Since Y has *property* $B(D,\omega)$ and $K(\sigma)$ is closed in Y, $K(\sigma)$ has *property* $B(D,\omega)$. Therefore there is a cover $\mathcal{K}_\sigma = \bigcup_{m<\omega} \mathcal{K}_{\sigma,m}$ of $K(\sigma)$ such that for each $m < \omega$,

$(i)_\sigma$. $\mathcal{K}_{\sigma,m} \prec \mathcal{U}_\sigma$.

$(ii)_\sigma$. $\mathcal{K}_{\sigma,m}$ is a discrete collection of closed subsets of $K(\sigma) \setminus \bigcup_{i<m}(\bigcup \mathcal{K}_{\sigma,i})$.

We may assume that $\mathcal{K}_{\sigma,m} = \{K(\sigma,m,\xi) | \xi \in \Xi\}$ with $K(\sigma,m,\xi) \subset U(\sigma;\xi)$ for each ξ.

Let $\phi : \omega \times \omega \to \omega$ be the bijection defined in Lemma 4. For each $k = \phi(n,m) \in \omega$ and each $\sigma \in \Omega^n$, let $L(k,\sigma,\xi) = B(\sigma) \times K(\sigma,m,\xi)$ and put $\mathcal{L}_k = \{L(k,\sigma,\xi) | \sigma \in \Omega^n, \xi \in \Xi\}$. Then

(i) $\mathcal{L} = \bigcup_{k<\omega} \mathcal{L}_k$ is a cover of $X \times Y$, $\mathcal{L} \prec \mathcal{G}$.

(ii) For each $k < \omega$, $\mathcal{L}_k | X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$ is a discrete collection of closed subsets of $X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$.

Proof of (i). Let $(x,y) \in X \times Y$. Let $\sigma \in \Omega^\omega$ such that $\{B(\sigma \restriction n) : n \in \omega\}$ is a local base of x. Then, by (4) and (5), $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$. Thus $y \in K(\sigma \restriction n)$ for some $n \in \omega$ and so $(x,y) \in B(\sigma \restriction n) \times K(\sigma \restriction n, m, \xi)$ for some m and ξ. Let us put $k = \phi(n,m)$. Then $(x,y) \in \bigcup \mathcal{L}_k$.

Since $B(\sigma) \times K(\sigma,m,\xi) \subset B(\sigma) \times U(\sigma,\xi) \subset G_\xi$ for each $\sigma \in \Omega^{<\omega}$ and $\xi \in \Xi, \mathcal{L} \prec \mathcal{G}$.

Proof of (ii). Let $k = \phi(n,m)$. Put $H = X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$. We shall prove the following.

(a) $L \cap H$ is closed in H for each $L \in \mathcal{L}_k$.

(b) $\{L \cap H | L \in \mathcal{L}_k\}$ is discrete in H.

Proof of (a). Let $L = B(\sigma) \times K(\sigma,m,\xi), \sigma \in \Omega^n, \xi \in \Xi$. For a moment, to simplify the notation, let us put $B(\sigma) = B, K(\sigma,m,\xi) = K$ and $A = \bigcup_{i<m}(\bigcup \mathcal{K}_{\sigma,i})$. Since K is a closed subset of $K(\sigma) \setminus A, B \times K$ is a closed subset of $B \times K(\sigma) \setminus B \times A$. Therefore $(B \times K) \cap H$ is a closed subset of $(B \times K(\sigma)) \cap H \setminus (B \times A) \cap H$. Since $\phi(n,i) < k$ for each $i < m$, $B \times A \subset \bigcup_{l<k}(\bigcup \mathcal{L}_l)$. Therefore $(B \times A) \cap H = \emptyset$. Thus $(B \times K) \cap H$ is a closed subset of $(B \times K(\sigma)) \cap H$. Hence $L \cap H = (B \times K) \cap H$ is a closed subset of H.

Proof of (b). Let $(x,y) \in H$. Since $\mathcal{B}_n$ is a discrete cover of X, there is only element σ of Ω^n such that $x \in B(\sigma)$. For each $i < m$, since $\phi(n,i) < k, (x,y) \notin L(\phi(n,i),\sigma,\xi) = B(\sigma) \times K(\sigma,i,\xi)$. Thus $y \notin K(\sigma,i,\xi)$ for each $\xi \in \Xi$. Therefore $y \notin \bigcup_{i<m}(\bigcup \mathcal{K}_{\sigma,i})$. If $y \notin K(\sigma)$, there is a neighborhood V of y in Y such that $V \cap K(\sigma) = \emptyset$. If $y \in K(\sigma)$, then $y \in K(\sigma) \setminus \bigcup_{i<m}(\bigcup \mathcal{K}_{\sigma,i})$. By $(ii)_\sigma$, there is a neighborhood V of y in Y such that $V \cap K(\sigma,m,\xi) \neq \emptyset$ for at most one $\xi \in \Xi$. Put $W = B(\sigma) \times V$. Then W is a neighborhood of (x,y) in $X \times Y$ such that $W \cap L \neq \emptyset$ for at most one $L \in \mathcal{L}_k$. □

