KADISON'S SCHWARZ INEQUALITY AND NONCOMMUTATIVE KANTOROVICH INEQUALITY

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Received September 20, 2005

Abstract. Kadison's Schwarz inequality implies the arithmetic-harmonic (operator) mean inequality and the Ando-Mond-Pečarić reverse inequality of Kadison's Schwarz one implies the noncommutative Kantorovich inequality.

Let Φ be a unital positive linear map on $B(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. Then Kadison's Schwarz inequality asserts

$$\Phi(A^{-1})^{-1} \leq \Phi(A) \tag{1}$$

for all positive invertible $A \in B(H)$.

If Φ is defined on $B(H) \oplus B(H)$ by

$$\Phi(A \oplus B) = \frac{1}{2}(A + B) \quad \text{for } A, B \in B(H), \tag{2}$$

then Φ satisfies

$$\Phi((A \oplus B)^{-1})^{-1} = A \,!\, B, \quad \Phi(A \oplus B) = A \nabla B \tag{3}$$

for all positive invertible $A, B \in B(H)$, where $A \,!\, B$ is the harmonic operator mean and $A \nabla B$ is the arithmetic operator mean in the sense of Kubo-Ando [3], so that (1) implies

Theorem 1. *Kadison's Schwarz inequality implies the arithmetic-harmonic mean inequality, i.e.,* $A \,!\, B \leq A \nabla B$.

On the other hand, the Ando-Mond-Pečarić reverse of Kadison's Schwarz inequality asserts that

$$\Phi(A) \leq \frac{(M+m)^2}{4Mm} \Phi(A^{-1})^{-1} \tag{4}$$

if A satisfies $0 < m \leq A \leq M$ for some constants $m < M$, cf. [2, Theorem 1.32]. Thus it follows from (3) that

$$A \nabla B \leq \frac{(M+m)^2}{4Mm} A!B \tag{5}$$

for A, B with $0 < m \leq A, B \leq M$. It is nothing but the noncommutative Kantorovich inequality introduced in [1]. That is,

2000 *Mathematics Subject Classification.* 47A63, 47A64 and 47B15.

Key words and phrases. Kadison's Schwarz inequality and its reverse one, operator mean, Kantorovich inequality, noncommutative Kantorovich inequality and Bochner integral.

Theorem 2. *The reverse of Kadison's Schwarz inequality implies the noncommutative Kantorovich inequality (5).*

In addition, Theorem 2 can be rephrased as follows:

Corollary 3. *The noncommutative Kantorovich inequality is a complement of the arithmetic-harmonic mean inequality.*

In [1], a difference version of the noncommutative Kantorovich inequality is also introduced by

$$A\nabla B - A\ !\ B \leq (\sqrt{M} - \sqrt{m})^2 \tag{6}$$

for all positive invertible $A, B \in B(H)$ with $0 < m \leq A, B \leq M$, cf. [1, Theorem 6], whereas it has already known in [2, Theorem 1.32] that

$$\Phi(A) - \Phi(A^{-1})^{-1} \leq (\sqrt{M} - \sqrt{m})^2 \tag{7}$$

for all positive invertible $A \in B(H)$ with $0 < m \leq A \leq M$. So the following is obtained:

Theorem 4. *The difference noncommutative Kantorovich inequality is a consequence of the difference version of Kadison's Schwarz inequality.*

At this end, we explain that the noncommutative Kantorovich inequality is reformed as follows: If $F(t)$ is an operator-valued continuous function on a closed interval I satisfying $0 < m \leq F(t) \leq M$ for all $t \in I$, then

$$\int_I F(t)^{-1} d\mu(t) \leq \frac{(M+m)^2}{4Mm} \left(\int_I F(t) d\mu(t) \right)^{-1}, \tag{8}$$

for all probability measures μ on I, where the integral is Bochner's sense.

References

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