A NOTE ON QUASI P-INJECTIVE MODULES

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Abstract. Let R be a ring. In this note we study some properties of finitely generated quasi p-injective Kasch $R-$modules and show that if M_R is a finitely generated quasi p-injective Kasch module, then $M/RadM$ is semisimple if and only if S is left finite dimensional, where $S = end(M_R)$. This generalizes the result obtained by Weimin Xue.

Throught R is an associative ring with identity and modules are unitary. A right $R-$module M is called quasi p-injective if every homomorphism from an M-cyclic submodule of M to M can be extended to an endomorphism of M. These modules are studied in [5] and [6]. Clearly, R is right p-injective(principally injective) if and only if R_R is quasi p-injective. Following Albu and Wisbauer [1, 2.6], a module M_R is called Kasch if any simple module in $\sigma[M]$ embeds in M. Here $\sigma[M]$ is the category consisting of all $M-$subgenerated right $R-$modules. It is easy to see that a ring R is right Kasch if and only if R_R is Kasch. In this note we study finitely generated quasi p-injective Kasch R-modules, and some properties of p-injective Kasch rings are extended to these modules.

As usual, we denote the socle and the Jacobson radical of a module N by $Soc(N)$ and $Rad(N)$ respectively. The Goldie dimension and the lenth of a module N are denoted by $G(N)$ and $c(N)$ respectively. Let M be a right $R-$module, let $S = end(M_R), X \subseteq M$ and $Y \subseteq S$. Then we write $l_S(X) = \{s \in S \mid sx = 0, \forall x \in X\}$ and $r_M(Y) = \{m \in M \mid ym = 0, \forall y \in Y\}$.

Lemma 1. Let M_R be a Kasch module with $S = end(M_R)$. Then $l_S(T) \neq 0$ for any maximal submodule T of M.

Proof: By hypothesis, there exists a monomorphism $\varphi : M/T \to M$. Define $\alpha : M \to M$ by $x \mapsto \varphi(x + T)$. Then $0 \neq \alpha \in S, \alpha T = \varphi(0) = 0$, and so $l_S(T) \neq 0$.

Our next result extends Theorem 1.2 in [2].

Theorem 2. Let M_R be a finitely generated, quasi p-injective Kasch module with $S = end(M_R)$. Then the maps

$$K \mapsto r_M(K) \text{ and } T \mapsto l_S(T)$$

are mutually inverse bijections between the set of all minimal left ideals K of S and the set of all maximal submodules T of M. In particular

(1) $l_S r_M(K) = K$ for all minimal left ideals K of S.

(2) $r_M l_S(T) = T$ for all maximal submodules T of M.

Proof: (1) follows from [5, Theorem 2.10] because M is quasi p-injective. Observe that always $T \subseteq r_M l_S(T)$ and that $r_M l_S(T) \neq M$ by Lemma 1. Hence (2) holds by the maximality of T. The proof is completed by establishing the following claims:

Claim 1. $r_M(K)$ is a maximal submodule of M_R for all minimal left ideals K of S.

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Proof. Since M is finitely generated, $r_M(K) \subseteq T$ for some maximal submodule T of M. By Lemma 1 and (1), we have $0 \neq l_S(T) \subseteq l_S r_M(K) = K$, and hence $l_S(T) = K$ by the minimality of K. Therefore, $r_M(K) = r_M l_S(T) = T$ by (2).

Claim (2). $l_S(T)$ is a minimal left ideal of S for all maximal submodules T of M.

Proof. By Lemma 1, we can choose $0 \neq a \in l_S(T)$. Then $T = r_M(a)$, and hence $l_S(T) = l_S r_M(a) = Sa$ because M is quasi p-injcetive. It follows that $l_S(T)$ is minimal.

Proposition 3. If M_R is a finitely generated, quasi p-injective Kasch module with $S = end(M_R)$, then
(1) $l_S(RadM) \trianglelefteq_S S$,
(2) $Soc(_S S) \trianglelefteq_S S$.

Proof: (1) If $0 \neq a \in S$, choose a maximal submodule T of the right $R-$moudle aM. Since M is Kasch, there exists a monomorphism $f : aM/T \to M$. Define $g : aM \to M$ by $g(x) = f(x + T)$. As M is quasi p-injective, $g = s|aM$ for some $s \in S$. Take $y \in M$ such that $ay\overline{\in}T$. Then $say = g(ay) = f(ay + T) \neq 0$, and thus $sa \neq 0$. If $a(RadM) \nsubseteq T$, then $a(RadM) + T = aM$. But $a(RadM) << aM$ because M is finitely generated. It follows that $T = aM$, a contradiction. Hence $a(RadM) \subseteq T$. Thus, $(sa)(RadM) = g(a(RadM)) = f(0) = 0$, whence $0 \neq sa \in Sa \cap l_S(RadM)$. This implies that $l_S(RadM) \trianglelefteq_S S$.

(2) Let $0 \neq a \in S$ and let $r_M(a) \subseteq T$ for some maximal submodule T of M. Since M is quasi p-injective, $Sa = l_S r_M(a) \supseteq l_S(T)$. But $l_S(T)$ is minimal, so $Soc(_S S) \cap Sa \neq 0$, and hence $Soc(_S S) \trianglelefteq_S S$.

Corollary 4. If R is a right p-injective Kasch ring with $J = J(R)$, then
(1) [3, Lemma 2.3] $l_R(J) \trianglelefteq_R R$,
(2) [2, Corollary 1.1] $Soc(_R R) \trianglelefteq_R R$.

Lemma 5. Given a right $R-$module M_R with $S = end(M_R)$. Let $I = l_S(X)$ for some subset X of M and let K be a left ideal of S. If $r_M(I) \subseteq r_M(K)$, then $I \supseteq l_S r_M(K)$.