3. *Property* $B(LF,\omega)$

Definition 2. *(Smith [15]). A space X is called to have property $B(LF,\omega)$ if for every open cover $\mathcal{G}$ of X, there is a cover $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$ of X such that $\mathcal{H}$ is a refinement of $\mathcal{G}$ such that for each $n < \omega$,*

(i) $\mathcal{H}_n$ is a locally finite collection of closed subsets of $X \setminus \bigcup_{m<n}(\bigcup \mathcal{H}_m)$,

(ii) $\bigcup_{m<n}(\bigcup \mathcal{H}_m)$ is closed in X.

It is obvious that *property* $B(LF,\omega)$ is closed hereditary.

Definition 3. *(Chaber [3] or cf. [15]). A space X is called to have property b_1 if for every open cover $\mathcal{G}$ of X, there is a cover $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$ of X such that $\mathcal{H}$ is a refinement of $\mathcal{G}$ such that for each $n < \omega$, $\mathcal{H}_n$ is a locally finite collection of closed subsets of $X \setminus \bigcup_{m<n}(\bigcup \mathcal{H}_m)$.*

Property b_1 is the same notion of *property* $B(LF,\omega)$. This fact is shown by the similar proof of Lemma 5.

Each regular σ-space is a strong Σ-space, each strong Σ-space is submetacompact and each submetacompact space has *property* $B(LF,\omega)$ ([15, Theorem 1.4 (2)]).

Theorem 2. *Let X be a regular σ-space and Y be a P-space. If Y has property $B(LF,\omega)$, then $X\times Y$ has property $B(LF,\omega)$.*

Proof. We shall show that $X\times Y$ has *property* b_1. Let $\mathcal{F}=\bigcup_{n\in\omega}\mathcal{F}_n$ be a net of X satisfying the conditions in Lemma 2. Let $\mathcal{G}=\{G_\xi|\xi\in\Xi\}$ is an open cover of $X\times Y$. For each $\sigma\in\Omega^{<\omega}$ and each $\xi\in\Xi$, let $U(\sigma;\xi)=\bigcup\{U|U$ is an open set in $Y, F(\sigma)\times U\subset G_\xi\}$. Then $U(\sigma;\xi)$ is an open set in Y and $F(\sigma)\times U(\sigma;\xi)\subset G_\xi$. Put $U(\sigma)=\bigcup_{\xi\in\Xi}U(\sigma;\xi)$. Then

(1) $\{U(\sigma)|\sigma\in\Omega^{<\omega}\}$ is an open cover of Y.

(2) $U(\sigma)\subset U(\sigma\vee\alpha)$ for each σ and α.

Since Y is a P-space, there is a closed cover $\{K(\sigma)|\sigma\in\Omega^{<\omega}\}$ of Y such that
(3) $K(\sigma)\subset U(\sigma)$ for each $\sigma\in\Omega^{<\omega}$,
(4) for each $\sigma\in\Omega^{\omega}$, if $\bigcup_{n\in\omega}U(\sigma\restriction n)=Y$, then $\bigcup_{n\in\omega}K(\sigma\restriction n)=Y$.

The following holds.

(5) Let x be an arbitrary element of X and let $\sigma\in\Omega^{\omega}$ such that $\{F(\sigma\restriction n):n<\omega\}$ is a net of x. Then $\bigcup_{n<\omega}U(\sigma\restriction n)=Y$.

For each $\sigma\in\Omega^{<\omega}$, $\mathcal{U}_\sigma=\{U(\sigma;\xi)|\xi\in\Xi\}$ is an open cover of $U(\sigma)$. Since Y has *property* $B(LF,\omega)$ and $K(\sigma)$ is closed in Y, $K(\sigma)$ has *property* $B(LF,\omega)$. Therefore there is a cover $\mathcal{K}_\sigma=\bigcup_{m<\omega}\mathcal{K}_{\sigma,m}$ of $K(\sigma)$ such that for each $m<\omega$,

$(i)_\sigma$ $\mathcal{K}_{\sigma,m}=\{K(\sigma,m,\xi)|\xi\in\Xi\}$ such that $K(\sigma,m,\xi)\subset U(\sigma,m,\xi)$ for each ξ,

$(ii)_\sigma$ $\mathcal{K}_{\sigma,m}$ is a locally finite collection of closed subsets of $K(\sigma)\setminus\bigcup_{i<m}(\bigcup\mathcal{K}_{\sigma,i})$.

