NOOR ITERATIONS WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE IN A BANACH SPACE

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Abstract. In this paper, we approximate common fixed points of two asymptotically nonexpansive mappings in the intermediate sense by three-step iteration process with random errors. Our weak and strong convergence results extend, generalize and improve the corresponding results of Kim and Kim in 2001, Kim, Kiuchi and Takahashi in 2004, Khan and Takahashi in 2000, Rhoades in 1994, Tan and Xu in 1993, and Xu and Noor in 2002.

1 Introduction Let C be a nonempty convex subset of a Banach space E and let $T : C \to C$ be a mapping. Then T is (i) asymptotically nonexpansive [2] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \text{ for all } x, y \in C \text{ and } n \ge 1;$$

if $k_n = 1$ for all $n \ge 1$, then T becomes nonexpansive and (ii) asymptotically nonexpansive in the weak sense (cf. Kirk [6]) if

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for each $x \in C$ and that T^N is continuous for some $N \ge 1$.

Consider a definition somewhere between these two: T is said to be asymptotically nonexpansive in the intermediate sense [1] if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

It is remarkable that every asymptotically nonexpansive mapping with bounded domain is asymptotically nonexpansive in the intermediate sense but the converse is not true in general; see [4].

Recall that a Banach space E is uniformly convex if for each $\epsilon \in [0, 2]$, the modulus of convexity of E given by:

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon \right\},$$

satisfies the inequality $\delta(\epsilon) > 0$ for all $\epsilon > 0$. The classical definition of the Opial property [7] states that whenever $x_n \rightharpoonup x$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

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for all $y \neq x$, where $\rightharpoonup$ denotes the weak convergence. Denote by τ, a Hausdorff linear topology on E. The τ-Opial property [1] is defined analogous to the classical Opial property replacing weak convergence by τ-sequential convergence. Moreover, E has the uniform τ-Opial property [1] provided for every $c > 0$, there exists $r > 0$ such that $1 + r \leq \limsup_{n\to\infty} \|x_n - x\|$ for each $x \in E$ with $\|x\| \geq c$ and each sequence $\{x_n\}$ with $x_n \xrightarrow{\tau} 0$ as $n \to \infty$, $\limsup_{n\to\infty} \|x_n\| \geq 1$; see Prus [9]. Note that a uniformly convex Banach space with τ-Opial property always has the uniform τ-Opial property. A mapping $T : C \to C$ is compact (or completely continuous) if every bounded sequence $\{x_n\}$ in C implies that $\{Tx_n\}$ has a convergent subsequence. Moreover, a mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Recently, for a mapping $T : C \to C$, Xu and Noor [13] constructed three-step iteration process in C as:

$$\tag{1.1} \begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad \text{for all } n \geq 1, \end{cases}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$.

In particular, if $\gamma_n = 0$ in (1.1), then the sequence $\{x_n\}$ becomes Ishikawa iteration process. Mann iteration process is obtained by taking $\beta_n = 0 = \gamma_n$ in (1.1). Xu and Noor proved the following result in [13].

Theorem A (Theorem 2.1, Xu-Noor[13]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define a sequence $\{x_n\}$ as given in* (1.1) *where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$ such that* (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Then $\{x_n\}, \{y_n\}, \{z_n\}$ converge to the same fixed point of T.*

In [13], Xu and Noor also pointed out that Ishikawa type convergence follows directly from Theorem A by taking $\gamma_n = 0$ while the Mann type convergence fails. This motivated them to prove Theorem 2.2 in [13] which unifies Ishikawa as well as Mann type convergence.

Motivated by the work of Xu and Noor in [13], we construct three-step iteration process $\{x_n\}$ with errors for two nonlinear mappings $S, T : C \to C$ given by:

$$\tag{1.2} \begin{cases} x_1 \in C, \\ z_n = \alpha''_n x_n + \beta''_n T^n x_n + \gamma''_n w_n, \\ y_n = \alpha'_n x_n + \beta'_n S^n z_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C.

This scheme is named as Noor iterations with errors. Mann, Ishikawa and Xu-Noor iteration processes can be obtained from the above scheme as special cases by suitably choosing the mappings and the parameters.

