

**RELATIVE COUNTABLE PARACOMPACTNESS, RELATIVE
EXPANDABILITY AND THEIR ABSOLUTE EMBEDDINGS**

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ABSTRACT. C.E. Aull [4] defined that a subspace Y of a space X is α -countably paracompact in X if for every countable collection \mathcal{U} of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection \mathcal{V} of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, \mathcal{V} is a partial refinement of \mathcal{U} and \mathcal{V} is locally finite in X . In this paper, we prove that a Tychonoff space Y is α -countably paracompact in every larger Tychonoff space if and only if Y is countably compact. Moreover, by introducing some notions of relative expandability, we develop our study to characterize their absolute embeddings. Finally, a negative answer to the question on relative discrete expandability in [6] is also given.

1. Introduction. Throughout this paper all spaces are assumed to be T_1 topological spaces and the symbol γ denotes an infinite cardinal.

Arhangel'skiĭ and Genedi [3] introduced the notions of relative topological properties of a subspace Y of a space X . Let us recall some preliminary notions and facts on relative (countable) paracompactness. Arhangel'skiĭ and Genedi [3] (respectively, Yasui [14], [15]) defined that a subspace Y of a space X is *1-paracompact* (respectively, *1-countably paracompact*) in X if for every open (respectively, for every countable open) cover \mathcal{U} of X , there exists a collection \mathcal{V} of open subsets of X with $X = \bigcup \mathcal{V}$ such that \mathcal{V} is a partial refinement of \mathcal{U} and \mathcal{V} is locally finite at each point of Y . Here, \mathcal{V} is said to be a *partial refinement* of \mathcal{U} if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V ([1], [2]). Being 1-paracompact of Y in X was earlier defined by Lupiañez-Outerele [10], where it is stated as “ X is A -paracompact in Y ”. It is clear that if Y is 1-paracompact in X , then Y is countably 1-paracompact in X .

Aull [4] defined that a subspace Y of a space X is *α -paracompact* (respectively, *α -countably paracompact*) in X if for every collection (respectively, for every countable collection) \mathcal{U} of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection \mathcal{V} of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, \mathcal{V} is a partial refinement of \mathcal{U} and \mathcal{V} is locally finite in X . It is obvious that if Y is α -paracompact in X , then Y is α -countably paracompact in X .

Recall that 1- and α -paracompactness do not imply each other in general, but for a closed subspace Y of a regular space X , Y is 1-paracompact in X if and only if Y is α -paracompact in X ([10, Theorem 1.3]). Hence, this fact and [9, 1.1 Proposition] immediately induce the following (see also [7]).

Theorem 1.1 (Lupiañez [9], Lupiañez-Outerele [10]). *For a Tychonoff (respectively, regular) space Y , the following statements are equivalent.*

- (a) Y is 1-paracompact in every larger Tychonoff (respectively, regular) space.
- (b) Y is α -paracompact in every larger Tychonoff (respectively, regular) space.
- (c) Y is compact.

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For the case of 1-countable paracompactness, Matveev [11] actually proved the following (see also [2, Theorem 13.5]).

Theorem 1.2 (Matveev [11]). *A Tychonoff (respectively, regular) space Y is 1-countably paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is Lindelöf.*

In view of these theorems, it seems to be natural to consider the case of α -countable paracompactness. In this paper, we obtain the following.

Theorem 1.3. *A Tychonoff (respectively, regular) space Y is α -countably paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is countably compact.*

Krajewski [8] defined that a space X is γ -*expandable* if for every locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X , there exists a locally finite collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. A space X is *expandable* if X is γ -expandable for every γ . It is known that every paracompact and every countably compact space are expandable. Moreover, it is also known that a space X is countably paracompact if and only if X is ω -expandable ([8]).