Let $\phi:\omega\times\omega\to\omega$ be the bijection defined in Lemma 4. For each $k=\phi(n,m)\in\omega$ and each $\sigma\in\Omega^n$, let $L(k,\sigma,\xi)=F(\sigma)\times K(\sigma,m,\xi)$ and put $\mathcal{L}_k=\{L(k,\sigma,\xi)|\sigma\in\Omega^n,\xi\in\Xi\}$. Then

(i) $\mathcal{L}=\bigcup_{k<\omega}\mathcal{L}_k$ is a cover of $X\times Y$ and $\mathcal{L}\prec\mathcal{G}$,

(ii) for each $k<\omega$, $\mathcal{L}_k|X\times Y\setminus\bigcup_{l<k}(\bigcup\mathcal{L}_l)$ is a locally finite collection of closed subsets of $X\times Y\setminus\bigcup_{l<k}(\bigcup\mathcal{L}_l)$.

Proof of (i). This proof is similar to that of (i) in Theorem 1.

Proof of (ii). Put $H=X\times Y\setminus\bigcup_{l<k}(\bigcup\mathcal{L}_l)$ where $k=\phi(n,m)\in\omega$. We shall prove the following.
(a) $L\cap H$ is closed in H for each $L\in\mathcal{L}_k$.
(b) $\{L\cap H|L\in\mathcal{L}_k\}$ is locally finite in H.

Proof of (a). This proof is similar to that of (a) in Theorem 1.

Proof of (b). Let $(x,y)\in H$. Since $\mathcal{F}_n$ is locally finite in X, there is a neighborhood U of x and a finite subset $\{\sigma_j|j=1,2,...,p\}$ of Ω^n such that $U\cap F(\sigma)\neq\emptyset\iff\sigma\in\{\sigma_j|j=1,2,...,p\}$ and $x\in F(\sigma_j)$ for each $j=1,2,...,p$. For each $i<m$, since $\phi(n,i)<k,(x,y)\notin L(\phi(n,i),\sigma,\xi)=F(\sigma)\times K(\sigma,i,\xi)$ for each $\sigma\in\Omega^n$. Since $x\in F(\sigma_j),y\notin K(\sigma_j,i,\xi)$ for each $\xi\in\Xi$. Therefore $y\notin\bigcup_{i<m}\mathcal{K}_{\sigma_j,i}$. If $y\notin K(\sigma_j)$, there is a neighborhood V_j of y in Y such that $V_j\cap K(\sigma_j)=\emptyset$. If $y\in K(\sigma_j)$, then $y\in K(\sigma_j)\setminus\bigcup_{i<m}\mathcal{K}_{\sigma_j,i}$. By $(ii)_{\sigma_j}$, there is a neighborhood V_j of y in Y such that $V_j\cap K(\sigma_j,m,\xi)\neq\emptyset$ is at most finite finite number of $\xi\in\Xi$. Put $V=\bigcap_{j=1}^{p}V_j$ and $W=U\times V$. Then W is a neighborhood of (x,y) in $X\times Y$ such that $W\cap L\neq\emptyset$ for at most finite number of $L\in\mathcal{L}_k$. $\square$

Theorem 3. *Let X be a normal strong Σ-space and Y be a P-space. If Y has property $B(LF,\omega)$, then $X \times Y$ has property $B(LF,\omega)$.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a spectral Σ-net of X, i.e., for some set Ω, $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}$ is a locally finite closed cover of X for each $n \in \omega$ satisfying the conditions in Lemma 3.

We shall show that $X \times Y$ has *property* b_1. Let $\mathcal{G} = \{G_\xi|\xi \in \Xi\}$ is an open cover of $X \times Y$. For each $\sigma \in \Omega^{<\omega}$, let $\mathcal{W}_\sigma$ is the maximal family of $U_\lambda \times V_\lambda$ satisfying the following conditions:
(1) U_λ is an open set in X, $U_\lambda \supset F(\sigma)$,
(2) V_λ is an open set in Y,
(3) there is a finite open cover $\mathcal{U}_{\sigma,\lambda}$ of U_λ such that $\{U \times V_\lambda|U \in \mathcal{U}_{\sigma,\lambda}\} \prec \mathcal{G}$.

