POLYNOMIAL HULLS OF GRAPHS ON THE TORUS IN $\mathbf{C^2}$

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Received May 24, 2005; revised July 15, 2005

ABSTRACT. We describe the polynomial hulls of graphs on the torus which are defined by the complex conjugate functions of polynomials in $\mathbb{C}^2$.

1. Introduction. Let X be a compact subset in $\mathbb{C}^N$ and $\hat{X}$ the polynomial hull of X. We denote by $C(X)$ the Banach algebra of all continuous functions on X with sup-norm $\| \|_X$ and by $P(X)$ the closure in $C(X)$ of the polynomials in the coordinates.

Let $p(z,w)$ be an arbitrary polynomial in $\mathbb{C}^2$ and f the restriction of the complex conjugate of p to the unit torus $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$. Let $G(f)$ denote the graph in $\mathbb{C}^3$ of f on $\mathbb{T}^2$, i.e.,

$$G(f) = \{(z,w,f(z,w)) \in \mathbb{C}^3 : (z,w) \in \mathbb{T}^2\}.$$

H. Alexander([1]) and P. Ahern - W. Rudin ([2]) studied the structure of polynomial hulls of graphs on the unit sphere in $\mathbb{C}^n$. In this paper we consider the structure of polynomial hulls of graphs on $\mathbb{T}^2$ which are defined by the complex conjugates of polynomials in $\mathbb{C}^2$.

Assume that the degrees of $p(z,w) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} z^i w^j$ in z and w respectively are m and n. We consider a polynomial $k(z,w) = \sum_{i=0}^m \sum_{j=0}^n \overline{a_{ij}} z^{m-i} w^{n-j}$ and rational function $h(z,w) = z^{-m} w^{-n} k(z,w)$. We have, for $(z,w) \in \mathbb{T}^2$,

$$\sum_{i=0}^m \sum_{j=0}^n \overline{a_{ij}} \frac{1}{z^i} \frac{1}{w^j} = \frac{1}{z^m w^n} k(z,w) = h(z,w)$$

We set

$$\Delta(z,w) = \begin{vmatrix} \dfrac{\partial p}{\partial z}(z,w) & \dfrac{\partial p}{\partial w}(z,w) \\ \dfrac{\partial h}{\partial z}(z,w) & \dfrac{\partial h}{\partial w}(z,w) \end{vmatrix}.$$

We can write as a product

$$\Delta(z,w) = \frac{1}{z^{m+1} w^{n+1}} \prod_{i=1}^t q_i(z,w)^{n_i}$$

where $q_i(z,w)$ are irreducible polynomials. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, $\mathbb{T}$ its boundary and $\mathbb{D}^2$ the open unit polydisk in $\mathbb{C}^2$. For each $q_i(z,w)$ put

$$Z(q_i) = \{(z,w) \in \mathbb{C}^2 : q_i(z,w) = 0\},$$

$$Q_i = Z(q_i) \cap \mathbb{T}^2, \quad R_i = Z(q_i) \cap \overline{\mathbb{D}^2}.$$

We put $L = (\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{D}})$ and

$$V = \{(z,w) \in \overline{\mathbb{D}^2} \setminus (\mathbb{T}^2 \cup L) : \overline{p(z,w)} = h(z,w)\}.$$

2000 *Mathematics Subject Classification.* Primary 32E20.
Key words and phrases. Polynomial hulls.

Let $[z,w,f;\mathbb{T}^2]$ be the uniform algebra genarated by the coodinate functions z,w and f on T^2. Our result is that the polynomial hull of the graph $G(f)$ can be deteremined as follows.

Theorem. Assume that $\Delta(z,w)\not\equiv 0$ on $\mathbb{D}^2\setminus L$. We put

$$J=\{i\in\{1,2,\cdots,t\}:\emptyset\neq Q_i\neq\widehat{Q_i},\ \widehat{Q_i}\setminus(\mathbb{T}^2\cup L)\subset V\}.$$

(a) If $J\neq\emptyset$, then we have $\widehat{G(f)}=\bigcup_{i\in J}\{(z,w,\overline{p(z,w)}):(z,w)\in\widehat{Q_i}\}\cup G(f)$.

In this case $\overline{p(z,w)}=\overline{c_i}$ (*constant*) on $\widehat{Q_i}$.

