COMMON FIXED POINT THEOREMS IN SMALL SELF DISTANCE QUASI-SYMMETRIC DISLOCATED METRIC SPACE

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Abstract. In this paper we introduce common fixed point theorems in a new type of generalized metric space so called a small self distance quasi-symmetric dislocated metric space (ssd-q-s-d-metric space for short). Our results are generalizations of Theorem 2.1 [1] due to Mohamed Aamri and Driss El Moutawakil.

1 Introduction and Preliminaries There have been a number of generalizations of metric space. One such generalization is symmetric space. M. Aamri and D. El Moutawakil [1] introduced the following theorem in symmetric space.

Theorem 2.1. Let d be a symmetric for X that satisfies $(W.3)$ and (H_E). Let A and B be two weakly compatible selfmappings of (X, d) such that (1) $d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})$ for all $(x, y) \in X^2$, (2) A and B satisfy the property $(E.A)$, and (3) $AX \subseteq BX$. If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point.

The aim of the present paper is to give generalizations of Theorem 2.1 [1] in a type of generalized metric space weaker than symmetric space so called small self distance quasi-symmetric dislocated metric space

Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function. The pair (X, d) is called a distance space [3].
We need the following conditions:

(d_1) $\forall x \in X, d(x, x) = 0$,
(d_2) $\forall x, y \in X, d(x, y) = 0 \Rightarrow x = y$,
$(d_2)'$ $\forall x, y \in X, d(x, y) = d(y, x) = 0 \Rightarrow x = y$,
(d_3) $\forall x, y \in X, d(x, y) = d(y, x)$,
(d_4) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y)$.
(d_5) $\forall\, x, y \in X.\ d(x, x) \leq \min\{d(x, y), d(y, x)\}$

for all $x, y, z \in X$. If d satisfies conditions $(d_1) - (d_4)$, then (X, d) is called a metric space. If it satisfies conditions $(d_2) - (d_4)$, then (X, d) is called a dislocated metric space [3]. Also (X, d) is called a symmetric space if satisfies $(d_1) - (d_3)$.

Definition 1.2 [2]. Let A and B be two selfmappings of a metric space (X, d). We say that A and B satisfy the property (E.A) if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$$

for some $t \in X$.

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2 Main results

Definition 2.1. A distance space (X,d) is called a small self distance quasi-symmetric-dislocated metric space (ssd-q-s-d-metric space, for short) if d satisfies $(d_2)'$ and (d_5).

Example 2.1. Let X be a nonempty set and $d : X \times X \to [0,\infty)$ defined by $d(x,y) = \frac{1}{3}$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$. Then (X,d) is a small self distance quasi-symmetric-dislocated metric space.

Definition 2.2. Let (X,d) be a ssd quasi-symmetric dislocated metric space and let $Y \subset X$. Y said to be $l-$closed (resp. r-closed) if $d(x,Y) = 0$ (resp. $d(Y,x) = 0$), then $x \in Y$.

Definition 2.3. Two selfmapping A and B of ssd-q-s-d-metric X are said to be weakly compatible if they commute at there coicidence points; i.e., if $Bu = Au$ for some $u \in X$, then $BAu = ABu$.

Definition 2.4. Let (X,d) a ssd-q-sd-metric space. Then (X,d) satisfies $(\ell w3)$ if for every sequence (x_n) in X and $x, y \in X$, if $\lim_{n\to\infty} d(x,x_n) = \lim_{n\to\infty} d(y,x_n) = 0$, then $x = y$; and satisfies $(rw3)$ if for every sequence (x_n) in X and $x, y \in X$, $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x_n,y) = 0$, then $x = y$.

Definition 2.5. Let (X,d) be a ssd-q-s-d-metric space. Two self mappings A and B of (X,d) are said to have the property $(\ell - E.A - H_E)$ if

(a) $AX \subseteq BX$,

(b) there exists a sequence (x_n) such that

$\lim_{n\to\infty} d(t,Ax_n) = \lim_{n\to\infty} d(t,Bx_n) = \lim_{n\to\infty} d(Bx_n,Ax_n) = 0$ for some $t \in X$.

Also, A and B are said to have the property $(r - E.A - H_E)$ if

(a') $AX \subseteq BX$,

(b') there exists a sequence (x_n) such that

$\lim_{n\to\infty} d(Ax_n,t) = \lim_{n\to\infty} d(Bx_n,t) = \lim_{n\to\infty} d(Bx_n,Ax_n) = 0$ for some $t \in X$.

In the sequel, we need a function $\phi : R^+ \to R^+$ satisfying the condition $0 < \phi(t) < t$ for each $t > 0$.

Theorem 2.1. Let (X,d) be a ssd-q-sd-metric space that satisfies $(\ell w3)$. Let A and B be two weakly compatible selfmappings of (X,d) such that

(1) $d(Ax,Ay) \leq \phi(\max\{d(Bx,By), d(Bx,Ay), d(Ay,By)\})\ \forall\, x,y \in X$;

(2) A and B satisfies $(\ell - E.A - H_E)$. If AX or BX is $l-$closed. Then A and B have a unique common fixed point.