In this paper, we approximate the common fixed points of two asymptotically nonexpansive mappings in the intermediate sense by using Noor iterations with errors. Our results, not only improve and generalize the corresponding results of Kim and Kim [4], Kim et al. [5], Khan and Takahashi [3], Rhoades [10], Schu [11] and Xu and Noor [13] but also unify Xu-Noor type, Ishikawa type and Mann type convergence results. It is remarked that we have considered more general iterations as well as a wider class of mappings than that studied by Xu and Noor [13].

In the sequel, we shall need the following well-known results.

Lemma 1 (Tan-Xu[12]). *Let $\{a_n\}, \{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq a_n + b_n \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2 (Xu[14]). *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda) \|y\|^p - \pi_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_r[0]$, where $B_r[0] = \{x \in E : \|x\| \leq r\}$ and $\pi_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ for all $\lambda \in [0, 1]$.

Theorem B (Bruck-Kuczumow-Reich[1]). *Suppose a Banach space E has the uniform τ-Opial property, C is a norm-bounded sequentially τ-compact subset of E and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in C such that $\lim_{n\to\infty} \|y_n - z\|$ exists for each fixed point z of T and if $\{y_n - T^k y_n\}$ is τ-convergent to 0 for each $k \geq 1$, then $\{y_n\}$ is τ-convergent to a fixed point of T.*

2 Common Fixed Point Theorems Denote by $F(T)$ and $F(S, T)$, the set of fixed points of T and the set of common fixed points of S and T respectively. In case of $S = T$, $F(S, T) = F(S) = F(T)$.

First, we prove a few lemmas which will be needed in the main results.

Lemma 3. *Let C be a nonempty convex subset of a Banach space E and let $S, T : C \to C$ be mappings with $F(S, T) \neq \emptyset$. Put*

$$r_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Then for the sequence $\{x_n\}$ generated by the scheme (1.2), $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S, T)$.

Proof. Set $H = \sup_{n\geq 1} \|u_n - p\| \vee \sup_{n\geq 1} \|v_n - p\| \vee \sup_{n\geq 1} \|w_n - p\|$ for $p \in F(S, T)$. Since

$$\begin{aligned}
\|z_n - p\| &\leq \alpha''_n \|x_n - p\| + \beta''_n \|T^n x_n - p\| + \gamma''_n \|w_n - p\| \\
&\leq \alpha''_n \|x_n - p\| + \beta''_n (\|x_n - p\| + r_n) + \gamma''_n H \\
&= (\alpha''_n + \beta''_n) \|x_n - p\| + \beta''_n r_n + \gamma''_n H \\
&\leq \| x_n - p\| + r_n + \gamma''_n H
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &\leq \alpha'_n \|x_n - p\| + \beta'_n \|S^n z_n - p\| + \gamma'_n \|v_n - p\| \\
&\leq \alpha'_n \|x_n - p\| + \beta'_n (\|z_n - p\| + r_n) + \gamma'_n H,
\end{aligned}$$

we have

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n\|x_n-p\|+\beta_n\|T^ny_n-p\|+\gamma_n\|u_n-p\|\\ &\leq \beta_n(\|y_n-p\|+r_n)+\alpha_n\|x_n-p\|+\gamma_nH\\ &\leq \beta_n[\alpha_n'\|x_n-p\|+\beta_n'(\|x_n-p\|+2r_n+\gamma_n''H)+r_n]\\ &\quad+\beta_n\gamma_n'H+\alpha_n\|x_n-p\|+\gamma_nH\\ &\leq (\alpha_n+\alpha_n'\beta_n+\beta_n\beta_n')\|x_n-p\|+3r_n+(\beta_n\beta_n'\gamma_n''+\beta_n\gamma_n'+\gamma_n)H\\ &\leq \|x_n-p\|+3r_n+(\beta_n\beta_n'\gamma_n''+\beta_n\gamma_n'+\gamma_n)H.\end{aligned}$$

The conclusion follows from Lemma 1. ☐

Remark 1. Lemma 2.4 in [5], Lemma 3 in [8] are special cases of Lemma 3.