Put $\mathcal{W}_\sigma = \{U_\lambda \times V_\lambda|\lambda \in \Lambda_\sigma\}$. Since $\mathcal{U}_{\sigma,\lambda}$ is a finite open cover of $F(\sigma)$ and $F(\sigma)$ is normal, there is a finite closed cover $\mathcal{F}_{\sigma,\lambda} = \{F_U|U \in \mathcal{U}_{\sigma,\lambda}\}$ of $F(\sigma)$ such that $F_U \subset U$ for each $U \in \mathcal{U}_{\sigma,\lambda}$.

For each $\sigma \in \Omega^{<\omega}$, put $V(\sigma) = \bigcup_{\lambda\in\Lambda_\sigma} V_\lambda$. Then

(4) Let $\sigma \in \Omega^\omega$. If $\{F(\sigma \restriction n)|n \in \omega\}$ is a K-net of $C(x)$ for a point $x \in X$, then $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$.

Proof. Let y be an arbitrary element of Y. Then $(x,y) \in G_\xi$ for some $\xi \in \Xi$. Then, since $C(x)$ is compact, there is a finite set $\{U_i|i = 1,2,...,k\}$ of open sets in X and an open set V of Y such that $C(x) \subset \cup_{i=1}^k U_i, y \in V, \{U_i \times V|i = 1,2,...,k\} \prec \mathcal{G}$. Then there is an n such that $C(x) \subset F(\sigma \restriction n) \subset U$. By the definition of $V(\sigma \restriction n), V \subset V(\sigma \restriction n)$. Thus $y \in V(\sigma \restriction n)$.

(5) $V(\sigma) \subset V(\sigma \vee \alpha)$ for each $\sigma \in \Omega^{<\omega}$ and each $\alpha \in \Omega$.

Since Y is a P-space, there is a closed cover $\{K(\sigma)|\sigma \in \Omega^{<\omega}\}$ of Y such that
(6) $K(\sigma) \subset V(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,
(7) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{V}_\sigma = \{V_\lambda|\lambda \in \Lambda_\sigma\}$ is an open cover of $K(\sigma)$. Since Y has *property* $B(LF,\omega)$ and $K(\sigma)$ is closed in Y, $K(\sigma)$ has *property* $B(LF,\omega)$. Therefore there is a cover $\mathcal{K}_\sigma = \bigcup_{m<\omega} \mathcal{K}_{\sigma,m}$ of $K(\sigma)$ such that for each $m < \omega$,

$(i)_\sigma$ $\mathcal{K}_{\sigma,m} = \{K(\sigma,m,\lambda)|\lambda \in \Lambda_\sigma\}$ such that $K(\sigma,m,\lambda) \subset V_\lambda$ for each $\lambda \in \Lambda_\sigma$,

$(ii)_\sigma$ $\mathcal{K}_{\sigma,m}$ is a locally finite collection of closed subsets of $K(\sigma) \setminus \bigcup_{i<m}(\bigcup \mathcal{K}_{\sigma,i})$.

Let $\phi : \omega \times \omega \to \omega$ be the bijection defined in Lemma 4. For each $k = \phi(n,m) \in \omega$, let $\mathcal{L}_k = \{F \times K(\sigma,m,\lambda)|\sigma \in \Omega^n, \lambda \in \Lambda_\sigma, F \in \mathcal{F}_{\sigma,\lambda}\}$. Then

(i) $\mathcal{L} = \bigcup_{k<\omega} \mathcal{L}_k$ is a cover of $X \times Y$ and $\mathcal{L} \prec \mathcal{G}$,

(ii) for each $k < \omega$, $\mathcal{L}_k|X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$ is a locally finite collection of closed subsets of $X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$.

Proof of (i). Let $(x,y) \in X \times Y$. Let $\sigma \in \Omega^\omega$ such that $\{F(\sigma \restriction n) : n \in \omega\}$ is a K-net of $C(x)$. Then, by (4) and (7), $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$. Thus $y \in K(\sigma \restriction n, m, \lambda)$ for some $n \in \omega$ and $m \in \omega, \lambda \in \Lambda_\sigma$. Since $\bigcup \mathcal{F}_{\sigma\restriction n,\lambda} = F(\sigma \restriction n)$, $x \in F$ for some $F \in \mathcal{F}_{\sigma\restriction n,\lambda}$. Thus $(x,y) \in F \times K(\sigma \restriction n, m, \lambda)$. Therefore $(x,y) \in \bigcup \mathcal{L}_k$.