(b) If $J=\emptyset$, then we have

$$\widehat{G(f)}=G(f),\ \textit{and}\ [z,w,f;\mathbb{T}^2]=C(\mathbb{T}^2).$$

2. Facts and lemmas. Let M be a C^∞ real submanifold of an open set U in $\mathbb{C}^N$. For a point $\eta\in M$ we denote by $T_\eta M$ the real tangent space of M at η. M is called totally real at η if $T_\eta M$ contains no non-trivial complex subspaces. M is called totally real if M is totally real at every point of M. For a subset S of $\mathbb{C}^2$ and a continuous function g on S, we denote by $G(g;S)$ the graph of g on S, i.e.,

$$G(g;S)=\{(z,w,g(z,w))\in\mathbb{C}^3:(z,w)\in S\}.$$

When M is a totally real submanifold of U in $\mathbb{C}^2$ and g is a C^∞ function in U, it is known that the graph $G(g;M)$ is totally real. For the graph $G(f)=G(\bar{p};\mathbb{T}^2)$ we have that $\widehat{G(f)}$ is connected and so it does not contain any isolated points, since the polynomial hull of a compact connected set is connected. We need several facts and lemmas to decide the polynomial hull of $\widehat{G(f)}$.

Theorem 2.1. ([4], [7]). Let M be a C^∞ totally real submanifold of U in $\mathbb{C}^N$.

(a) If X is a compact polynomialy convex subset of M, then $P(X)=C(X)$.

(b) For a point $\eta\in M$ there exsists a small ball B_0 centered at η such that $\bar{B}_0\cap M$ is polynomially convex.

Lemma 2.2. ([5]). If (z^0,w^0) is a point in V with $\Delta(z^0,w^0)\neq 0$, then there is an open ball B_0 centered at (z^0,w^0) such that $B_0\cap V$ is totally real in B_0.

Lemma 2.3. ([5]). Let X be a compact connected subset of $\mathbb{C}^N$ and U an open subset of $\mathbb{C}^N$ with $U\cap X=\emptyset$. If $\hat{X}\cap U$ is contained in a totally real submanifold M of U, then we have $\hat{X}\cap U=\emptyset$.

The proof of next lemma is obtained by the same way ([2]) in the case of the unit ball.

Lemma 2.4. Let g be a continuous function on $\mathbb{T}^2$. If $(z^0,w^0)\in\mathbb{T}^2$ and $(z^0,w^0,\zeta^0)\in\widehat{G(g;\mathbb{T}^2)}$, then $\zeta^0=g(z^0,w^0)$.

Next lemma is a special case of Lemma 1 in [6]. By using the results of uniform algebras it is also proved as follows.

Lemma 2.5. Let g_1 and g_2 be holomorphic functions on $\overline{\mathbb{D}^2}$ and $f = (\bar{g}_1 + g_2)|_{\mathbb{T}^2}$. Then we have

$$\widehat{G(f)} \subset G(\bar{g}_1 + g_2; \overline{\mathbb{D}^2}).$$

Proof. Let $A = [z, w, \bar{g}_1 + g_2; \mathbb{T}^2] = [z, w, \bar{g}_1; \mathbb{T}^2]$ and M_A the maximal ideal space of A. We denote by X the joint spectrum of $z, w, \bar{g}_1 + g_2$. Since a point evaluation of $\mathbb{T}^2$ belongs M_A, $G(f)$ is contained in X, and so $\widehat{G(f)} \subset \hat{X} = X$ (cf.[3]). For a point (z_0, w_0, ζ_0) in $\widehat{G(f)}$ there is a $\varphi \in M_A$ such that $z_0 = \varphi(z)$, $w_0 = \varphi(w)$ and $\zeta_0 = \varphi(\bar{g}_1 + g_2)$. Then $|z_0| = |\varphi(z)| \leq \|z\|_{\mathbb{T}^2} = 1$ and similarly $|w_0| \leq 1$. By using the polynomial approximation of g_i we have that $\varphi(g_i) = g_i(z_0, w_0),\ i = 1, 2$. Let μ be the representing measure on $\mathbb{T}^2$ for φ. Then

$$\varphi(\bar{g}_1) = \int_{\mathbb{T}^2} \bar{g} d\mu = \overline{\int_{\mathbb{T}^2} g_1 d\mu} = \overline{\varphi(g_1)}.$$

Thus we have that $\varphi(\bar{g}_1 + g_2) = \overline{\varphi(g_1)} + \varphi(g_2) = \overline{g_1(z_0, w_0)} + g_2(z_0, w_0)$ and (z_0, w_0, ζ_0) is contained in $G(\bar{g}_1 + g_2; \overline{\mathbb{D}^2})$.

