# APPROXIMATING COMMON FIXED POINTS OF NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. In this paper, we prove the following theorem: Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable, let $\Im = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$ such that $F(\Im) = \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and let $P$ be the sunny nonexpansive retraction from $C$ onto $F(\Im)$. For some $u \in C$, define a sequence $\{x_n\}$ in $C$ by $x_n = (1-\alpha_n)T(t_n)x_n + \alpha_n u$, where $0 < \alpha_n < 1$, $t_n > 0$ for all $n \geq 1$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Then $\{x_n\}$ converges strongly to $Pu$.

**1 Introduction** Let $E$ be a real Banach space and let $C$ be a nonempty closed convex subset of $E$. Then a mapping $T$ of $C$ into itself is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. For a given $u \in C$ and each $r \in (0,1)$, we define a contraction $T_r : C \to C$ by

$$T_r x = (1-r)Tx + ru \quad \text{for all} \quad x \in C,$$

where $T : C \to C$ is a nonexpansive mapping. Then, there exists a unique fixed point $x_r$ of $T_r$ in $C$, that is, we have a unique point $x_r$ such that

$$x_r = (1-r)Tx_r + ru.$$

A question naturally arises to whether $\{x_r\}$ converges strongly as $r \to 0$ to a fixed point of $T$. This question has been investigated for nonexpansive self-mappings(or nonself-mappings) by several authors; see, for example, Browder [2], Halpern [5], Singh and Watson [9], Xu-Yin [14], Kim-Takahashi [6], Takahashi-Kim [12] and others.

Recently, Suzuki [10] proved the following theorem: Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\Im = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$ such that $F(\Im) \neq \emptyset$. For a fixed $u \in C$, define a sequence $\{x_n\}$ in $C$ by

$$x_n = (1-\alpha_n)T(t_n)x_n + \alpha_n u \quad \text{for all} \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$ satisfy $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Then $\{x_n\}$ converges strongly to the element of $F(\Im)$ nearest to $u$.

In this paper, using Banach limits, we prove a strong convergence theorem for a strongly continuous semigroup of nonexpansive mappings in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. This extends Suzuki's result [10] in a Hilbert space to a Banach space.

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**2 Preliminaries** Throughout this paper we denote by $E$ and $E^*$ a Banach space and the dual space of $E$, respectively. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let $C$ be a nonempty closed convex subset of $E$ and let $T$ be a mapping from $C$ into itself. Then we denote by $F(T)$ the set of all fixed points of $T$, i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by $\mathbb{N}$ and $\mathbb{R}^+$ the sets of positive integers and nonnegative real numbers, respectively. When $\{x_n\}$ is a sequence in $E$, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to $x$. A Banach space $E$ is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of $E$ is said to be *Gâteaux differentiable* (and $E$ is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for each $x$ and $y$ in $S(E)$. It is also said to be *uniformly Gâteaux differentiable* if for each $y \in S(E)$, the limit (1) is attained uniformly for $x$ in $S(E)$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Then $J : E \to E^*$ is said to be the *duality mapping*. It is well known if $E$ is smooth, then the duality mapping $J$ is single-valued and strong-$weak^*$ continuous. It is also known if $E$ has a uniformly Gâteaux differentiable norm, $J$ is uniformly continuous on bounded sets when $E$ has its strong topology while $E^*$ has its weak star topology; for more details, see Diestel [4] and Takahashi [11]. Let $\mu$ be a continuous, linear functional on $l^\infty$ and let $(a_1, a_2, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_1, a_2, \cdots))$. We call $\mu$ a *Banach limit* [1] when $\mu$ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_1, a_2, \cdots) \in l^\infty$. For a Banach limit $\mu$, we know that

$$\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n \quad \text{for all} \quad (a_1, a_2, \cdots) \in l^\infty.$$

So, we have that if $a_n \to 0$, then $\mu_n(a_n) \to 0$; see [11] for more details. Let $C$ be a convex subset of $E$, let $K$ be a nonempty subset of $C$ and let $P$ be a *retraction* from $C$ onto $K$, i.e., $Px = x$ for each $x \in K$. $P$ is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny nonexpansive retraction from $C$ onto $K$, $K$ is said to be a *sunny nonexpansive retract* of $C$. Let $\Im = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings on a closed convex subset $C$ of a Banach space $E$, i.e.,

(1) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on $C$;

(2) $T(0)x = x$ for all $x \in C$;

(3) $T(s + t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;

(4) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into $C$ is continuous.