Let $L = F \times K(\sigma,m,\lambda) \in \mathcal{L}_k$ where $k = \phi(n,m)$. Then $F \subset U$ for some $U \in \mathcal{U}_{\sigma,\lambda}$ and $K(\sigma,m,\lambda) \subset V_\lambda$. By (3), $U \times V_\lambda \subset G_\xi$ for some $\xi \in \Xi$. Thus $L \subset G_\xi$.

Proof of (ii). Put $H = X \times Y \setminus \bigcup_{l<k}(\bigcup \mathcal{L}_l)$ where $k = \phi(n,m)$. We shall prove the following.
(a) $L \cap H$ is closed in H for each $L \in \mathcal{L}_k$.

(b) $\{L\cap H | L\in\mathcal{L}_k\}$ is locally finite in H.

Proof of (a). This proof is similar to that of (a) in Theorem 1.

Proof of (b). Let $(x,y)\in H$. Since $\mathcal{F}_n$ is locally finite in X, there is a neighborhood U of x and a finite subset $\{\sigma_j|j=1,2,...,p\}$ of Ω^n such that $U\cap F(\sigma)\neq\emptyset \iff \sigma\in\{\sigma_j|j=1,2,...,p\}$ and $x\in F(\sigma_j)$ for each $j=1,2,...,p$. For each $i<m$, since $\phi(n,i)<k, (x,y)\notin L(\phi(n,i),\sigma,\lambda)=F(\sigma)\times K(\sigma,i,\lambda)$ for each $\sigma\in\Omega^n$. Since $x\in F(\sigma_j), y\notin K(\sigma_j,i,\lambda)$ for each $\lambda\in\Lambda_\sigma$. Therefore $y\notin\bigcup_{i<m}\mathcal{K}_{\sigma_j,i}$. If $y\notin K(\sigma_j)$, there is a neighborhood V_j of y in Y such that $V_j\cap K(\sigma_j)=\emptyset$. If $y\in K(\sigma_j)$, then $y\in K(\sigma_j)\setminus\bigcup_{i<m}\mathcal{K}_{\sigma_j,i}$. By $(ii)_{\sigma_j}$, there is a neighborhood V_j of y in Y such that $\Lambda_j=\{\lambda\in\Lambda_{\sigma_j}|V_j\cap K(\sigma_j,m,\xi)\neq\emptyset\}$ is a finite set. Put $V=\bigcap_{j=1}^p V_j$ and $W=U\times V$. Then W is a neighborhood of (x,y) in $X\times Y$ such that $W\cap L\neq\emptyset$ for at most finite number of $L\in\mathcal{L}_k$.

To show this, let L be an arbitrary element of $\mathcal{L}_k$. Then $L=F\times K(\sigma,m,\lambda)$ for each $\sigma\in\Omega^n$, $\lambda\in\Lambda_\sigma$ and $F\in\mathcal{F}_{\sigma,\lambda}$. Suppose $W\cap L\neq\emptyset$. Then $\sigma=\sigma_j$ for some $j=1,2,...,p$. And $V_j\cap K(\sigma_j,m,\lambda)\neq\emptyset$. Therefore $\lambda\in\Lambda_{\sigma_j}$. Since $\mathcal{F}_{\sigma_j,\lambda}$ is finite, such L is at most finite. □

Remark. Price and Smith [12] introduced the notion of *property*$B(CP,\omega)$ which is weaker than *property*$B(LF,\omega)$.

Definition 4. *([12]). A space X is called to have property $B(CP,\omega)$ if for every open cover $\mathcal{G}$ of X, there is a cover $\mathcal{H}=\bigcup_{n<\omega}\mathcal{H}_n$ of X such that $\mathcal{H}$ is a refinement of $\mathcal{G}$ such that for each $n<\omega$,*

(i) $\mathcal{H}_n$ is a closure preserving collection of closed subsets of $X\setminus\bigcup_{m<n}(\bigcup\mathcal{H}_m)$,

(ii) $\bigcup_{m<n}(\bigcup\mathcal{H}_m)$ is closed in X.

The condition (ii) follows from (i). It is obvious that *property* $B(CP,\omega)$ is closed hereditary.

The following theorem is shown by the similar proof of Theorem 2.