Lemma 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S,T:C\to C$ be mappings with $F(S,T)\neq\emptyset$ and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by* (1.2). *Put*

$$r_n=\sup_{x,y\in C}(\|S^nx-S^ny\|-\|x-y\|)\vee\sup_{x,y\in C}(\|T^nx-T^ny\|-\|x-y\|)\vee 0$$

and suppose that $\sum_{n=1}^{\infty}r_n<\infty$. Then (i) *and* (ii) *hold.*

(i) *If* $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, *then*

$$\lim_{n\to\infty}\|T^ny_n-x_n\|=0;$$

(ii) *If* $\liminf_{n\to\infty}\beta_n>0$ *and* $0<\liminf_{n\to\infty}\beta_n'\leq\limsup_{n\to\infty}\beta_n'<1$, *then*

$$\lim_{n\to\infty}\|S^nz_n-x_n\|=0.$$

Proof. Let $p\in F(S,T)$. Since $\lim_{n\to\infty}\|x_n-p\|$ exists as proved in Lemma 3, we can obtain a closed ball $B_r[0]$ such that $\{x_n-p,y_n-p,z_n-p,T^nx_n-p,T^ny_n-p,S^nz_n-p,u_n-x_n,v_n-x_n,w_n-x_n\}\subset B_r[0]\cap C$. Let Q denote the possible constant terms appearing in the following estimates: With the help of Lemma 2 and the scheme (1.2), we have

$$\begin{aligned}\|z_n-p\|^2 &= \|\beta_n''(T^nx_n-p)+(1-\beta_n'')(x_n-p)+\gamma_n''(w_n-x_n)\|^2\\ &\leq \|\beta_n''(T^nx_n-p)+(1-\beta_n'')(x_n-p)\|^2+\gamma_n''Q\\ &\leq \beta_n''\|T^nx_n-p\|^2+(1-\beta_n'')\|x_n-p\|^2\\ &\quad-\pi_2(\beta_n'')g(\|x_n-T^nx_n\|)+\gamma_n''Q\\ &\leq \beta_n''\left(\|x_n-p\|^2+2r_n\|x_n-p\|+r_n^2\right)\\ &\quad+(1-\beta_n'')\|x_n-p\|^2+\gamma_n''Q\\ &= \|x_n-p\|^2+2\beta_n''r_n\|x_n-p\|+\beta_n''r_n^2+\gamma_n''Q\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\|y_n-p\|^2 &= \|\beta_n'(S^nz_n-p)+(1-\beta_n')(x_n-p)+\gamma_n'(v_n-x_n)\|^2\\ &\leq \|\beta_n'(S^nz_n-p)+(1-\beta_n')(x_n-p)\|^2+\gamma_n'Q\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\leq \beta'_n \left\|S^n z_n - p\right\|^2 + (1-\beta'_n)\left\|x_n - p\right\|^2 \\
&\quad - \pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) + \gamma'_n Q \\
&\leq \beta'_n \left(\left\|z_n - p\right\|^2 + 2r_n\left\|z_n - p\right\| + r_n^2\right) + (1-\beta'_n)\left\|x_n - p\right\|^2 \\
&\quad - \pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) + \gamma'_n Q.
\end{aligned}$$

Substituting (2.1) into (2.2), it follows that

$$\begin{aligned}
\left\|y_n - p\right\|^2 &\leq \left\|x_n - p\right\|^2 + 2\beta'_n\beta''_n r_n \left\|x_n - p\right\| + 2\beta'_n r_n \left\|z_n - p\right\| \\
&\quad + \beta'_n(1+\beta''_n) r_n^2 - \pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) + (\gamma'_n + \gamma''_n) Q.
\end{aligned} \tag{2.3}$$