We also set $F(\Im) = \bigcap_{t \in \mathbb{R}^+} F(T(t))$.

**3 Strong convergence theorem** For proving our main theorem, we need the following lemmas.

**Lemma 1 ([8]).** *Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $\{x_n\}$ be a bounded sequence of $E$ and let $\mu$ be a Banach limit. Let $g$ be a real valued function on $C$ defined by*

$$g(y) = \mu_n \|x_n - y\|^2 \quad \textit{for every} \quad y \in C.$$

*Then $g$ is continuous and convex, and $g$ satisfies $\lim_{\|y\| \to \infty} g(y) = \infty$. Moreover, for each $R > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$g\Big(\frac{y+z}{2}\Big) \le \frac{g(y) + g(z)}{2} - \delta$$

*for all $y, z \in C \cap B_R$ with $\|y - z\| \ge \epsilon$, where $B_R$ is the closed ball with center $0$ and radius $R$.*

**Lemma 2 ([13]).** *Let $C$ be a nonempty convex subset of a Banach space $E$ whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded subset of $E$, let $z$ be an element of $C$ and let $\mu$ be a Banach limit. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

*if and only if*

$$\mu_n \langle y - z, J(x_n - z) \rangle \le 0 \quad \textit{for all} \quad y \in C,$$

*where $J$ is the duality mapping on $E$.*

**Lemma 3 ([3], [7]).** *Let $C$ be a convex subset of a smooth Banach space, let $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$. Then $P$ is sunny and nonexpansive if and only if*

$$\langle x - Px, J(y - Px) \rangle \le 0 \quad \textit{for all} \quad x \in C \quad \textit{and} \quad y \in K.$$

We extend Theorem 3 of Suzuki [10] to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

**Theorem.** *Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$. Let $\Im = \{T(t) : t \ge 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$ such that $F(\Im) \ne \emptyset$ and let $P$ be the sunny nonexpansive retraction from $C$ onto $F(\Im)$. For some $u \in C$, define a sequence $\{x_n\}$ in $C$ by*

$$x_n = (1 - \alpha_n) T(t_n) x_n + \alpha_n u \quad \textit{for all} \quad n \ge 1,$$

*where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$ satisfy $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0$. Then $\{x_n\}$ converges strongly to $Pu$.*

*Proof.* Let $x$ be an element of $F(\Im)$. Then we have

$$\begin{aligned} \|x_n - x\| &= \|(1 - \alpha_n) T(t_n) x_n + \alpha_n u - x\| \\ &\le (1 - \alpha_n) \|T(t_n) x_n - x\| + \alpha_n \|u - x\| \\ &\le (1 - \alpha_n) \|x_n - x\| + \alpha_n \|u - x\| \end{aligned}$$

and hence

$$\alpha_n\|x_n - x\| \le \alpha_n\|u - x\|.$$

So, we have $\|T(t_n)x_n - x\| \le \|x_n - x\| \le \|u - x\|$. Hence, setting $r = \|u - x\|$ and $D = C \cap B_r$, we obtain, for any $v \in D$ and $s \in \mathbb{R}^+$, $\|T(s)v - x\| \le \|v - x\| \le \|u - x\|$ and hence $T(s)D \subset D$. Further, $x$, $u$ and $Pu$ are in $D$. So, without loss of generality, we can assume that $C$ is bounded. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. To prove the theorem, it is sufficient to show that there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\{x_{n_{i_j}}\}$ converges strongly to $Pu$. Put $w_i = x_{n_i}$, $\beta_i = \alpha_{n_i}$ and $s_i = t_{n_i}$ for $i \in \mathbb{N}$. For a Banach limit $\mu$, we can define a real valued function $g$ on $C$ given by

$$g(y) = \mu_i\|w_i - y\|^2 \quad \text{for every} \quad y \in C.$$

From Lemma 1, we see that there exists a unique element $z$ of $C$ satisfying

$$g(z) = \min_{y \in C} g(y).$$

We shall first prove that $z \in F(\Im)$. To prove $z \in F(\Im)$ it sufficies to show $\lim_{t\to\infty} T(t)z = z$. In fact, for any $s \in \mathbb{R}^+$, we have $T(s)z = T(s)\lim_{t\to\infty} T(t)z = \lim_{t\to\infty} T(s+t)z = z$. Suppose $\lim_{t\to\infty} T(t)z \ne z$. Then there exists $\epsilon > 0$ such that for each $s > 0$, there exists $t \ge s$ satisfying $\|T(t)z - z\| \ge \epsilon$. Take $t \in \mathbb{R}^+$ with $t > s_i (i \in \mathbb{N})$ and $\|T(t)z - z\| \ge \epsilon$. Then, we have

