MULTIPLIERS ON SECOND DUAL ALGEBRAS OF WEIGHTED GROUP ALGEBRAS

ALIREZA BAGHERI SALEC AND ABDOLHAMID RIAZI

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ABSTRACT. Let G be a locally compact group. In this paper we show by an example that compactness of G is not necessarily equivalent to the existence of a non-zero compact (or weakly compact) multiplier on the second dual of weighted group algebra $L^1(G,\omega)$.

Let G be a locally compact group with a fixed left Haar measure λ. A weight function ω on G is a strictly positive sub-additive λ-measurable function on G. The weighted group algebra $L^1(G,\omega)$ is the set of all λ-measurable functions f such that:

$$\| f \|_{1,\omega}= \int_G |f|(x)\omega(x)d\lambda(x) < \infty.$$

Let also

$$L^\infty(G,\omega) = \{f : f \;\; is \;\; \lambda - measurable \;\; and \;\; \| f \|_{\infty,\omega}=\| f/\omega \|_\infty < \infty\}.$$

Then $(L^1(G,\omega), \| \, . \, \|_{1,\omega})$ and $(L^\infty(G,\omega), \| \, . \, \|_{\infty,\omega})$ are Banach spaces and $L^\infty(G,\omega)$ is the dual of $L^1(G,\omega)$ by the pairing,

$$< f, g >= \int_G f(x)g(x)d\lambda(x) \;\; (f \in L^\infty(G,\omega) \;\; , \;\; g \in L^1(G,\omega)).$$

A.T. Lau and F. Gharamani proved that G is compact if and only if there is a compact (or weakly compact) left multiplier T on $L^1(G)^{**}$ with $< T(n), 1 >\neq 0$ for some $n \in L^1(G)^{**}$ [3,4]. Recently V. Losert proved that G is compact if and only if there is a $m \in M(G)^{**}$ such that $n \mapsto mn$ is weakly compact on $M(G)^{**}$ [7]. The last statement is equivalent to the existence a compact (or weakly compact) left multiplier on $M(G)^{**}$ or $L^1(G)^{**}$. In this paper we give a counter example for this equivalence in weighted group algebras. We recall that a Banach algebra A is Arens regular if the first and second Arens products are coincide on A^{**}, or equivalently left multipliers are $weak^*$ continuous on A^{**} (see [2] for example). So also if A is commutative, then A^{**} is commutative under either Arens product if and only if A is Arens regular [2, proposition 1].

Let G be a abelian group and ω be a weight on G with $\omega(s) \geq 1 (s \in G)$, such that $\ell^1(G,\omega)$ be Arens regular and $\ell^\infty(G,\omega)$ has a topological left invariant mean, such as m. Let . be the first Arens product on $\ell^1(G,\omega)^{**}$ and $\diamond$ be the first Arens product on $\ell^1(G,\omega)^{***}$ where $\ell^\infty(G,\omega)$ is considered with pointwise multiplication, and define the maps $\Phi : \ell^1(G,\omega) \longrightarrow \ell^1(G,\omega)^*$ and $\Psi : \ell^1(G,\omega)^{***} \longrightarrow \ell^1(G,\omega)^{**}$ by $\Phi(f) = 1.f$ and $\Psi(\mathcal{F}) = \mathcal{F} \diamond m$. Then for $n, p \in \ell^1(G,\omega)^{**}$ we have $\Phi^*(n) = n.1$ and $< \Phi^{**}(n), p >=< 1, n >< 1, p >$, because 1 is a