□

3. Proof of Theorem. @ We write $I = \{1, 2, \cdots, t\}$,

$$E_i = \{(z, w) \in R_i \setminus (\mathbb{T}^2 \cup L) : \frac{\partial q_i}{\partial z}(z, w) = 0,\ or\ \ \frac{\partial q_i}{\partial w}(z, w) = 0\},$$

$$F_i = \bigcup_{j \in I \setminus \{i\}} (R_i \cap R_j) \setminus (\mathbb{T}^2 \cup L),$$

$$R_i^* = R_i \setminus (\mathbb{T}^2 \cup L \cup E_i \cup F_i),$$

$$\Sigma = \bigcup_{i \in I} R_i \setminus (\mathbb{T}^2 \cup L).$$

It is known that the sets E_i and $R_i \cap R_j$ $(i \neq j)$ are finite at most, respectively, and $Z(q_i) \setminus (E_i \cup F_i)$ is a connected set in $\mathbb{C}^2$.

Step I. $\widehat{G(f)} \setminus G(\bar{p}; \mathbb{T}^2 \cup L) \subset G(\bar{p}; \Sigma \cap V)$.

Proof. Let ζ be the third coordinate of $\mathbb{C}^3$. By Lemma 2.5 we have that

$$\widehat{G(f)} \subset \{(z, w, \zeta) :\ (z, w) \in \overline{\mathbb{D}^2},\ \zeta = \overline{p(z, w)}\ \}$$

and by the definition of $k(z, w)$

$$\widehat{G(f)} \subset \{(z, w, \zeta) :\ (z, w) \in \overline{\mathbb{D}^2},\ |\zeta| \leq \|p\|_{\mathbb{T}^2},\ z^m w^n \zeta - k(z, w) = 0\ \}.$$

Hence we have $\widehat{G(f)} \setminus G(\bar{p}; \mathbb{T}^2 \cup L) \subset G(\bar{p}; V)$. If a point $(z^0, w^0) \in V \setminus \Sigma$, then $\Delta(z^0, w^0) \neq 0$. By Lemma 2.2 there is a ball B_0 centered at (z^0, w^0) such that $B_0 \cap (\mathbb{T}^2 \cup L) = \emptyset$ and $B_0 \cap V$ is a totally real submanifold of B_0. Thus the graph $G(\bar{p}; B_0 \cap V)$ is also totally real and $(B_0 \times \mathbb{C}) \cap G(f) = \emptyset$. It follows from Lemma 2.3 that

$$G(\bar{p}; B_0 \cap V) \cap \widehat{G(f)} = \emptyset,$$

and so

$$G(\bar{p}; V \setminus \Sigma) \cap \widehat{G(f)} = \emptyset,$$

which proves Step I.

Note. It is sufficient to investigate $G(\bar{p}; V \cap \Sigma)$, since the graph $\widehat{G(f)}$ is connected and $\widehat{G(f)} \subset G(\bar{p}; \mathbb{T}^2) \cup G(\bar{p}; V \cap \Sigma) \cup G(\bar{p}; L)$.

Assume that for some $i \in I$, $V \cap R_i^* \neq \emptyset$. For a point (z^0, w^0) in $V \cap R_i^*$, there exist a neighborhood $\overline{U_0}$ of (z^0, w^0) in R_i^* and holomorphic functions $\varphi(\lambda)$ and $\psi(\lambda)$ on $\bar{\mathbb{D}}$ such that $(z^0, w^0) = (\varphi(0), \psi(0))$ and

$$U_0 = \{(\varphi(\lambda), \psi(\lambda)) : \lambda \in \mathbb{D}\}.$$

Step II. The case that $\varphi(\lambda)$ and $\psi(\lambda)$ satisfy the condition

$$\overline{p(\varphi(\lambda), \psi(\lambda))} - h(\varphi(\lambda), \psi(\lambda)) \equiv 0 \text{ on } \overline{\mathbb{D}}. \tag{1}$$