In this paper, we further introduce notions of 1- and α -expandability of a subspace in a space. We define that Y is 1- γ -*expandable in X* if for each locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X . If Y is 1- γ -expandable in X for every γ , Y is said to be 1-*expandable in X* . A subspace Y of a space X is said to be α - γ -*expandable in X* if for each collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X which is locally finite at every point of Y in X , there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite in X . If Y is α - γ -expandable in X for every γ , Y is said to be α -*expandable in X* . As we show below, 1- (respectively, α -) expandability lies between 1- (respectively, α -) paracompactness and 1- (respectively, α -) countable paracompactness (see Proposition 2.1).

The following theorems are also our main results.

Theorem 1.4. *For a Tychonoff (respectively, regular) space Y , the following are equivalent.*

- (a) *Y is 1-expandable in every larger Tychonoff (respectively, regular) space.*
- (b) *Y is 1- ω_1 -expandable in every larger Tychonoff (respectively, regular) space.*
- (c) *Y is compact.*

Theorem 1.5. *A Tychonoff (respectively, regular) space Y is α -expandable in every larger Tychonoff (respectively, regular) space if and only if Y is countably compact.*

In Section 4, we will give a negative answer to the question on relative discrete expandability asked by E. Grabner et. al. [6].

Other undefined notations and terminology are used as in [5]. For general surveys, see Arhangel'skiĭ's papers [1], [2].

2. Preliminaries.

First let us notice the following facts.

Proposition 2.1. *For a subspace Y of a space X , the following statements hold.*

- (a) *If Y is 1-paracompact in X , then Y is 1-expandable in X .*
- (b) *If Y is α -paracompact in X , then Y is α -expandable in X .*
- (c) *Y is 1-countably paracompact in X if and only if Y is 1- ω -expandable in X .*
- (d) *Y is α -countably paracompact in X if and only if Y is α - ω -expandable in X .*

PROOF. (a) Assume that Y is 1-paracompact in X . Let $\{F_\alpha \mid \alpha \in \Omega\}$ be a locally finite collection of closed subsets of X . For each $\delta \in [\Omega]^{<\omega}$, put $U_\delta = X \setminus \bigcup_{\beta \notin \delta} F_\beta$, and let $\mathcal{U} = \{U_\delta \mid \delta \in [\Omega]^{<\omega}\}$. Then, \mathcal{U} is an open cover of X . Since Y is 1-paracompact in X , there exists an open refinement \mathcal{V} of \mathcal{U} such that \mathcal{V} is locally finite at each point of Y in X . For every $\alpha \in \Omega$, put $G_\alpha = \text{St}(F_\alpha, \mathcal{V})$. Then, $\{G_\alpha \mid \alpha \in \Omega\}$ is a collection of open subsets of X which is locally finite at each point of Y and satisfies $F_\alpha \subset G_\alpha$ for every $\alpha \in \Omega$. Therefore, Y is 1-expandable in X . (b) can be proved similarly to (a).

(c) The “only if” part can be proved similarly to (a). To prove the “if” part, assume that Y is 1- ω -expandable in X . Let $\{U_n \mid n \in \mathbb{N}\}$ be a countable open cover of X . For every $n \in \mathbb{N}$, put $V_n = \bigcup_{j \leq n} U_j$. Let $E_1 = V_1$ and $E_n = V_n \setminus V_{n-1}$ for every $n \geq 2$. Then, $\{\overline{E_n} \mid n \in \mathbb{N}\}$ is a locally finite collection of closed subsets of X . Hence, there exists a collection $\{G_n \mid n \in \mathbb{N}\}$ of open subsets of X such that $\overline{E_n} \subset G_n$ for every $n \in \mathbb{N}$ and $\{G_n \mid n \in \mathbb{N}\}$ is locally finite at every point of Y in X . For each $n \in \mathbb{N}$, set $W_n = U_n \cap G_n$. Then, $\{W_n \mid n \in \mathbb{N}\}$ is an open refinement of \mathcal{U} which is locally finite at every point of Y in X . Therefore, Y is 1-countably paracompact in X . (d) can be proved similarly to (c). \square

The following examples show that 1-expandability and α -expandability of Y in X need not imply each other even if Y is a closed subspace of a regular space X .

