CHARACTERIZATIONS OF δ–ORDER ASSOCIATED WITH KANTOROVICH TYPE OPERATOR INEQUALITIES

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Received October 18, 2004; revised October 24, 2005

Abstract. In this note, we obtain more precise estimations than the constants are given in the paper by M.Fujii, E.Kamei and Y.Seo, *Kantorovich type operator inequalities via grand Furuta inequality*, Sci. Math., **3** (2000), 263–272. Among other, we show that the following statements are mutually equivalent for each $\delta \in (0,1]$:

(i) $K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} A^p \geq B^p$
for any $n > 0$, $s \geq 1$, $p \geq \delta$ with $(p-\delta)s \geq n\delta$.

(ii) $K(m^\delta, M^\delta, \frac{p}{\delta}) A^p \geq B^p$ for any $p \geq \delta$.

For each $\delta \in (0,1]$

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} \geq K(m^\delta, M^\delta, \frac{p}{\delta})$$

holds for any $n > 0$, $s \geq 1$, $p \geq \delta$ such that $(p-\delta)s \geq n\delta$.

1 Introduction.

Let $\mathcal{B}(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H and $\mathcal{B}_{++}(H)$ be the set of all positive invertible operators of $\mathcal{B}(H)$. An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for any $x \in H$. We denote by $\mathsf{Sp}(A)$ the spectrum of the operator A. The order between operators $A, B \in \mathcal{B}_{++}(H)$ defined by $\log A \geq \log B$ is said to be the chaotic order $A \gg B$.

First of all, we recall the celebrated Kantorovich inequality: If a positive operator $A \in \mathcal{B}_{++}(H)$ satisfies $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$, then

$$\frac{(m+M)^2}{4mM}(Ax,x)^{-1} \geq (A^{-1}x, x)$$

for every unit vector $x \in H$. The number $\frac{(m+M)^2}{4mM}$ is called the Kantorovich constant. Related to an extension of the Kantorovich inequality, Furuta [4] showed the following Kantorovich type operator inequality:

Theorem A *If $A \geq B \geq 0$ and $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K(m, M, p) A^p \geq B^p \qquad \textit{holds for any } p \geq 1,$$

where a generalized Kantorovich constant $K(m, M, p)$ [4, 5] is defined as

$$(\star) \qquad K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p \qquad \textit{for all } p \in \mathbf{R}.$$

Next, we cite the grand Furuta inequality which interpolates the Furuta inequality [3] and the Ando-Hiai inequality [1].

2000 *Mathematics Subject Classification.* Primary 47A63.
Key words and phrases. Operator order, chaotic order, Kantorovich inequality, grand Furuta inequality.

Theorem G (The grand Furuta inequality) *If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,*

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}A^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for any $s \geq 0$, $p \geq 0$, $q \geq 1$ and $r \geq t$ with $(s-1)(p-1) \geq 0$ and $(1-t+r)q \geq (p-t)s+r$.

In [2] Fujii et al. consider the class of orders $A^\delta \geq B^\delta$ for $A, B \in \mathcal{B}_{++}(H)$ and $\delta \in [0,1]$, where the case $\delta = 0$ means the chaotic order. This class of orders interpolates the usual order $A \geq B$ and the chaotic order $A \gg B$ continuously. As applications of Theorem A and the grand Furuta inequality, they obtained in [2, Theorem 2] the following Kantorovich type order preserving operator inequalities by means of the generalized Kantorovich constant ($\star$).

Theorem B *Let $A, B \in \mathcal{B}_{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(i) $A^\delta \geq B^\delta$.

(ii) *For each $n \in \mathbf{N}$ and $\alpha \in [0,1]$*

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1)A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq \delta$ and $u \geq \delta$ with $(p-\delta+\alpha u)s \geq (n+\alpha)u$.

(iii) *For each $n \in \mathbf{N}$*

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}}A^p \geq B^p$$

holds for $s \geq 1$ and $p \geq \delta$ with $(p-\delta)s \geq n\delta$.