In this case, $q_i(z, w)$ is a common factor of $p(z, w) - p(z^0, w^0)$ and $k(z, w) - z^m w^n \overline{p(z^0, w^0)}$, and so

$$R_i \setminus (\mathbb{T}^2 \cup L) \subset V. \tag{2}$$

Proof. We obtain the power series on $\bar{\mathbb{D}}$

$$p(\varphi(\lambda), \psi(\lambda)) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots,$$

$$h(\varphi(\lambda), \psi(\lambda)) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots.$$

It follows from the assumption that for every polynomial $q(\lambda)$

$$\begin{aligned} 0 &= \int_{|\lambda|=1} \{\overline{p(\varphi(\lambda), \psi(\lambda))} - h((\varphi(\lambda), \psi(\lambda))\} q(\lambda) d\lambda \\ &= \int_{|\lambda|=1} \{(\bar{a}_0 + \bar{a}_1\bar{\lambda} + \bar{a}_2\bar{\lambda}^2 + \cdots) - b_0\} q(\lambda) d\lambda. \end{aligned}$$

Thus $\bar{a}_1 = \bar{a}_2 = \cdots = 0$, $\bar{a}_0 = \overline{p(z^0, w^0)} = b_0$ and $\overline{a}_0 - h(\varphi(\lambda), \psi(\lambda)) \equiv 0$ on $\bar{\mathbb{D}}$. Since a_0 depends on q_i, we put $c_i = a_0$. Then we can write that

$$k(z, w) - \bar{c}_i z^m w^n = q_i(z, w) k_i(z, w),$$

$$\overline{p(z, w)} - \overline{c_i} = \overline{q_i(z, w) p_i(z, w)}$$

for some polynomials $p_i(z, w)$ and $k_i(z, w)$. Thus (2) follows.

Step III. The case that (1) does not holds, i.e.,

$$\overline{p(\varphi(\lambda), \psi(\lambda))} - h(\varphi(\lambda), \psi(\lambda)) \not\equiv 0 \text{ on } \overline{\mathbb{D}}. \tag{3}$$

In this case, we have

$$\widehat{G(f)} \setminus G(\bar{p}; \mathbb{T}^2 \cup L) \subset G(\bar{p}; \Sigma_i \cap V) \tag{4}$$

where $\Sigma_i = \bigcup_{j \in I \setminus \{i\}} R_j \setminus (\mathbb{T}^2 \cup L)$.

To show this we consider the condition (3) from two viewpoints of (5), (6) of Step IV and V.

Step IV. If

$$\overline{p(\varphi(\lambda), \psi(\lambda))} - c_i \equiv 0 \text{ on } \mathbb{D}, \tag{5}$$

then we have $G(\bar{p}; (V \cap R_i) \setminus \Sigma_i) \cap \widehat{G(f)} = \emptyset$.

Proof. Since $q_i(z, w)$ is an irreducible polynomial, it is a factor of $p(z, w) - c_i$. Thus $p(z, w) - c_i \equiv 0$ on R_i and $\overline{c_i} - h(z, w) \not\equiv 0$ on $\overline{\mathbb{D}^2} \setminus (\mathbb{T}^2 \cup L)$. Thus the set

$$V \cap R_i = \{(z, w) \in \overline{\mathbb{D}^2} \setminus (\mathbb{T}^2 \cup L) : \overline{c_i} - h(z, w) = 0,\ q_i(z, w) = 0\}$$

is finite. Thus $G(\bar{p}; V \cap R_i)$ is the set of isolated points. Since $\widehat{G(f)}$ does not contain any isolated points, we have $G(\bar{p}; V \cap R_i \setminus \Sigma_i) \cap \widehat{G(f)} = \emptyset$, which proves (5).

Step V. Now let $(z^0, w^0) \in R_i^*$. Assume that

$$\overline{p(\varphi(\lambda), \psi(\lambda))} - \overline{p(z^0, w^0)} \not\equiv 0 \text{ on } \mathbb{D}. \qquad (6)$$

We can assume that $\varphi(\lambda) = z_0 + \lambda$ in $\rho\mathbb{D}$ for some positive $\rho\mathbb{D}$. We put

$$W_0 = \{(\varphi(\lambda), \psi(\lambda)) : \lambda \in \rho\mathbb{D}\}$$

and

$$W_0^* = \{(z_0 + \lambda, \psi(\lambda)) : \lambda \in \rho\mathbb{D},\ \frac{\partial p}{\partial z}(\varphi(\lambda), \psi(\lambda)) + \frac{\partial p}{\partial w}(\varphi(\lambda), \psi(\lambda))\frac{d\psi(\lambda)}{d\lambda} \neq 0\}.$$