Proof: If $a \in l_S r_M(K)$, then $r_M(K) \subseteq r_M(a)$, and so $a \in l_S r_M(a) \subseteq l_S r_M l_S(X) = l_S(X)$, as required.

Proposition 6. Let M_R be a finitely generated Kasch module with $S = end(M_R)$. If S is left finite dimensional, then $M/RadM$ is semisimple.

Proof: Let $\Omega = \{I \mid 0 \neq I = l_S(X) \text{ for some } X \subseteq M\}$. Since S is left finite dimensional, so there exist some minimal members $I_1, I_2, \cdots, I_n$ in Ω such that $I = \oplus_{i=1}^n I_i$ is a maximal direct sum of minimal members in Ω. The proof is completed by establishing the following claims:

Claim 1. $r_M(I_i)$ is a maximal submodule of M for each i.

Proof. Since M is finitely generated and Kasch, so $r_M(I_i) \subseteq T_i = r_M l_S(T_i)$ for some maximal submodule T_i. By Lemma 5 and Lemma 1, $I_i \supseteq l_S r_M l_S(T_i) = l_S(T_i) \neq 0$, and so $I_i = l_S(T_i)$ by the minimality of I_i in Ω. Now we choose $0 \neq a_i \in l_S(T_i)$. Then $T_i = r_M(a_i)$, and thus $r_M(I_i) = r_M l_S(T_i) = r_M l_S r_M(a_i) = r_M(a_i) = T_i$.

Claim 2. $RadM = \cap_{i=1}^n r_M(I_i)$.

Proof. Clearly, $RadM \subseteq \cap_{i=1}^n r_M(I_i)$. If T is a maximal submodule of M, then $l_S(T)$ is minimal in Ω. In fact, if $l_S(T) \supseteq l_S(X) \neq 0$, where $X \subseteq M$, then $T \subseteq r_M l_S(X) \neq M$. So $T = r_M l_S(X)$, and hence $l_S(T) = l_S(X)$. Thus $l_S(T) \cap I \neq 0$. Taking some $0 \neq b \in l_S(T) \cap I$, we have $T = r_M(b) \supseteq \cap_{i=1}^n r_M(I_i)$. This gives that $\cap_{i=1}^n r_M(I_i) \subseteq RadM$, and the claim follows.

Lemma 7. Given a right $R-$Module M_R with $S = end(M_R)$. If $K_R \leq M_R$, then ${}_SHom_R(M/K, {}_SM_R) \cong l_S(K)$.

Proof: Let $\pi : M \to M/K$ be the natural epimorphism and define $\sigma : Hom_R(M/K, M) \to l_S(K)$ by $f \mapsto f\pi$. It is easy to see that σ is a left S-monomorphism. For any $s \in l_S(K)$, let $f_s : M/K \to M; x + K \mapsto s(x)$. Then $f_s \in Hom_R(M/K, M)$ with $\sigma(f_s) = s$, so σ is epic and hence an isomorphism.

Lemma 8. Let M be a right $R-$module. If $M/RadM$ is a finitely generated none zero semisimple module, then $M/RadM \cong M/T_1 \oplus M/T_2 \oplus \cdots \oplus M/T_n$ for some maximal submodules $T_1, T_2, \cdots, T_n$ of M.

Proof: It is obvious that $M/RadM$ is Artinian and hence $RadM = T_1 \cap T_2 \cap \cdots \cap T_l$ for some maximal submodules $T_1, T_2, \cdots, T_l$. Let $\varphi : M/RadM \to \oplus_{i=1}^{l} M/T_i; x + RadM \mapsto (x + T_1, x + T_2, \cdots, x + T_l)$, then φ is a monomorphism, and so there exist some members in $\{T_1, T_2, \cdots T_l\}$, say, $T_1, T_2, \cdots, T_n$ such that $M/RadM \cong \oplus_{i=1}^{n} M/T_i$.

Now we give the main result of this paper.

Theorem 9. Let M_R be a finitely generated and quasi p-injective Kasch module with $S = end(M_R)$. Then $M/RadM$ is semisimple if and only if S is left finite dimensional. In this case, $Soc({}_SS) = l_S(RadM)$, and $G({}_SS) = c({}_SSoc(S_S)) = c(M/RadM)$.

Proof: ($\Rightarrow$). It is trival in case $M = 0$. If $M \neq 0$, then $M/RadM \neq 0$ because M is finitely generated. As $M/RadM$ is semisimple, by Lemma 8, there exist maximal submodules $T_1, T_2, \cdots, T_n$ such that $M/RadM \cong \oplus_{i=1}^{n} M/T_i$. Hence, by Lemma 7 and Theorem 2, $l_S(RadM) \cong {}_SHom_R(M/RadM, {}_SM_R) \cong {}_SHom_R(\oplus_{i=1}^{n} M/T_i, {}_SM_R) \cong \oplus_{i=1}^{n} l_S(T_i)$ is semisimple. This implies that $l_S(RadM) = Soc({}_SS) \trianglelefteq_S S$ by Proposition 3, and therefore S is left finite dimensional and $G({}_SS) = n = c({}_SSoc({}_SS))$.
($\Leftarrow$). See Proposition 6.

Corollary 10. [7, Theorem 1] Let R be right p-injective and right Kasch. Then R is semilocal if and only if R is left finite dimensional. In this case, $Soc({}_RR) = Soc(R_R)$, and $G({}_RR) = c({}_RSoc({}_RR)) = c(\overline{R}_R)$, where $\overline{R} = R/J(R)$.

Proof: This is immediate from Theorem 9 and [4, Proposition 1.4].

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