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character on $\ell^1(G,\omega)^{**}$ [1, proposition 6.12]. Now let $T = \Psi \circ \Phi^{**}$ then for $n, p \in \ell^1(G,\omega)^{**}$ and $f \in \ell^\infty(G,\omega)$ we have,

$$\begin{aligned} < T(n.p), f > &= < \Psi \circ \Phi^{**}(n.p), f > \\ &= < \Phi^{**}(n.p) \diamond m, f > \\ &= < \Phi^{**}(n.p), m \diamond f > \\ &= < 1, n.p >< 1, m \diamond f > \\ &= < 1, n >< 1, p >< m, f > \\ &= < 1, n >< p.m, f > \end{aligned}$$

on the other hands

$$\begin{aligned} < T(n)p, f > &= < T(n), p.f > \\ &= < \Psi \circ \Phi^{**}(n), p.f > \\ &= < \Phi^{**}(n) \diamond m, p.f > \\ &= < \Phi^{**}(n), m \diamond p.f > \\ &= < 1, n >< 1, m \diamond p.f > \\ &= < 1, n >< m, 1 \diamond p.f > \\ &= < 1, n >< m, p.f > \\ &= < 1, n >< m.p, f > \\ &= < 1, n >< p.m, f > . \end{aligned}$$

Hence T is a left multiplier on $\ell^1(G,\omega)^{**}$. In addition T is weakly compact because by Arens regularity of $\ell^1(G,\omega)$, Φ is weakly compact [2, Theorem 1]. Now if $< m, 1 >\neq 0$ we have,

$$\begin{aligned} < T(m), 1 > &= < \Psi \circ \Phi^{**}(m), 1 > \\ &= < \Phi^{**}(m) \diamond m, 1 > \\ &= < \Phi^{**}(m), m \diamond 1 > \\ &= < m, 1 >^2 \neq 0. \end{aligned}$$

Similarly we have,

$$< T(m), \omega >=< m, \omega >< m, 1 >\neq 0.$$

Thus we have following proposition.

Proposition 1: Let $\ell^1(G,\omega)$ be commutative and Arens regular and $\ell^\infty(G,\omega)$ has a topological left invariant mean, such as m, with $< m, 1 >\neq 0$. Then $\ell^1(G,\omega)^{**}$ has a weakly compact left multiplier T, with $< T(n), \omega >\neq 0$ (or $< T(n), 1 >\neq 0$) for some $n \in \ell^1(G,\omega)^{**}$.

Let $\Omega(x,y) = \omega(xy)/\omega(x)\omega(y)$ for $x, y \in G$. Then Ω is called zero cluster if

$$lim_n lim_m \Omega(x_n, y_m) = 0 = lim_m lim_n \Omega(x_n, y_m)$$

for all sequences $\{x_n\}$ and $\{y_n\}$ in G with distinct element, whenever both iterated limits of $\Omega(x_n, y_m)$ exists. By [8] $L^1(G,\omega)$ is Arens regular if and only if $L^1(G,\omega)^{**}$ is Arens regular if and only if G is finite or G is discrete and Ω is zero cluster. Hence we have following example by proposition 1.

Example 2: Let $\omega_\alpha(n) = (1 + |n|)^\alpha, \quad (n \in \mathbb{Z}, \quad \alpha > 0)$. Then for each $\alpha > 0$ and $G = (\mathbb{Z}, +)$, there is a weakly compact left multiplier T on $\ell^1(\mathbb{Z}, \omega_\alpha)^{**}$ with $< T(n), \omega_\alpha >\neq 0$

(or $< T(n), 1 >\neq 0$) for some $n \in \ell^1(\mathbb{Z}, \omega_\alpha)^{**}$.

Note that by [4] $L^1(G)$ is amenable if and only if there is a non-zero compact (or weakly compact) right multiplier on $L^1(G)^{**}$. Similarly if $L^1(G,\omega)$ is amenable then by [9, corollary 2] $L^1(G,\omega)$ and $L^1(G)$ are isomorphic, and hence by [4, theorem 2.1] there is a non-zero compact (or weakly compact) right multiplier on $L^1(G,\omega)^{**}$. Example 2 show that the converse is not necessarily true because ω_α^* is not bounded (see [6]).

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Faculty of Mathematics and Computer Science, Amir kabir University, 424 Hafez Avenue, Tehran 15914, Iran.

E-mail address: a.r.bagheri_s@aut.ac.ir, riazi@aut.ac.ir