Example 2.2. There exist a Tychonoff space X and its subspace Y such that Y is 1-expandable in X , but not α -countably paracompact in X .

PROOF. Let $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$ be the Tychonoff plank and $Y = \{\omega_1\} \times \omega$. Then Y is a closed subspace of X . Since $\omega_1 \times (\omega + 1)$ is countably compact, it is clear that Y is 1-expandable in X . Moreover, it is immediate from the definition that Y is not α -countably paracompact in X . \square

Example 2.3. There exist a Tychonoff space X and its subspace Y such that Y is α -expandable in X , but not 1-countably paracompact in X .

PROOF. Let $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$ be the Tychonoff plank and $Y = \omega_1 \times \{\omega\}$. Then Y is a closed subspace of X . Since $\omega_1 \times \{\omega\}$ is countably compact, it is obvious that Y is α -expandable in X . Moreover, Y is not 1-countably paracompact in X . Indeed, put $\mathcal{U} = \{(\omega_1 + 1) \times \{n\} \mid n \in \mathbb{N}\} \cup \{\omega_1 \times (\omega + 1)\}$. Then \mathcal{U} is a countable open cover of X . Let \mathcal{V} be an open refinement of \mathcal{U} . For every $n \in \mathbb{N}$, there exist an $\alpha_n < \omega_1$ and a $V_n \in \mathcal{V}$ such that $\langle \omega_1, n \rangle \in (\alpha_n, \omega_1] \times \{n\} \subset V_n$. By the definition of \mathcal{U} , we necessarily have $V_n \subset (\omega_1 + 1) \times \{n\}$ for every $n \in \mathbb{N}$. Put $\alpha^* = \sup\{\alpha_n \mid n \in \mathbb{N}\}$. Then, \mathcal{V} is not locally finite at $\langle \alpha^* + 1, \omega \rangle \in Y$. Hence, Y is not 1-countably paracompact in X . \square

Remark 2.4. By Examples 2.2 and 2.3, we also have that 1-countable paracompactness and α -countable paracompactness need not imply each other.

3. Proofs of main theorems. First, we prove Theorems 1.3 and 1.5.

PROOF OF THEOREMS 1.3 AND 1.5. It suffices to show that the following statements are equivalent:

- (a) Y is α -expandable in every larger Tychonoff (respectively, regular) space.
- (b) Y is α -countably paracompact in every larger Tychonoff (respectively, regular) space.
- (c) Y is countably compact.

We only prove (b) \Rightarrow (c). Suppose Y is not countably compact. Then Y has a countable closed discrete subset $F = \{y_n \mid n \in \mathbb{N}\}$. Let $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank, X the quotient space obtained from $Y \oplus T$ by identifying y_n with $\langle \omega_1, n \rangle$ for each $n \in \mathbb{N}$ and $q : Y \oplus T \rightarrow X$ the quotient map.

It is easy to see that if Y is Tychonoff (respectively, regular), then X is also Tychonoff (respectively, regular). Since Y is homeomorphic to $q(Y)$, Y is viewed as a closed subspace of X . We show that Y is not α -countably paracompact in X . To prove this, put $p_n = q(y_n) = q(\langle \omega_1, n \rangle)$ and $U_n = (q(Y) \setminus \{p_m \mid m \neq n\}) \cup q((\omega_1 + 1) \times \{n\})$ for each $n \in \mathbb{N}$. Then the collection $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is a countable collection of open subsets of X with $Y \subset \bigcup \mathcal{U}$, but \mathcal{U} has no partial open refinement \mathcal{V} satisfying $Y \subset \bigcup \mathcal{V}$ and \mathcal{V} is locally finite in X . Hence, Y is not α -countably paracompact in X . \square

A space X is said to be *linearly Lindelöf* if every uncountable subset of regular cardinality has a complete accumulation point in X . It is known that a space X is compact if and only if X is countably compact and linearly Lindelöf ([12]).