Step VI. If (6) holds, then $G(\bar{p}; W_0^*)$ is totally real, and so

$$G(\bar{p}; R_i \setminus (\mathbb{T}^2 \cup L \cup \Sigma_i)) \cap \widehat{G(f)} = \emptyset. \qquad (7)$$

Proof. We put $\lambda = x + iy$ and $p = u + iv$ $(x, y, u, v$ real). The real tangent vectors at $(z_0 + \lambda, \psi(\lambda), \overline{p(z_0 + \lambda, \psi(\lambda))})$ to $G(\bar{p}; W_0)$ for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are as follows.

$$v_1 = \big(1, 0, \frac{\partial \text{Re}\psi}{\partial x}(\lambda), \frac{\partial \text{Im}\psi}{\partial x}(\lambda), \frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x}\big),$$

$$v_2 = \big(0, 1, \frac{\partial \text{Re}\psi}{\partial y}(\lambda), \frac{\partial \text{Im}\psi}{\partial y}(\lambda), \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial y}\big).$$

The rank of the matrix defined by components of v_1, v_2, iv_1, iv_2 is 4, since

$$\begin{vmatrix} 1 & 0 & u_x & -v_x \\ 0 & 1 & u_y & -v_y \\ 0 & 1 & v_x & u_x \\ -1 & 0 & v_y & u_y \end{vmatrix} = -4(u_x^2 + v_x^2) = -4\left|\frac{dp}{d\lambda}\right|^2.$$

Thus $G(\bar{p}; W_0^*)$ is a totally real manifold. It follows from Lemma 2.3 that

$$G(\bar{p}; W_0^* \setminus \Sigma_i) \cap \widehat{G(f)} = \emptyset.$$

Since $W_0 \setminus W_0^*$ is a set of isolated points, by connectivity of $\widehat{G(f)}$ we have

$$G(\bar{p}; W_0 \setminus (W_0^* \cup \Sigma_i)) \cap \widehat{G(f)} = \emptyset.$$

When points (z_0, w_0) run in R_i^*, the coresponding neighborhoods U_0 cover R_i^*. Thus $G(\bar{p}; R_i^* \setminus (\Sigma_i \cup \mathbb{T}^2 \cup L) \cap \widehat{G(f)} = \emptyset$. Since the set $G(\bar{p}; R_i \setminus (R_i^* \cup \mathbb{T}^2 \cup L))$ is finite, we have

$$G(\bar{p}; R_i \setminus (R_i^* \cup \Sigma_i \cup \mathbb{T}^2 \cup L)) \cap \widehat{G(f)} = \emptyset,$$

and the assertion (7) is proved. From (5) and (7) we obtain (4) of Step III.

By the above facts we obtain the following:

Step VII. If we put

$$I_0 = \{i \in \{1, 2, \cdots, t\} : \emptyset \neq R_i \setminus (\mathbb{T}^2 \cup L) \subset V\},$$

then

$$\widehat{G(f)} \setminus G(\bar{p}; \mathbb{T}^2 \cup L) \subset G(\bar{p}; \cup_{i \in I_0} R_i \cap V).$$

For $i \in I_0$, we consider the following cases:

(i). $Q_i = \emptyset,\ R_i \neq \emptyset$. (ii). $\emptyset \neq Q_i = \hat{Q}_i \neq R_i$.
(iii). $\emptyset \neq Q_i \neq \hat{Q}_i = R_i$. (iv). $\emptyset \neq Q_i \neq \hat{Q}_i \neq R_i$.

Step VIII. Assume that (ii) holds for $i \in I_0$, then

$$G(\bar{p}; R_i \setminus (\mathbb{T}^2 \cup L \cup \Lambda_i) \cap \widehat{G(f)} = \emptyset, \tag{8}$$

where $\Lambda_i = \bigcup_{j \in I_0 \setminus \{i\}} R_j$.