Theorem 4. *Let X be a regular σ-space and Y be a P-space. If Y has property $B(CP,\omega)$, then $X\times Y$ has property $B(CP,\omega)$.*

4. Weak $\overline{\theta}$-refinability and weak $\overline{\delta\theta}$-refinability

A space X is said to be *weakly $\overline{\theta}$-refinable* ([13]) (resp. *$\overline{\delta\theta}$-refinable* ([14])) if for any open cover $\mathcal{G}$ of X there is an open refinement $\mathcal{H}=\bigcup_{n\in\omega}\mathcal{H}_n$ of $\mathcal{G}$ such that if $x\in X$ there is some n with $1\leq \text{ord}(x,\mathcal{H}_n)<\omega$ (resp. $1\leq\text{ord}(x,\mathcal{H}_n)\leq\omega$) and $\{\bigcup\mathcal{H}_n|n\in\omega\}$ is point finite at each $x\in X$. Such cover $\mathcal{H}$ is said to be a weak $\overline{\theta}$ cover (resp. $\overline{\delta\theta}$ cover).

For an open cover $\mathcal{G}$ of X, define $\mathcal{G}^{<\omega}=\{\bigcup\mathcal{G}'|\mathcal{G}'$ is a finite subfamily of $\mathcal{G}\}$.

The following lemma can be easily proved.

Lemma 6. *Let $\mathcal{G}$ be an open cover of X. If $\mathcal{G}^{<\omega}$ has an open refinement which is a weak $\overline{\theta}$-cover (resp. a weak $\overline{\delta\theta}$-cover), then $\mathcal{G}$ has an open refinement which is a weak $\overline{\theta}$-cover (resp. a weak $\overline{\delta\theta}$-cover).*

Theorem 5. *Let X be a strong Σ-space and Y be a P-space.*
(a) If Y is weakly $\overline{\theta}$-refinable, then $X\times Y$ is weakly $\overline{\theta}$-refinable.
(b) If Y is weakly $\overline{\delta\theta}$-refinable, then $X\times Y$ is weakly $\overline{\delta\theta}$-refinable.

We only give the proof of part (b).

Proof of (b). Let $\mathcal{G}=\{G_\xi|\xi\in\Xi\}$ be an open cover of $X\times Y$. Let us define $\mathcal{F},\mathcal{W}_\sigma,\mathcal{U}_{\sigma,\lambda},V(\sigma)$ and $K(\sigma)$ as in the proof of Theorem 3.

Let us put $M_n = \bigcup\{F(\sigma) \times K(\sigma) | \sigma \in \Omega^n\}$ for each $n \in \omega$. Then
(8) M_n is a closed subset of $X \times Y$ and $X \times Y = \bigcup_{n\in\omega} M_n$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{V}_\sigma = \{V_\lambda | \lambda \in \Lambda_\sigma\}$ is a collection of open sets in Y, cover $K(\sigma)$ and $\mathcal{V}_\sigma' = \mathcal{V}_\sigma \cup \{Y \setminus K(\sigma)\}$ is an open cover of Y.

Since Y is weakly $\overline{\delta\theta}$-refinable, there is an open cover $\mathcal{K}_\sigma' = \bigcup_{m\in\omega} \mathcal{K}_{\sigma,m}'$ such that
$(i)_\sigma$ for each $y \in Y$, there is an m_y with $0 <$ord$(y, \mathcal{K}_{\sigma,m_y}') \leq \omega$,
$(ii)_\sigma$ $y \in \bigcup \mathcal{K}_{\sigma,m}'$ for at most finitely many $m \in \omega$.

Put $\mathcal{K}_{\sigma,m} = \{K \in \mathcal{K}_{\sigma,m}' | K \cap K(\sigma) \neq \emptyset\}$ and $\mathcal{K}_\sigma = \bigcup_{m\in\omega} \mathcal{K}_{\sigma,m}$. Then $\mathcal{K}_{\sigma,m}$ are collections of open sets in Y, $\mathcal{K}_\sigma$ covers $K(\sigma)$, and
(9) for each $y \in Y$, there is an m_y with ord$(y, \mathcal{K}_{\sigma,m_y}) \leq \omega$; for each $y \in K(\sigma)$, there is an m_y with $0 <$ord$(y, \mathcal{K}_{\sigma,m_y}) \leq \omega$. And $y \in \bigcup \mathcal{K}_{\sigma,m}$ for at most finitely many $m \in \omega$.

We can represent $\mathcal{K}_{\sigma,m} = \{K_{\sigma,m,\lambda} | \lambda \in \Lambda_\sigma\}$ with $K_{\sigma,m,\lambda} \subset V_\lambda$ for each λ.

Let Ω^n well order by $\prec$ and put $\Gamma_{n,k} = \{(\sigma_0, \sigma_1, ..., \sigma_{k-1}) | \sigma_0, \sigma_1, ..., \sigma_{k-1} \in \Omega^n, \sigma_0 \prec \sigma_1 \prec ... \prec \sigma_{k-1}\}$ for each $k \in \omega$.