(iv) $(\frac{M}{m})^{p-\delta}A^p \geq B^p$ *holds for $p \geq \delta$.*

Moreover, Hashimoto and Yamazaki in [6, Theorem 4] showed the following Kantorovich type order preserving operator inequalities under the chaotic order.

Theorem C *Let $A, B \in \mathcal{B}_{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \gg B$ (*i.e.* $\log A \geq \log B$).

(ii) *For each $n > 0$ and $\alpha \in [0,1]$*

$$K(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1)A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq 0$ and $u \geq 0$ with $(p+\alpha u)s \geq (n+\alpha)u$.

In this note, we shall show more precise estimations than the constants (ii) and (iii) of Theorem B and the constant (ii) of Theorem C.

2 Results. In this section $K(m,M,p)$ denotes the generalized Kantorovich constant ($\star$) and $S(h,p)$ denotes the Specht ratio [7, 6] defined for all $p \in \mathbf{R}$ as

$$(\star\star) \qquad S(h,p) = \frac{(h^p-1)\ h^{\frac{p}{h^p-1}}}{p\ e\ \log h} \quad \text{for } h>0,\ h\neq 1 \qquad \text{and} \qquad S(1,p)=1.$$

We need the following properties [5, 7]:

(a) $K(m,M,1) = \lim_{p\to 1} K(m,M,p) = 1$ and $S(h,0) = \lim_{p\to +0} S(h,p) = 1$,

(b) $\lim_{r\to +0} K(m^r, M^r, \frac{p}{r}+1) = S(h,p)$, where $h = \frac{M}{m}$,

(c) $\lim_{p\to +0} S(h,p)^{\frac{1}{p}} = 1$

and the following proposition [7, Proposition 4] proven by T.Yamazaki and M.Yanagida:

Proposition P *Let $K(m,M,p)$ be defined in ($\star$). Then*

$$F(p,r,m,M) = K(m^r, M^r, \frac{p}{r}+1)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p>0$, $r>0$ and $M>m>0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geq K(m^r, M^r, \frac{p}{r}+1) \geq 1 \qquad \text{for all } p>0,\ r>0 \text{ and } M>m>0.$$

We begin by stating the following theorem, which gives more precise estimations than the constants (ii) of Theorems B and C.

Theorem 1 *Let $A, B \in \mathcal{B}_{++}(H)$ and M, m some scalars such that $M>m>0$. Then the following statements are mutually equivalent for each $\delta \in [0,1]$:*

(ii) $K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1) A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s$
holds for any $n>0$, $\alpha \in [0,1]$, $s\geq 1$, $p\geq\delta$ and $u\geq\delta$ with $(p-\delta+\alpha u)s \geq (\alpha+n)u$.

(ii)$_0$ $K(m^u, M^u, \frac{(p-\delta+\alpha u)s-\alpha u}{u}+1) A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s$
holds for any $\alpha\in[0,1]$, $s\geq 1$, $p\geq\delta$ and $u\geq\delta$.

For each $\delta\in[0,1]$

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1) \geq K(m^u, M^u, \frac{(p-\delta+\alpha u)s-\alpha u}{u}+1)$$

holds for any $n>0$, $\alpha\in[0,1]$, $s\geq 1$, $p\geq\delta$, $u\geq\delta$ such that $(p-\delta+\alpha u)s\geq(\alpha+n)u$.

Proof. First in the case of $u\neq 0$ we prove that for each $\delta\in[0,1]$

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1) \geq K(m^u, M^u, \frac{(p-\delta+\alpha u)s-\alpha u}{u}+1) \geq 1$$

holds for any $n>0$, $\alpha\in[0,1]$, $s\geq1$, $p\geq\delta$, $u\geq\delta$, $u\neq0$ such that $(p-\delta+\alpha u)s\geq(\alpha+n)u$. We replace r_1 by $\frac{(p-\delta+\alpha u)s-\alpha u}{n}$, r_2 by u and p by $(p-\delta+\alpha u)s-\alpha u$ in Proposition P.

