CONTINUOUS SELECTIONS OF SIMPLEX-VALUED MAPPINGS

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Abstract. It is proved that for a T_1-space X the following statements are equivalent: (1) X is PF-normal, (2) for every simplicial complex K, every locally selectionable simplex-valued mapping $\varphi : X \to 2^{|K|_w}$ admits a continuous selection $f : X \to |K|_w$, (3) for every simplicial complex K, every lower semicontinuous simplex-valued mapping $\varphi : X \to 2^{|K|_m}$ admits a continuous selection $f : X \to |K|_m$. We also characterize finite-dimensional spaces, pseudofinitistic spaces, strongly countable-dimensional spaces, spaces with strong large transfinite dimension and locally finite-dimensional spaces in terms of simplex-valued mappings.

1 Introduction. Throughout this paper, all spaces are assumed to be T_1. Let 2^Y denote the set of all non-empty subsets of a set Y. For sets X, Y and a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* of φ if $f(x) \in \varphi(x)$ for every $x \in X$. For spaces X and Y, a mapping $\varphi : X \to 2^Y$ is called *locally selectionable* ([12]) if for each point $x \in X$, there exist a neighborhood U of x and a continuous selection $f : U \to Y$ of the restriction $\varphi|_U : U \to 2^Y$. Note that $\varphi : X \to 2^Y$ is locally selectionable if and only if there is an open cover $\{U_\alpha \mid \alpha \in A\}$ of X and continuous mappings $f_\alpha : U_\alpha \to Y$, $\alpha \in A$ such that $f_\alpha(x) \in \varphi(x)$ for each $x \in U_\alpha$.

Let K be a simplicial complex and $\varphi : X \to K$ a mapping. Then φ is naturally viewed as a mapping $\varphi : X \to 2^{|K|}$ of X, where $|K|$ is the geometric realization of K. Such a mapping is said to be a simplex-valued mapping. For a simplicial complex K, let $|K|_w$ and $|K|_m$ denote its geometric realizations with the weak topology and the metric topology, respectively. For simplex-valued mappings to simplicial complexes with the weak topology, in [8, Theorem 1.2], I. Ivanšić and L. R. Rubin proved the following selection theorem.

Theorem 1.1 (I. Ivanšić and L. R. Rubin [8]). *Let X be a hereditarily normal paracompact Hausdorff space, K a simplicial complex and $\varphi : X \to K$ a mapping. If $\varphi : X \to 2^{|K|_w}$ is locally selectionable, then φ admits a continuous selection $f : X \to |K|_w$.*

In order to mention a result for simplicial complexes with the metric topology, let us recall the following theorem due to T. Kandô [9, Theorem IV] and S. Nedev [11, Theorem 4.1]. Let λ be an infinite cardinal. A space X is λ-*PF-normal* if every point-finite open cover $\mathcal{U}$ of X of cardinality $\leq \lambda$ is normal; a space X is *PF-normal* ([14]) if X is λ-PF-normal for every λ. Every collectionwise normal space is PF-normal, and a space is ω-PF-normal if and only if it is normal, where ω denotes the first infinite cardinal (see [3]).

Theorem 1.2 (T. Kandô [9] and S. Nedev [11]). *A space X is λ-PF-normal if and only if for every Banach space Y of weight $\leq \lambda$, every lower semicontinuous mapping $\varphi : X \to \mathcal{C}_c(Y)$ admits a continuous selection, where $\mathcal{C}_c(Y)$ is the set of all compact convex subsets of Y.*

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Let K be a simplicial complex. Then $|K|_m$ is naturally embedded in the Banach space $l_1(V)$ generated by the set V of all vertices of K. Thus Theorem 1.2 implies that if a space X is λ-PF-normal, then for every simplicial complex K of cardinality $\leq \lambda$, every lower semicontinuous simplex-valued mapping $\varphi : X \to 2^{|K|_m}$ admits a continuous selection $f : X \to |K|_m$.

The purpose of this paper is to characterize some topological properties in terms of simplex-valued mappings. In section 3 we establish the following theorem, which shows that λ-PF-normality is essentially described by Theorem 1.1 and the fact above.

Theorem 1.3. *For a T_1-space X and an infinite cardinal λ, the following are equivalent.*

(a) *X is λ-PF-normal.*

(b) *For every simplicial complex K with $\operatorname{Card} K \leq \lambda$, every locally selectionable mapping $\varphi : X \to K$ admits a continuous selection $f : X \to |K|_w$.*

(c) *For every simplicial complex K with $\operatorname{Card} K \leq \lambda$, every lower semicontinuous simplex-valued mapping $\varphi : X \to 2^{|K|_m}$ admits a continuous selection $f : X \to |K|_m$.*

In section 2, we present some properties of continuity for simplex-valued mappings, which are preliminary in our discussion.

Section 3 is devoted to prove Theorem 1.3. In particular, we show that if a mapping $\varphi : X \to K$ admits a continuous selection $f : X \to |K|_m$, then φ also has a continuous selection $g : X \to |K|_w$.

In section 4, we describe characterizations of finite-dimensional spaces, pseudofinitistic spaces, strongly countable-dimensional spaces, spaces with strong large transfinite dimension and locally finite-dimensional spaces in terms of simplex-valued mappings.

For undefined terminology, we refer to [3] and [13].