Proof. We denote m_i by the maximal order of an irreducible factor $q_i(z, w)$ in $p(z, w)$, and we define a polynomial $p_1(z, w)$ by

$$p(z, w) - c_i = p_1(z, w) q_i(z, w)^{m_i}.$$

By using $p_1(z, w)$ we put $K = \{(z, w) \in \overline{\mathbb{D}^2} : p_1(z, w) = 0\}$. For a point $(z^0, w^0) \in R_i \setminus (K \cup \mathbb{T}^2 \cup L)$, we put

$$p_2(z, w) = \frac{1}{p_1(z^0, w^0)} p_1(z, w).$$

Since Q_i and $\{(z^0, w^0)\}$ are disjoint polynomially convex sets, there exist a polynomial $p_0(z, w)$, a neighborhood U of Q_i and a neighborhood W of K in $\mathbb{T}^2$ such that

$$p_0(z^0, w^0) = 1, \text{ and } \quad |p_0(z, w) p_2(z, w)| < \frac{1}{2} \text{ on } U,$$

$$|p_0(z, w) p_2(z, w)| < \frac{1}{2} \text{ on } W.$$

If we put $M = \|p - c_i\|_{\mathbb{T}^2}$, $K_1 = \{(z, w) \in \overline{\mathbb{D}^2} : p(z, w) - c_i = 0\}$, and put

$$g_1(z, w, \zeta) = 1 - \frac{1}{2M^2} (\zeta - \overline{c_i})(p(z, w) - c_i),$$

then we have

$$g_1(z, w, \zeta) = 1 \text{ on } G(\bar{p}; K_1).$$

Since $|g_1| < 1$ on $G(\bar{p}; \mathbb{T}^2 \setminus (U \cup W))$, there exists a positive integer k such that

$$|p_2(z, w) p_0(z, w) g_1(z, w, \zeta)^k| < \frac{1}{2} \text{ on } G(\bar{p}; \mathbb{T}^2 \setminus (U \cup W)).$$

If we put $g(z, w, \zeta) = p_2(z, w) p_0(z, w) g_1(z, w, \zeta)^k$, then

$$|g(z, w, \zeta)| < \frac{1}{2} \text{ on } G(f), \text{ and } g(z^0, w^0, \overline{p(z^0, w^0)}) = 1.$$

Thus $(z^0, w^0, \overline{p(z^0, w^0)}) \notin \widehat{G(f)}$ and so $G(\bar{p}; R_i \setminus (K \cup \mathbb{T}^2 \cup L)) \cap \widehat{G(f)} = \emptyset$. Since a set $(R_i \cap K) \setminus (\mathbb{T}^2 \cup L)$ is finite, by connectivity of $G(f)$ we have

$$G(\bar{p}; R_i \setminus (\Lambda_i \cup \mathbb{T}^2 \cup L) \cap \widehat{G(f)} = \emptyset.$$

which proves (8).

In the case (i), if we choose a point (z^*, w^*) in $\mathbb{T}^2 \setminus \Lambda_i$, and put $Q_i = \{(z^*, w^*)\}$, then we similarly obtain the proof of (i).

Step IX. Assume the (iii) holds, then

$$G(\bar{p}; R_i) \subset \widehat{G(f)}. \tag{9}$$

Proof. Since $G(\bar{p}; Q_i) \subset G(f) = G(\bar{p}; \mathbb{T}^2)$ and $G(\bar{p}; Q_i) \subset \{(z, w, \zeta) \in \mathbb{C}^3 : \zeta = c_i\}$, then we obtain (9).

Step X. Assume that (iv) holds. Then we have

$$G(\bar{p}; R_i \setminus (L \cup \mathbb{T}^2 \cup \hat{Q}_i)) \cap \widehat{G(f)} = \emptyset. \qquad (10).$$

Proof. Let (z^0, w^0) be a point of $R_i \setminus (L \cup \mathbb{T}^2 \cup \hat{Q}_i)$. If Q_i in (ii) is replaced by $\hat{Q}_i$, we similarly have (10).