Since $(p-\delta+\alpha u)s \geq (\alpha+n)u$ then we have $r_1 \geq r_2 > 0$ and $(p-\delta+\alpha u)s - \alpha u = (p-\delta)s + \alpha u(s-1) \geq 0$ and it follows from Proposition P that

$$K(m^{r_1}, M^{r_1}, \frac{(p-\delta+\alpha u)s-\alpha u}{r_1}+1) \geq K(m^{r_2}, M^{r_2}, \frac{(p-\delta+\alpha u)s-\alpha u}{r_2}+1) \geq 1,$$

i.e.,

$$\begin{aligned} & K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1) \\ (1) \qquad & = K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, \tfrac{(p-\delta+\alpha u)s-\alpha u}{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}+1) \\ & \geq K(m^u, M^u, \tfrac{(p-\delta+\alpha u)s-\alpha u}{u}+1) \geq 1, \end{aligned}$$

which is the desired result in the case of $u \neq 0$. Letting $u \to +0$ in (1) and using (b) and that $u \geq \delta \geq 0$ we obtain

$$K(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, n+1) \geq S(h, ps) \geq 1,$$

which is the desired result in the case $\delta = u = 0$.

(ii) $\Longrightarrow$ (ii)$_0$. Put $n = \frac{(p-\delta+\alpha u)s-\alpha u}{u}$ for $u \neq 0$ and $n \to +\infty$ for $u = 0$ in (ii).

(ii)$_0$ $\Longrightarrow$ (ii). Let $n > 0$ such that $(p-\delta+\alpha u)s \geq (\alpha+n)u$ holds. We have from (1) and (ii)$_0$ that

$$\begin{aligned} & K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1)A^{(p-\delta+\alpha u)s} \\ & \geq K(m^u, M^u, \tfrac{(p-\delta+\alpha u)s-\alpha u}{u}+1)A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s \end{aligned}$$

holds.

So the proof of theorem is complete. □

Next, we give more precise estimations than the constants (iii) of Theorem B.

Theorem 2 *Let $A, B \in \mathcal{B}_{++}(H)$ and M, m some scalars such that $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(iii) $\quad K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} A^p \geq B^p$
holds for any $n > 0$, $s \geq 1$ and $p \geq \delta$ with $(p-\delta)s \geq n\delta$.

(iii)$_0$ $\quad K(m^\delta, M^\delta, \frac{p}{\delta})A^p \geq B^p$
holds for any $p \geq \delta$.

For each $\delta \in (0,1]$

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} \geq K(m^\delta, M^\delta, \frac{p}{\delta})$$

holds for any $n > 0$, $s \geq 1$, $p \geq \delta$ such that $(p-\delta)s \geq n\delta$.

Proof. First we prove again that for each $\delta \in (0,1]$

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} \geq K(m^\delta, M^\delta, \frac{p}{\delta})$$

holds for any $n > 0$, $s \geq 1$, $p \geq \delta$ such that $(p-\delta)s \geq n\delta$. As in [7] we define a function

$$g(p,r,h) := \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{r}} \qquad \text{where } h > 1,\ p > 0,\ r > 0.$$

If we put $h = \frac{M}{m} > 1$, then we have (see [7, The proof of Proposition 4])

$$K(m^r, M^r, \frac{p}{r}+1) = \left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p. \tag{2}$$

It follows that

$$\begin{aligned} K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} &= K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, \tfrac{(p-\delta)s}{\frac{(p-\delta)s}{n}}+1)^{\frac{1}{s}} \\ &= \left\{\tfrac{1}{h}\cdot g((p-\delta)s, \tfrac{(p-\delta)s}{n}, h)\cdot g(\tfrac{(p-\delta)s}{n}, (p-\delta)s, h)\right\}^{p-\delta}. \end{aligned} \tag{3}$$