Again by Lemma 2, the scheme (1.2) and inequality (2.3), we infer that

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &\leq \left\|\beta_n(T^n y_n - p) + (1-\beta_n)(x_n - p)\right\|^2 + \gamma_n Q \\
&\leq \beta_n \left\|T^n y_n - p\right\|^2 + (1-\beta_n)\left\|x_n - p\right\|^2 \\
&\quad - \pi_2(\beta_n) g\left(\left\|T^n y_n - x_n\right\|\right) + \gamma_n Q \\
&\leq \beta_n \left[\left\|y_n - p\right\|^2 + 2r_n\left\|y_n - p\right\| + r_n^2\right] + (1-\beta_n)\left\|x_n - p\right\|^2 \\
&\quad - \pi_2(\beta_n) g\left(\left\|T^n y_n - x_n\right\|\right) + \gamma_n Q \\
&\leq \beta_n \left\|x_n - p\right\|^2 + 2\beta_n\beta'_n\beta''_n r_n \left\|x_n - p\right\| + 2\beta_n\beta'_n r_n \left\|z_n - p\right\| \\
&\quad + \beta_n\beta'_n(1+\beta''_n) r_n^2 + \beta_n(\gamma'_n + \gamma''_n) Q + 2r_n\beta_n \left\|y_n - p\right\| + \beta_n r_n^2 \\
&\quad - \beta_n\pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) + (1-\beta_n)\left\|x_n - p\right\|^2 \\
&\quad - \pi_2(\beta_n) g\left(\left\|T^n y_n - x_n\right\|\right) + \gamma_n Q \\
&\leq \left\|x_n - p\right\|^2 - \pi_2(\beta_n) g\left(\left\|T^n y_n - x_n\right\|\right) - \beta_n\pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) \\
&\quad + (r_n + \gamma_n + \gamma'_n + \gamma''_n) Q.
\end{aligned}$$

¿From this, we obtain the following two inequalities:

$$\begin{aligned}
\pi_2(\beta_n) g\left(\left\|T^n y_n - x_n\right\|\right) &\leq \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&\quad + (r_n + \gamma_n + \gamma'_n + \gamma''_n) Q
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\beta_n\pi_2(\beta'_n) g\left(\left\|S^n z_n - x_n\right\|\right) &\leq \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&\quad + (r_n + \gamma_n + \gamma'_n + \gamma''_n) Q.
\end{aligned} \tag{2.5}$$

Suppose that (i) holds. Then there exists $\delta > 0$ and a positive integer n_0 such that $\pi_2(\beta_n) \geq \delta$ for all $n \geq n_0$. Let m be any positive integer satisfying $m \geq n_0$. Summing up the terms from n_0 to m on both sides in the inequality (2.4), we have

$$\begin{aligned}
\delta \sum_{n=n_0}^{m} g\left(\left\|T^n y_n - x_n\right\|\right) &\leq \left\|x_{n_0} - p\right\|^2 - \left\|x_{m+1} - p\right\|^2 \\
&\quad + Q \sum_{n=n_0}^{m} (r_n + \gamma_n + \gamma'_n + \gamma''_n) \\
&\leq \left\|x_{n_0} - p\right\|^2 + Q \sum_{n=n_0}^{m} (r_n + \gamma_n + \gamma'_n + \gamma''_n).
\end{aligned}$$

When $m \to \infty$ in the above inequality, we get that

$$\delta \sum_{n=1}^{\infty} g\left(\|T^n y_n - x_n\|\right) < \infty$$

and hence

$$\lim_{n\to\infty} g\left(\|T^n y_n - x_n\|\right) = 0.$$

By virtue of the properties of g, we conclude that

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0.$$

Next assume (ii). So there exists $\eta > 0$ and a positive integer n_1 such that $\beta_n \pi_2(\beta'_n) > \eta$ for all $n \geq n_1$. Therefore inequality (2.5) becomes

$$\eta \sum_{n=n_1}^{\infty} g\left(\|S^n z_n - x_n\|\right) \leq \|x_{n_1} - p\|^2 + Q \sum_{n=n_1}^{\infty} \left(r_n + \gamma_n + \gamma'_n + \gamma''_n\right) < \infty$$

and hence $\lim_{n\to\infty} \|S^n z_n - x_n\| = 0$. ☐

Remark 2. Lemma 4 generalizes Lemma 2.2 in [13].