PROOF OF THEOREM 1.4. We only prove (b) \Rightarrow (c). Suppose Y is not compact. If Y is not countably compact, Y has a countable closed discrete subset $F = \{y_n \mid n \in \mathbb{N}\}$. Let $D(\omega_1)$ be the discrete space with the cardinality ω_1 and $A(\omega_1) = D(\omega_1) \cup \{\infty\}$ its one-point compactification. Let $T^* = A(\omega_1) \times (\omega + 1) \setminus \{(\infty, \omega)\}$ and let X be the quotient space obtained from $Y \oplus T^*$ by identifying y_n with $\langle \infty, n \rangle$ for each $n \in \mathbb{N}$ and $q : Y \oplus T^* \rightarrow X$ the quotient map.

It is easy to see that if Y is Tychonoff (respectively, regular), then X is also Tychonoff (respectively, regular). Since Y is homeomorphic to $q(Y)$, Y is viewed as a closed subspace of X . We show that Y is not $1-\omega_1$ -expandable in X . To prove this, put $p_\alpha = q(\langle \alpha, \omega \rangle)$ for each $\alpha \in D(\omega_1)$. Then $\{\{p_\alpha\} \mid \alpha \in D(\omega_1)\}$ is a discrete collection of closed subsets of X . But there is no collection $\mathcal{G} = \{G_\alpha \mid \alpha \in D(\omega_1)\}$ of open subsets of X such that $p_\alpha \in G_\alpha$ for every $\alpha \in D(\omega_1)$ and \mathcal{G} is locally finite at every $y \in Y$.

Assume that Y is not linearly Lindelöf. Then the proof is based on [11]. Y has an uncountable subset Z , of regular cardinality κ , which has no complete accumulation point in Y . Enumerate $Z = \{z_\alpha \mid \alpha < \kappa\}$. Put $Z^* = Z \cup \{z^*\}$, where $z^* \notin Y$. Let $X = (Z^* \times \omega) \cup (Y \times \{\omega\})$ as a set. Define a topology on X as follows.

- (i) A neighborhood base at $\langle z^*, n \rangle$ is the collection of all sets of the form $\{\langle z_\lambda, n \rangle \mid \lambda > \alpha\} \cup \{\langle z^*, n \rangle\}$, where $n \in \omega$ and $\alpha < \omega$.
- (ii) A neighborhood base at $\langle y, \omega \rangle$ is the collection of all sets of the form $((U \cap Z) \times (n, \omega)) \cup (U \times \{\omega\})$, where U is a usual neighborhood of y in Y and $n < \omega$.
- (iii) Each point in $Z \times \omega$ is isolated in X .

Since Y is homeomorphic to $Y \times \{\omega\}$, Y is viewed as a closed subspace of X . Since Z has no complete accumulation point in Y , we can prove by the same argument as in [11] that if Y is Tychonoff (respectively, regular), then so is X . Then, Y is not $1-\omega_1$ -expandable in X . Actually, $\mathcal{F} = \{\{z^*, n\} \mid n \in \omega\}$ is a discrete collection of closed subsets of X . But \mathcal{F} cannot be expanded to a collection of open subsets of X which is locally finite at every $y \in Y$. \square

Remark 3.1. Lupiañez [9] proved that a Hausdorff space Y is α -paracompact in every larger Hausdorff space if and only if Y is compact. Moreover, in [7], it was proved that a Hausdorff space Y is 1-paracompact in every larger Hausdorff space if and only if $Y = \emptyset$.

Remark 3.2. Similarly to the proof of [7, Proposition 3.19], we have that for a Hausdorff space Y , the following statements are equivalent.