Since $g(p,r,h)$ is the increasing function of p and r by [7, Lemma 8], we have that

$$\begin{aligned} g((p-\delta)s, \tfrac{(p-\delta)s}{n}, h) &\geq g(p-\delta, \tfrac{p-\delta}{n}, h), \\ g(\tfrac{(p-\delta)s}{n}, (p-\delta)s, h) &\geq g(\tfrac{p-\delta}{n}, p-\delta, h) \end{aligned}$$

hold if $s \geq 1$. Then

$$\begin{aligned} &\tfrac{1}{h}\cdot g((p-\delta)s, \tfrac{(p-\delta)s}{n}, h)\cdot g(\tfrac{(p-\delta)s}{n}, (p-\delta)s, h) \\ &\geq \tfrac{1}{h}\cdot g(p-\delta, \tfrac{p-\delta}{n}, h)\cdot g(\tfrac{p-\delta}{n}, p-\delta, h). \end{aligned} \tag{4}$$

Since $g(p,r,h)$ is a bounded function by [7, Lemma 10]: $h \geq g(p,r,h) \geq \sqrt{h}$, then we have

$$h \geq \frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h) \geq 1 \tag{5}$$

holds for $h > 1$, $p > 0$, $r > 0$.

Using (2), (3), (4) and (5) we obtain that

$$\begin{aligned} &K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} \\ &= \left\{\tfrac{1}{h}\cdot g((p-\delta)s, \tfrac{(p-\delta)s}{n}, h)\cdot g(\tfrac{(p-\delta)s}{n}, (p-\delta)s, h)\right\}^{p-\delta} \\ &\geq \left\{\tfrac{1}{h}\cdot g(p-\delta, \tfrac{p-\delta}{n}, h)\cdot g(\tfrac{p-\delta}{n}, p-\delta, h)\right\}^{p-\delta} = K(m^{\frac{p-\delta}{n}}, M^{\frac{p-\delta}{n}}, n+1) \end{aligned} \tag{6}$$

holds for any $n > 0$, $s \geq 1$, $p \geq \delta$.

The rest part of proof is similar to the proof of Theorem 1. For any $n > 0$, $s \geq 1$, $p \geq \delta$ such that $(p-\delta)s \geq n\delta$ we replace r_1 by $\frac{p-\delta}{n}$, r_2 by δ and p by $p-\delta$ in Proposition P. Since $r_1 \geq r_2 > 0$ and $p-\delta \geq 0$, it follows from Proposition P that

$$K(m^{r_1}, M^{r_1}, \frac{p-\delta}{r_1}+1) \geq K(m^{r_2}, M^{r_2}, \frac{p-\delta}{r_2}+1) \geq 1,$$

i.e.

$$K(m^{\frac{p-\delta}{n}}, M^{\frac{p-\delta}{n}}, n+1) \geq K(m^\delta, M^\delta, \frac{p}{\delta}) \geq 1. \tag{7}$$

By (6) and (7) we obtain

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} \geq K(m^{\delta}, M^{\delta}, \frac{p}{\delta}) \geq 1, \tag{8}$$

which is the desired result.

(iii) $\Longrightarrow$ (iii)$_0$. Put $n = \frac{p}{\delta} - 1$ and $s = 1$ in (iii).

(iii)$_0$ $\Longrightarrow$ (iii). Let $n > 0$ such that $(p-\delta)s \geq n\delta$. By (8) and (iii)$_0$ it follows that

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} A^p \geq K(m^{\delta}, M^{\delta}, \frac{p}{\delta}) A^p \geq B^p$$

holds.

So the proof of theorem is complete. □

Using Theorem B and Theorems 1 and 2 we obtain the following:

Theorem 3 *Let $A, B \in \mathcal{B}_{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0, 1]$:*

(i)$_0$ $A^{\delta} \geq B^{\delta}$.

(ii)$_0$ *For each $\alpha \in [0, 1]$*

$$K(m^u, M^u, \frac{(p-\delta+\alpha u)s - \alpha u}{u} + 1) A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u - \delta}{2}} B^p A^{\frac{\alpha u - \delta}{2}}\right)^s$$

holds for any $s \geq 1$, $p \geq \delta$ and $u \geq \delta$.