Lemma 5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be mappings with $F(S,T) \neq \emptyset$. Put*

$$r_n = \sup_{x,y\in C} \left(\|S^n x - S^n y\| - \|x - y\|\right) \vee \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. For the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by (1.2), where the sequences $\{\beta_n\}$ and $\{\beta'_n\}$ satisfy the additional restrictions: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \beta'_n \leq \limsup_{n\to\infty} \beta'_n < 1$, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Proof. As proved in Lemma 4, we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0 = \lim_{n\to\infty} \|S^n z_n - x_n\|.$$

Since

$$\|x_n - y_n\| \leq \beta'_n \|x_n - S^n z_n\| + \gamma'_n \|x_n - v_n\| \to 0$$

as $n \to \infty$, therefore

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\leq \|x_n - y_n\| + r_n + \|T^n y_n - x_n\| \to 0 \end{aligned} \tag{2.6}$$

as $n \to \infty$. On the other hand

$$\|x_n - z_n\| \leq \beta''_n \|x_n - T^n x_n\| + \gamma''_n \|x_n - w_n\| \to 0$$

as $n \to \infty$, gives

$$\begin{aligned} \|S^n x_n - x_n\| &\leq \|S^n x_n - S^n z_n\| + \|S^n z_n - x_n\| \\ &\leq \|x_n - z_n\| + r_n + \|S^n z_n - x_n\| \to 0 \end{aligned}$$

as $n \to \infty$. Also observe that

$$\begin{aligned}\|x_n - x_{n+1}\| &\le \|x_n - T^n x_n\| + \|T^n x_n - x_{n+1}\| \\ &\le \|x_n - T^n x_n\| + \alpha_n \|T^n x_n - x_n\| \\ &\quad + \beta_n \|T^n x_n - T^n y_n\| + \gamma_n \|u_n - T^n x_n\| \\ &\le \|x_n - T^n x_n\| + \alpha_n \|T^n x_n - x_n\| \\ &\quad + \beta_n (\|x_n - y_n\| + r_n) + \gamma_n \|u_n - T^n x_n\| \to 0\end{aligned}$$

as $n \to \infty$. From this, uniform continuity of T, (2.6) and the inequality

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| \\ &\quad + \left\|T^{n+1} x_{n+1} - T^{n+1} x_n\right\| + \left\|T^{n+1} x_n - Tx_n\right\| \\ &\le 2\|x_n - x_{n+1}\| + r_{n+1} + \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| + \left\|T^{n+1} x_n - Tx_n\right\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Similarly,

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

completing the proof. □

Remark 3. Lemma 5 extends Lemma 2.9 in [5] and Lemma 4 in [3].

We now establish the following weak convergence result based on Theorem B.

Theorem 1. *Let E be a uniformly convex Banach space satisfying the Opial property and let C be a nonempty bounded closed convex subset of E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense with $F(S,T) \neq \emptyset$. Put*

$$r_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be generated by (1.2), where the sequences $\{\beta_n\}$ and $\{\beta'_n\}$ satisfy the additional restrictions: $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \beta'_n \le \limsup_{n\to\infty} \beta'_n < 1$. Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to the same point p of $F(S,T)$.

Proof. It follows from Lemma 3 that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S,T)$ and

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$$

by Lemma 5. As $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$ and S is uniformly continuous, so for any $k \ge 1$, inductively, we have

$$\left\|S^k x_n - x_n\right\| \to 0 \text{ as } n \to \infty.$$

Similarly

$$\left\|T^k x_n - x_n\right\| \to 0 \text{ as } n \to \infty.$$

Now, we can use Theorem B with weak topology instead of τ-topology and hence, there exists $z_1 \in F(S)$, $z_2 \in F(T)$ such that $x_n \rightharpoonup z_1$ and $x_n \rightharpoonup z_2$. By uniqueness property of the limit, $z_1 = z_2 \in F(S,T)$. Further, the limits of the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ coincide by Lemma 5. □

The following corollary is an immediate consequence of the above theorem; this result improves Theorem 2.10 of Kim, Kiuchi and Takahashi [5] and includes as special case, Theorem 1 of Khan and Takahashi [3].