Proof. From (2), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} d(t,Ax_n) = \lim_{n\to\infty} d(t,Bx_n) = \lim_{n\to\infty} d(Bx_n,Ax_n) = 0$. Since BX is l-closed or AX is l-closed, then $t \in BX$ or $t \in AX$. Thus there exists $u \in X$ such that $Bu = t$. Now, we prove that $Au = Bu$. If $Au \neq Bu$, then from $(\ell w3)$, $\lim_{n\to\infty} d(Au,Ax_n) = \alpha > 0$. Thus for $0 < \epsilon < \alpha$, there exists $n_0(\epsilon) \in N$ such that $\forall\ n \geq n_0(\epsilon)$, $\mid d(Au,Ax_n) - \alpha \mid < \epsilon, i.e.,$
$\alpha - \epsilon < d(Au,Ax_n) < \epsilon + \alpha$. Thus $\forall\ n \geq n_0(\epsilon)$,

$$\begin{aligned} d(Au,Ax_n) &\leq \phi(\max\{d(Bu,Bx_n), d(Bu,Ax_n), d(Bx_n,Ax_n)\}) \\ &< \max\{d(Bu,Bx_n), d(Bu,Ax_n), d(Bx_n,Ax_n)\} \end{aligned}$$

Letting $n \to \infty$ we have $\lim_{n\to\infty} d(Au,Ax_n) = 0$. So ¿from $(\ell w3)$, $Au = Bu$. The weak compatibility of A and B implies that $ABu = BAu$ and then $AAu = ABu = BAu = BBu$. Let us show that Au is a common fixed of A and B. Suppose that $AAu \neq Au$, then

$d(AAu, Au) \neq 0$ or $d(Au, AAu) \neq 0$. First, if $d(AAu, Au) \neq 0$, then

$$\begin{aligned} d(AAu, Au) &\leq \phi(\max\{d(BAu, Bu), d(BAu, Au), d(Au, Bu)\}) = \phi(d(AAu, Au)) \\ &< d(AAu, Au), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = BAu = BBu$. Second if $d(Au, AAu) \neq 0$, then

$$\begin{aligned} d(Au, AAu) &\leq \phi(\max\{d(Bu, BAu), d(Bu, AAu), d(BAu, AAu)\}) = \phi(d(Au, AAu)) \\ &< d(Au, AAu), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = BAu$. Hence Au is a common fixed point of A and B. Suppose u and v are two fixed points of A and B and $u \neq v$. Then $d(u, v) > 0$ or $d(v, u) > 0$. If $d(u, v) > 0$, then

$$d(u, v) = d(Au, Av) \leq \phi(\max\{d(Bu, Bv), d(Bu, Av), d(Bv, Av)\} = \phi(d(u, v)) < d(u, v),$$

which is a contradiction. Also if $d(v, u) > 0$, one can deduce that $d(v, u) < d(v, u)$ which is a contradiction. Therefore $u = v$.

Theorem 2.2. Let (X, d) be a ssd-q-sd-metric space that satisfies $(r - w.3)$. Let A and B be two weakly compatible selfmappings of (X, d) such that

(1) $d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Ax, By), d(Bx, Ax)\}) \; \forall \, x, y \in X$;

(2) A and B satisfies $(r - E.A - H_E)$. If AX or BX is $r-$closed, then A and B have a unique common fixed point

Proof. From (2), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = \lim_{n\to\infty} d(Bx_n, Ax_n) = 0$. Since BX is $r-$closed or AX is $r-$closed, then $t \in BX$ or $t \in AX$. Thus there exists $u \in X$ such that $Bu = t$. Now, we prove that $Au = Bu$. If $Au \neq Bu$, then from $(rw3)$, $\lim_{n\to\infty} d(Ax_n, Bu) = \alpha > 0$. Thus for $0 < \epsilon < \alpha$, there exists $n_0(\epsilon) \in N$ such that $\forall \, n \geq n_0(\epsilon)$, $\mid d(Ax_n, Bu) - \alpha \mid < \epsilon, i.e.,$ $\alpha - \epsilon < d(Ax_n, Bu) < \epsilon + \alpha$. Thus $\forall \, n \geq n_0(\epsilon)$,

$$\begin{aligned} d(Ax_n, Au) &\leq \phi(\max\{d(Bx_n, Bu), d(Ax_n, Bu), d(Bx_n, Ax_n)\}) \\ &< \max\{d(Bx_n, Bu), d(Ax_n, Bu), d(Bx_n, Ax_n)\} \end{aligned}$$

Letting $n \to \infty$ we have $\lim_{n\to\infty} d(Ax_n, Au)) = 0$. So from $(rw3)$, $Au = Bu$. The weak compatibility of A and B implies that $ABu = BAu$ and then $AAu = ABu = BAu = BBu$. Let us show that Au is a common fixed of A and B. Suppose that $AAu \neq Au$, then $d(AAu, Au) \neq 0$ or $d(Au, AAu) \neq 0$. First, if $d(AAu, Au) \neq 0$, then

$$\begin{aligned} d(AAu, Au) &\leq \phi(\max\{d(BAu, Bu), d(AAu, Bu), d(BAu, AAu)\}) = \phi(d(AAu, Au)) \\ &< d(AAu, Au), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = BAu = BBu$. Second if $d(Au, AAu) \neq 0$, then

$$\begin{aligned} d(Au, AAu) &\leq \phi(\max\{d(Bu, BAu), d(Au, BAu), d(Bu, Au)\}) = \phi(d(Au, AAu)) \\ &< d(Au, AAu), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = BAu$. Hence Au is a common fixed of A and B. Suppose u and v are two fixed points of A and B and $u \neq v$. Then $d(u,v) > 0$ or $d(v,u) > 0$. If $d(u,v) > 0$, then

$$d(u,v) = d(Au, Av) \leq \phi(\max\{d(Bu,Bv), d(Au,Bv), d(Bu,Au)\} = \phi(d(u,v)) < d(u,v),$$

which is a contradiction. The same is obtained if $d(v,u) > 0$.Therefore $u = v$.

Conclusion. Since any symmetric space is ssd-q-s-d-metric space and the conditions in Theorem 2.1 [1] implies the conditions in Theorem 2.1 or in Theorem 2.2, then Theorem 2.1 [1] is obtained as a corollary of Theorem 2.1 or Theorem 2.2.

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