For each $\overline{m} = (m_0, m_1, ..., m_{k-1}) \in \omega^k$ and $\tau = (\sigma_0, \sigma_1, ..., \sigma_{k-1}) \in \Gamma_{n,k}$, put $\mathcal{L}(n, \overline{m}, \tau) = \{\bigcup_{i<k}((U_{\lambda_i} \setminus A_\tau) \times K_{\sigma_i,m_i,\lambda_i}) \setminus \bigcup_{j<n} M_j | \lambda_i \in \Lambda_{\sigma_i}, i < k\}$ where $A_\tau = \bigcup\{F(\sigma) | \sigma \in \Omega^n \setminus \{\sigma_i | i < k\}\}$. For each $n \in \omega$ and each $\overline{m} \in \omega^k$, let $\mathcal{L}(n, \overline{m}) = \bigcup\{\mathcal{L}(n, \overline{m}, \tau) | \tau \in \Gamma_{n,k}\}$.

Then $\mathcal{L} = \bigcup\{\mathcal{L}(n, \overline{m}) | n \in \omega, \overline{m} \in \omega^{<\omega}\}$ is an open refinement of $\mathcal{G}^{<\omega}$ and a weak $\overline{\delta\theta}$-cover of $X \times Y$.

It is obvious that each element of $\mathcal{L}$ is an open set of $X \times Y$. For each $i < k$, $(U_{\lambda_i} \setminus A_\tau) \times K_{\sigma_i,m_i,\lambda_i} \subset U_{\lambda_i} \times V_{\lambda_i} \subset G_i'$ for some $G_i' \in \mathcal{G}^{<\omega}$. Thus $\bigcup_{i<k}((U_{\lambda_i} \setminus A_\tau) \times K_{\sigma_i,m_i,\lambda_i}) \subset \bigcup_{i<k} Gi' \in \mathcal{G}^{<\omega}$.

It is sufficient to prove the following.
(10) For each $(x, y) \in X \times Y$, there are an $n \in \omega$ and $\overline{m} \in \omega^{<\omega}$ such that $0 <$ ord$((x, y), \mathcal{L}(n, \overline{m})) \leq \omega$.
(11) For each $(x, y) \in X \times Y$, $(x, y) \in \bigcup \mathcal{L}(n, \overline{m})$ for finitely many n and $\overline{m}$.

Proof of (10). Let $(x, y) \in X \times Y$. Let us choose an $n \in \omega$ with $(x, y) \in M_n \setminus \cup_{j<n} M_j$. Since $\mathcal{F}_n$ is locally finite, there is a finite subset $\{\sigma_i | i = 0, 1, ..., k-1\}$ of Ω^n such that $x \in F(\sigma) \iff \sigma \in \{\sigma_i | i = 0, 1, ..., k-1\}$. We may assume that $(x, y) \in F(\sigma_0) \times K(\sigma_0)$. Since $y \in K(\sigma_0)$, there is an m_0 such that $0 <$ ord$(y, \mathcal{K}_{\sigma_0,m_0}) \leq \omega$. There are $m_i, i = 1, ..., k-1$ such that ord$(x, \mathcal{K}_{\sigma_i,m_i}) \leq \omega$. Put $\overline{m} = (m_0, m_1, ..., m_{k-1})$ and $\tau = (\sigma_0, \sigma_1, ..., \sigma_{k-1})$. Then
(10-1) $0 <$ ord$((x, y), \mathcal{L}(n, \overline{m}, \tau)) \leq \omega$.
(10-2) ord$((x, y), \mathcal{L}(n, \overline{m}, \tau')) = 0$ for each $\tau' \in \Gamma_{n,k}$ with $\tau' \neq \tau$.

Proof of (10-1). Let us choose λ_0 with $y \in K_{\sigma_0,m_0,\lambda_0}$. Since $F(\sigma_0) \subset U_{\lambda_0}, x \in U_{\lambda_0} \setminus A_\tau$. Therefore $(x, y) \in (U_{\lambda_0} \setminus A_\tau) \times K_{\sigma_0,m_0,\lambda_0} \subset L$ for some $L \in \mathcal{L}(n, \overline{m}, \tau)$. Thus ord$((x, y), \mathcal{L}(n, \overline{m}, \tau)) > 0$.

Let $L \in \mathcal{L}(n, \overline{m})$ with $(x, y) \in L$. Then there are an $i \in \{0, 1, ..., k-1\}$ and a $\lambda_i \in \Lambda_{\sigma_i}$ such that $(x, y) \in U_{\lambda_i} \times K_{\sigma_i,m_i,\lambda_i}$. Such λ_i are at most countable. Hence ord$((x, y), \mathcal{L}(n, \overline{m}, \tau)) \leq \omega$.