$$\begin{aligned}
\|w_i - T(t)z\| \le & \sum_{k=0}^{[\frac{t}{s_i}]-1} \|T((k+1)s_i)w_i - T(ks_i)w_i\| \\
& + \|T([\frac{t}{s_i}]s_i)w_i - T([\frac{t}{s_i}]s_i)z\| + \|T([\frac{t}{s_i}]s_i)z - T(t)z\| \\
& \le [\frac{t}{s_i}]\|T(s_i)w_i - w_i\| + \|w_i - z\| + \|T(t - [\frac{t}{s_i}]s_i)z - z\| \\
& = [\frac{t}{s_i}]\beta_i\|T(s_i)w_i - u\| + \|w_i - z\| + \|T(t - [\frac{t}{s_i}]s_i)z - z\| \\
& \le \frac{t\beta_i}{s_i}\|T(s_i)w_i - u\| + \|w_i - z\| + \|T(t - [\frac{t}{s_i}]s_i)z - z\|
\end{aligned}$$

for $i \in \mathbb{N}$. Since $\frac{t\beta_i}{s_i} \to 0$ and $t - [\frac{t}{s_i}]s_i \to 0$ as $i \to \infty$, from the property of $\mu$, we have

$$\mu_i\|w_i - T(t)z\|^2 \le \mu_i\|w_i - z\|^2. \tag{2}$$

By Lemma 1, there exists $\delta > 0$ such that

$$\mu_i\left\|w_i - \frac{p+q}{2}\right\|^2 \le \frac{1}{2}(\mu_i\|w_i - p\|^2 + \mu_i\|w_i - q\|^2) - \delta \tag{3}$$

for all $p, q \in C \cap B_R$ with $\|p - q\| \ge \epsilon$. By using (2) and (3), we obtain

$$\begin{aligned}
\mu_i\left\|w_i - \frac{T(t)z + z}{2}\right\|^2 & \le \frac{1}{2}(\mu_i\|w_i - T(t)z\|^2 + \mu_i\|w_i - z\|^2) - \delta \\
& \le \frac{1}{2}(\mu_i\|w_i - z\|^2 + \mu_i\|w_i - z\|^2) - \delta \\
& < \mu_i\|w_i - z\|^2.
\end{aligned}$$

This is a contradiction. Hence $z \in F(\Im)$. Let $w \in F(\Im)$. Then, from $w_i = (1-\beta_i)T(s_i)w_i + \beta_i u$ and $T(s_i)w = w$, we have

$$\begin{aligned}\langle \frac{1}{1-\beta_i}w_i - \frac{\beta_i}{1-\beta_i}u - w, J(w_i - w)\rangle &= \langle T(s_i)w_i - T(s_i)w, J(w_i - w)\rangle \\ &\le \|T(s_i)w_i - T(s_i)w\|\|J(w_i - w)\| \\ &\le \|w_i - w\|^2 \\ &= \langle w_i - w, J(w_i - w)\rangle\end{aligned}$$

and hence $\frac{\beta_i}{1-\beta_i}\langle w_i - u, J(w_i - w)\rangle \le 0$. So, we obtain

$$\langle w_i - u, J(w_i - w)\rangle \le 0. \tag{4}$$

In particular, we obtain

$$\|w_i - z\|^2 \le \langle u - z, J(w_i - z)\rangle.$$

Using Lemma 2, we obtain

$$\mu_i\|w_i - z\|^2 \le \mu_i\langle u - z, J(w_i - z)\rangle \le 0.$$

Hence there exists a subsequence of $\{w_i\}$ converging strongly to $z \in F(\Im)$. Let $\{w_{i_j}\}$ be a subsequence of $\{w_i\}$ such that $\lim_{j\to\infty} w_{i_j} = z \in F(\Im)$. Then we obtain $z = Pu$. In fact, from (4), we obtain

$$\langle w_{i_j} - u, J(w_{i_j} - Pu)\rangle \le 0.$$

So, we obtain

$$\langle z - u, J(z - Pu)\rangle \le 0.$$

Using Lemma 3, we obtain

$$\|z - Pu\|^2 \le \langle u - Pu, J(z - Pu)\rangle \le 0.$$

Hence we obtain $z = Pu$. Therefore, we obtain $x_n \to Pu$. □

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