Corollary 1. *Let E be a uniformly convex Banach space satisfying the Opial property and let C be a nonempty bounded closed convex subset of E. Let $S,T : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense with $F(S,T) \neq \emptyset$. Put*

$$r_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x-y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Generate $\{x_n\}$, $\{y_n\}$ by:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n' x_n + \beta_n' S^n x_n + \gamma_n' v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad \textit{for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\}$ are real sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} \beta_n' \leq \limsup_{n\to\infty} \beta_n' < 1$ and $\{u_n\}, \{v_n\}$ are sequences in C. Then $\{x_n\}$, $\{y_n\}$ converge weakly to the same point p of $F(S,T)$.

Now we prove a strong convergence theorem; our result constitutes a generalization of Theorem 1 and Theorem 2 in [4], Theorem 3.6 in [5], Theorem 2.1 in [13] and Theorem 2 in [3].

Theorem 2. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S,T : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense with $F(S,T) \neq \emptyset$. Let r_n defined as in Theorem 1 satisfying $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be generated by (1.2), where the sequences $\{\beta_n\}$ and $\{\beta_n'\}$ satisfy the additional restrictions: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \beta_n' \leq \limsup_{n\to\infty} \beta_n' < 1$. Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to the same point p of $F(S,T)$, if one of the following conditions is satisfied:*

(i) *$\{x_n\}$ has a subsequence which converges strongly to a point in C;*

(ii) *S^m is compact for some $m \geq 1$.*

Proof. As obtained in Lemma 5, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|. \tag{2.7}$$

Assume that (i) holds. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \to z \in C$. Then (2.7) assures that z is a common fixed point of S and T. As $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S,T)$, therefore $x_n \to z$. Next suppose that (ii) is given. We have already shown in Theorem 1 that

$$\lim_{n\to\infty} \left\|S^k x_n - x_n\right\| = 0 = \lim_{n\to\infty} \left\|T^k x_n - x_n\right\| \text{ for all } k \geq 1.$$

By the compactness of S^m, we must have a convergent subsequence $\{S^m x_{n_j}\}$ of $\{S^m x_n\}$. Suppose that $\lim_{j\to\infty} S^m x_{n_j} = q$. Then

$$0 \leq \left\|x_{n_j} - q\right\| \leq \left\|x_{n_j} - S^m x_{n_j}\right\| + \left\|S^m x_{n_j} - q\right\| \to 0 \text{ as } j \to \infty.$$

Hence

$$\lim_{j\to\infty} \left\|x_{n_j} - q\right\| = 0.$$

That is, $x_{n_j} \to q$. The rest of the proof is similar to assumption (i). As obtained in Lemma 5 that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 = \lim_{n\to\infty} \|x_n - y_n\|,$$

the limits of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ coincide. This completes the proof. □

The following results are immediate consequences of Theorem 2.

Corollary 2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings in the intermediate sense with at least one common fixed point and let r_n be defined as in Theorem 1 satisfying $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$, $\{y_n\}$ be given by:*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha'_n x_n + \beta'_n S^n x_n + \gamma'_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} \beta'_n \leq \limsup_{n\to\infty} \beta'_n < 1$. Then $\{x_n\}, \{y_n\}$ converge strongly to the same point p of $F(S,T)$ provided that one of the following conditions is satisfied:

(i) *$\{x_n\}$ has a subsequence which converges strongly to a point in C;*

(ii) *S^m is compact for some $m \geq 1$.*

Corollary 3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be asymptotically nonexpansive mapping in the intermediate sense. Set*

$$r_n = \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \vee 0$$

and suppose that $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{x_n\}$ be defined by:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n, \quad \text{for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T provided that one of the following conditions is satisfied:

(i) *$\{x_n\}$ has a subsequence which converges strongly to a point in C;*

(ii) *T^m is compact for some $m \geq 1$.*

Remark 4. Xu-Noor type convergence, Ishikawa type convergence and Mann type convergence results are the direct consequences of Theorem 1 and Theorem 2.

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