Proof of (10-2). Let $\tau' = (\sigma_0', \sigma_1', ..., \sigma_{k-1}') \in \Gamma_k^n$. If $\tau \neq \tau'$, then there is an i such that $\sigma_i \notin \{\sigma_i' | i = 0, 1, ..., k-1\}$. Thus $F(\sigma_i) \cap A_\tau' = \emptyset$. Since $x \in F(\sigma_i), x \notin A_\tau'$ and so $(x, y) \notin \bigcup \mathcal{L}(n, \overline{m}, \tau')$.

Proof of (11). Let us choose an n with $(x, y) \in M_n \setminus \cup_{j<n} M_j$. For each $l \leq n$, let $x(l) = \{\sigma \in \Omega^l | x \in F(\sigma)\}$. Then $x(l)$ is finite. For each $\sigma \in x(l)$, there is a finite set $m(\sigma)$ of ω such that $y \in \bigcup \mathcal{K}_{\sigma,m} \iff m \in m(\sigma)$. Put $\Lambda(l) = \prod\{m(\sigma) | \sigma \in x(l)\}$. Then, if $(x, y) \in \bigcup \mathcal{L}(l, \overline{m})$, then $l \leq n$ and $\overline{m} \in \Lambda(l)$. Since $| \cup_{j<n} \Lambda(l)| < \omega$, such $\overline{m}$ are finite. □

It is known that any weakly $\delta\theta$-refinable, countably compact space is compact (cf. p. 414 in [2]). Therefore any weakly $\overline{\delta\theta}$-refinable, countably compact space is compact. Thus we obtain the following.

Corollary. *Let X be a Σ-space and Y be a P-space.*
(a) If X and Y are both weakly $\overline{\theta}$-refinable, then $X \times Y$ is weakly $\overline{\theta}$-refinable.
(b) If X and Y are both weakly $\overline{\delta\theta}$-refinable, then $X \times Y$ is weakly $\overline{\delta\theta}$-refinable.

References

[1] H. R. Bennet and D. J. Lutzer, A note on weak θ-refinability, General Topology and Appl., **2** (1972), 49-54.

[2] D. K. Burke, Covering properties, Handbook of Set Thoretic Topology, chapter 9, K. Kunen and J. Vaughan, editors, Horth-Holland, Amsterdam (1984), 347 - 422.

[3] J. Chaber, On θ-refinability of strict p-spaces, (1st version) (unpublished).

[4] M. Katětov, On the dimension of nonseparable spaces I (in Russian), Czechoslovak Math. J., **8** (1952), 333 - 368.

[5] D. J. Lutzer, Another property of the Sorgenfrey Line, Compositio Math., **24** (1972), 359 - 363.

[6] K. Morita, Normal families and dimension theory for metric spaces, Math. Ann., **128** (1954), 350 - 362.

[7] K. Morita, Products of normal spaces with metric spaces, Math. Ann., **154** (1964), 365 - 382.

[8] K. Nagami, Σ-spaces, Fund. Math., **65** (1969), 169 - 192.

[9] K. Nagami, σ-spaces and product spaces, Math. Ann., **181** (1969), 109 - 118.

[10] K. Nagami, Dimension Theory, Academic Press, New York and London, (1970).

[11] A. Okuyama, Some generalizations of metric spaces, their metrization theorems and product spaces, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A., **9** (1967), 236 -254.

[12] R. H. Price and J. C. Smith, Applications of $B(P,\alpha)$-refinability for generalized collectionwise normal spaces, Proc. Japan Acad. Ser. A Math. Sci. **65**, No. 7 (1989), 249-252.

[13] J. C. Smith, Properties of weak $\overline{\theta}$-refinable spaces, Proc. Amer. Math. Soc., **53**, No. 2, (1975), 511-517.

[14] J. C. Smith, Applications of shrinkable covers, Proc. Amer. Math. Soc., **73**, No. 3, (1979), 379 - 387.

[15] J. C. Smith, Irreducible spaces and property b_1, Topology Proc., **5**, (1980), 187-200.

[16] Y. Yajima, Special refinements and their applications on products, Topology and its Appl., **104**, (2000), 293-308.

Department of Mathematics, Faculty of Science, Shizuoka University, Ohya, Surugaku, Shizuoka, 422-8529 Japan

E-mail address: `smktiba@ipc.shizuoka.ac.jp`