- (a) Y is 1-expandable in every larger Hausdorff space.
- (b) Y is 1-countably paracompact in every larger Hausdorff space.
- (c) $Y = \emptyset$.

It suffices to prove (b) \Rightarrow (c). Indeed, assume $Y \neq \emptyset$. Take $y \in Y$ and let $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank. Consider the quotient space X obtained from $T \oplus Y$ by identifying the top edge $\omega_1 \times \{\omega\}$ of T and y to one point. Let $q : T \oplus Y \rightarrow X$ be the natural quotient map. Then, X is Hausdorff and Y is homeomorphic to $q(Y)$ which is a closed subspace of X . But, Y is not 1-countably paracompact in X .

Remark 3.3. The proof of Theorems 1.3 and 1.5 works to show that for a Hausdorff space Y , the following statements are equivalent:

- (a) Y is α -expandable in every larger Hausdorff space.
- (b) Y is α -countably paracompact in every larger Hausdorff space.
- (c) Y is countably compact.

4. Concluding remarks. Smith and Krajewski [13] defined that a space X is *discretely γ -expandable* if for every discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X , there exists a locally finite collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. A space X is *discretely expandable* if X is discretely γ -expandable for every γ . It is easy to see that every expandable space is discretely expandable, and every collectionwise normal space is discretely expandable ([13]).

For relative version of these notions, we define that a subspace Y of a space X is *1-discretely γ -expandable in X* if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X . We also define that Y is *α -discretely γ -expandable in X* if for each collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X which is discrete at every point of Y in X , there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite in X . Moreover, 1- and α -discretely expandability of a subspace in a space are now easy to be understood. It is easy to see that if Y is 1- (respectively, α -) γ -expandable in X , then Y is 1- (respectively, α -) discretely γ -expandable in X .

In Example 2.2, Y is 1-expandable but not α -discretely expandable in X . Moreover, in Example 2.3, Y is α -expandable but not 1-discretely expandable in X . Hence, 1-discrete expandability and α -discrete expandability of Y in X do not imply each other.

The proofs of Theorems 1.4 and 1.5 essentially show the following.

Theorem 4.1. *A Tychonoff (respectively, regular) space Y is 1-discretely expandable in every larger Tychonoff (respectively, regular) space if and only if Y is compact.*

Theorem 4.2. *A Tychonoff (respectively, regular, Hausdorff) space Y is α -discretely expandable in every larger Tychonoff (respectively, regular, Hausdorff) space if and only if Y is countably compact.*

Remark 4.3. As in Remark 3.2, we have that a Hausdorff space Y is 1-discretely expandable in every larger Hausdorff space if and only if $Y = \emptyset$.

Remark 4.4. In Theorems 1.1, 1.2, 1.3, 1.4, 1.5, 4.1 and 4.2, and Remarks 3.1, 3.2, 3.3 and 4.3, “in every larger Tychonoff (respectively, regular, Hausdorff) space” can be replaced by “in every larger Tychonoff (respectively, regular, Hausdorff) space containing Y as a closed subspace”.

Remark 4.5. (1) In [6], E. Grabner et. al. asked the following question; suppose that Y is a closed subspace of a regular space X . If Y is 1-discretely expandable in X and metacompact in itself, is Y 1-paracompact in X ? Notice that Example 2.2 actually provides a negative answer to this question.

(2) [6, Example 5.6] asserts that there exists a regular space X having a subspace which is 1-metacompact in X and strongly collectionwise normal in X but not 1-discretely expandable in X . But, since their example is incomplete, we give here another complete example as follows. Define X to be the space T^* which is constructed in the proof of Theorem 1.4. Let $Y = D(\omega_1) \times (\omega + 1)$ and $Z = D(\omega_1) \times \{\omega\}$. Then, Y is open in X and Z is closed in X . Moreover, both Y and Z are 1-metacompact and strongly collectionwise normal in X , but neither Y nor Z are 1-discretely expandable in X .

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