5. Examples.

Example 5.1. If $p(z,w) = \{(z+1) - (w+1)^2\}\{(z+1)w^2 - z(w+1)^2\}$ and $f = \bar{p}|_{\mathbb{T}^2}$, then $h(z,w) = \dfrac{1}{z^2w^4}p(z,w)$ and

$$\Delta(z,w) = \frac{2p(z,w)}{z^3w^5}g(z,w)$$

where

$$g(z,w) = wp_w(z,w) - 2zp_z(z,w)$$
$$= 2[(w+1)z^2 + w^2(2w+3)z - w^3(2w+3)].$$

The polynomial $g(z,w)$ is irreducible. The sets defined by the section 1 are as follows:

$Q_1 = \{(z,w) \in \mathbb{T}^2 : z - w^2 - 2w = 0\} = \{(-1,-1)\} = \hat{Q}_1$.
$R_1 = \{(z,w) \in \overline{\mathbb{D}}^2 : z - w^2 - 2w = 0\}$.
$Q_2 = \{(z,w) \in \mathbb{T}^2 : w^2 - z - 2zw = 0\} = \{(-1,-1)\} = \hat{Q}_2$.
$R_2 = \{(z,w) \in \overline{\mathbb{D}^2} : w^2 - z - 2zw = 0\}$.
$R_3 = \{(z,w) \in \overline{\mathbb{D}^2} : g(z,w) = 0\}$.

Then we have that $R_j \setminus (\mathbb{T}^2 \cup L) \subset V$ and $\emptyset \neq Q_j = \hat{Q}_j \neq R_j,\ j = 1,2$. Since $g(z,w)$ and $p(z,w) - c$ for every $c \in \mathbb{C}$ are relatively prime polynomials. Thus $R_3 \setminus (\mathbb{T}^2 \cup L)$ is not contained in V. Since $I_0 = \{1,2\}$ and $J = \emptyset$, by the theorem we have

$$\widehat{G(f)} = G(f).$$

Example 5.2. If $p(z,w) = (z+w)(w+2)(2w+1)$ and $f = \bar{p}|_{\mathbb{T}^2}$, then we have that $h(z,w) = \dfrac{1}{zw^3}(z+w)(w+2)(2w+1)$ and

$$\Delta(z,w) = \frac{2}{zw^3}(z+w)(w+2)(2w+1)g(z,w)$$

where $g(z,w) = -z(w^2+5w+3) + w(3w^2+5w+1)$. Since the polynomial $g(z,w)$ is irreducible, the sets $\{(z,w) \in \overline{\mathbb{D}^2} \setminus (\mathbb{T}^2 \cup L) : z + w = 0\}$ and $\{(z,w) \in \overline{\mathbb{D}^2} \setminus (\mathbb{T}^2 \cup L) : 2w + 1 = 0\}$ are contained in V, it follows from the theorem that

$$\widehat{G(f)} = G(f) \cup \{(z,w,0) \in \bar{\mathbb{D}}^2 : z + w = 0\}.$$

Example 5.3. ([5]). Let $p(z,w)$ be a homogeneous polynomial:

$$P(z,w) = cz^m w^n (z^k + a_1 z^{k-1} w + a_2 z^{k-2} w^2 + \cdots + a_k w^k)(a_k \neq 0)$$
$$= c(z - \lambda_1 w)(z - \lambda_2 w)\cdots(z - \lambda_k w)z^m w^n$$

where k is a positive integer, m and n are nonnegative integers, and $c, \lambda_1, \lambda_2, \cdots, \lambda_k$ are some constants with $c\lambda_1\lambda_2\cdots\lambda_k \neq 0$. We put

$$J = \{j \in \{1,2,\cdots,k\} : |\lambda_j| = 1\}.$$

(1) If $J \neq \emptyset$, then $\widehat{G(f)} = \bigcup_{j \in J}\{(z,w,0); z - \lambda_j w = 0, w \in D\} \cup G(f)$.
(2) If $J = \emptyset$, then $\widehat{G(f)} = G(f)$, and moreover $[z,w,f;\mathbb{T}^2] = C(\mathbb{T}^2)$.

Example 5.4. If $p(z,w) = (z^2-1)w + z$ and $f = \bar{p}|_{\mathbb{T}^2}$, then $h(z,w) = \frac{(1-z^2)+zw}{z^2w}$ and

$$\Delta(z,w) = \frac{1}{z^3w^2}(z^2-1)g(z,w)$$

where $g(z,w) = zw^2 + 2(z^2+1)w + z$. We have that $z-1$ is a factor of $p(z,w)-1$ and $z+1$ is a factor of $p(z,w)+1$ and $g(z,w)$ is an irreducible polynomial. Thus

$$\widehat{G(f)} = G(f) \cup \{(1,w,1) : w \in \mathbb{D}\} \cup \{(-1,w,-1) : w \in \mathbb{D}\}.$$

Acknowledgements

The author wishes to thank Professor Akira Sakai for variable suggestions for improving the manuscript.

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