(iii)$_0$ $K(m^{\delta}, M^{\delta}, \frac{p}{\delta}) A^p \geq B^p$ *holds for any $p \geq \delta$.*

(iv)$_0$ $(\frac{M}{m})^{p-\delta} A^p \geq B^p$ *holds for any $p \geq \delta$.*

These constants (ii)$_0$ *and* (iii)$_0$ *are more precise estimations than the constants* (ii) *and* (iii) *of Theorem B, respectively.*

Remark 4 *We remark that* (iii)$_0$ *in Theorem 3 follows directly from Theorem A if we replace A by A^{δ} and B by B^{δ}.*

In particular, if we put $\delta = 1$ in Theorem 3, then we obtain the following Kantorovich type order preserving operator inequalities under the usual order.

Theorem 5 *Let $A, B \in \mathcal{B}_{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i)$_0$ $A \geq B$.

(ii)$_0$ *For each $\alpha \in [0, 1]$*

$$K(m^u, M^u, \frac{(p-1+\alpha u)s - \alpha u}{u} + 1) A^{(p-1+\alpha u)s} \geq \left(A^{\frac{\alpha u - 1}{2}} B^p A^{\frac{\alpha u - 1}{2}}\right)^s$$

holds for any $s \geq 1$, $p \geq 1$ and $u \geq 1$.

(iii)$_0$ $K(m, M, p) A^p \geq B^p$ *holds for any $p \geq 1$.*

(iv)$_0$ $(\frac{M}{m})^{p-1} A^p \geq B^p$ *holds for any $p \geq 1$.*

These constants (ii)$_0$ *and* (iii)$_0$ *are more precise estimations than the constants* (ii) *and* (iii) *of [2, Theorem 3], respectively.*

Using Theorem C and Theorem 1 we obtain the following:

Theorem 6 *Let $A, B \in \mathcal{B}_{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

$(\mathrm{i})_0$ $A \gg B$ (*i.e.* $\log A \geq \log B$).

$(\mathrm{ii})_0$ *For each* $\alpha \in [0,1]$

$$K(m^u, M^u, \frac{(p+\alpha u)s - \alpha u}{u} + 1)A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}}\right)^s$$

holds for any $s \geq 1$, $p \geq 0$ *and* $u \geq 0$.

$(\mathrm{iii})_0$ $S(h,p)A^p \geq B^p$ *holds for any* $p \geq 0$.

This constant $(\mathrm{ii})_0$ *is more precise estimation than the constant* (ii) *of Theorem C.*

Proof.

$(\mathrm{i})_0 \Longrightarrow (\mathrm{ii})_0$. $(\mathrm{i})_0 \Longrightarrow$ [(ii) of Theorem C] $\Longrightarrow (\mathrm{ii})_0$ by Theorem 1.

$(\mathrm{ii})_0 \Longrightarrow (\mathrm{iii})_0$. Let be $u > 0$. If we put $\alpha = 0$ and $s = 1$ in $(\mathrm{ii})_0$, then we obtain that

$$K(m^u, M^u, \frac{p}{u} + 1)A^p \geq B^p$$

holds for any $p \geq 0$ and $u > 0$. Letting $u \to +0$ and using `(b)` we obtain $(\mathrm{iii})_0$. We remark that the statements $(\mathrm{ii})_0$ for $u = 0$ and $(\mathrm{iii})_0$ are identical.

$(\mathrm{iii})_0 \Longrightarrow (\mathrm{i})_0$. It is proved in [7, Theorem 5]. Indeed, if $p > 0$, we take logarithm of both sides $(\mathrm{iii})_0$ and obtain $\log(S(h,p)^{\frac{1}{p}}A) \geq \log B$. Then letting $p \to +0$ and using `(c)` we obtain $(\mathrm{i})_0$. We remark that using `(a)` the statements $(\mathrm{iii})_0$ for $p = 0$ and $(\mathrm{ii})_0$ are identical. □

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