2 Continuity of simplex-valued mappings. Let X and Y be spaces and K a simplicial complex. For a subset S of X, $\operatorname{Int} S$ (respectively, $\operatorname{Cl} S$) denotes the interior (respectively, closure) of S in X. For a mapping $\varphi : X \to 2^Y$ and $A \subset Y$, put $\varphi^{-1}[A] = \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}$. Let $\langle v_i \mid i = 0, 1, \ldots n\rangle$ or $\langle v_0, v_1, \ldots v_n\rangle$ denote the n-simplex spanned by the vertices $v_0, v_1, \ldots v_n$. For a simplex σ and a vertex v of K, let $\operatorname{int} \sigma$ denote the geometric interior of σ in K and $\operatorname{St}(v, K)$ the open star of v on K, that is, $\bigcup\{\operatorname{int} \sigma \mid v \in \sigma \in K\}$. By $K^{(n)}$ we denote the n-skeleton of K.

We will use the fact that for a mapping $\varphi : X \to K$,

$$\varphi^{-1}[\{v\}] = \varphi^{-1}[\operatorname{St}(v, K)] \tag{1}$$

holds for each vertex v of K. In particular, the collection $\{\varphi^{-1}[\{v\}] \mid v \in K^{(0)}\}$ covers X.

Recall that a mapping $\varphi : X \to 2^Y$ is said to have the *local intersection property* ([17]) if each $x \in X$ has a neighborhood U with $\bigcap\{\varphi(z) \mid z \in U\} \neq \emptyset$. It is obvious that every mapping $\varphi : X \to 2^Y$ with the local intersection property is locally selectionable. For simplex-valued mappings we have the following.

Proposition 2.1. *Let $\varphi : X \to K$ be a mapping of a space X to a simplicial complex K and $|K|$ the geometrical realization of K with the weak topology or the metric topology. Then the following are equivalent.*

(a) *The mapping φ has the local intersection property.*

(b) *The mapping $\varphi : X \to 2^{|K|}$ is locally selectionable.*

(c) *The collection* $\{\operatorname{Int}\varphi^{-1}[\{v\}] \mid v \in K^{(0)}\}$ *covers* X.

Proof. The implication (a) $\Rightarrow$ (b) is obvious. Let us show the implication (b) $\Rightarrow$ (c). For $x \in X$, take a neighborhood U of x and a continuous selection $f : U \to |K|$ of the restriction $\varphi|_U : U \to K$. Choose a vertex v of K so that $f(x) \in \mathrm{St}(v, K)$. Then $U \cap f^{-1}(\mathrm{St}(v, K))$ is a neighborhood of x contained in $\varphi^{-1}[\mathrm{St}(v, K)]$. By (1), we have that $x \in \operatorname{Int}\varphi^{-1}[\{v\}]$. The implication ($c$) $\Rightarrow$ (a) follows from the fact that a mapping $\varphi : X \to 2^Y$ has the local intersection property if and only if the collection $\{\operatorname{Int}\varphi^{-1}[\{y\}] \mid y \in Y\}$ covers X. □

For spaces X and Y, a mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* if for every open subset V in Y, the subset $\varphi^{-1}[V]$ is open in X. A mapping $\varphi : X \to 2^Y$ is *strongly lower semicontinuous* ([16]) if the set $\{x \in X \mid C \subset \varphi(x)\}$ is open in X for each compact subset C of Y. It is obvious that strong lower semicontinuity implies lower semicontinuity, but not conversely in general. Indeed, let $\mathbf{R}$ be the space of real numbers with the usual topology and let $\varphi : \mathbf{R} \to 2^{\mathbf{R}}$ be the mapping defined by $\varphi(x) = \{x\}$ for each $x \in \mathbf{R}$. Then φ is lower semicontinuous, but not strongly lower semicontinuous. For simplex-valued mappings these two notions coincide.

Proposition 2.2. *Let* $\varphi : X \to K$ *be a mapping of a space* X *to a simplicial complex* K *and* $|K|$ *the geometrical realization of* K *with the weak topology or the metric topology. Then the following are equivalent.*

(a) *The mapping* $\varphi : X \to 2^{|K|}$ *is strongly lower semicontinuous.*

(b) *The mapping* $\varphi : X \to 2^{|K|}$ *is lower semicontinuous.*

(c) *The set* $\varphi^{-1}[\{v\}]$ *is open in* X *for each vertex* v *in* K.

Proof. The implication (a) $\Rightarrow$ (b) is obvious. The implication (b) $\Rightarrow$ (c) follows from (1). To show the implication (c) $\Rightarrow$ (a), assume (c) and let C be a compact subset of $|K|$ such that $\{x \in X \mid C \subset \varphi(x)\}$ is non-empty. Then C is contained in some simplex in K. Let $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ be the smallest simplex in K containing C. Then we have that $\{x \in X \mid C \subset \varphi(x)\} = \{x \in X \mid \sigma \subset \varphi(x)\} = \bigcap\{\varphi^{-1}[\{v_i\}] \mid i = 1, 2, \ldots, n\}$, which is open in X. □

Propositions 2.1 and 2.2 yield that every lower semicontinuous simplex-valued mapping is locally selectionable, which will be used in section 3. The following example is a locally selectionable simplex-valued mapping $\varphi : X \to 2^{|K|}$ which is not lower semicontinuous

Example 2.3. Let K be the simplicial complex $\{v_1, v_2, v_3, \langle v_1, v_2\rangle, \langle v_2, v_3\rangle\}$ and define a mapping $\varphi : [0, 1] \to K$ as follows.

$$\varphi(x) = \begin{cases} \langle v_1, v_2\rangle & \text{if } x \in [0, 1) \\ \langle v_2, v_3\rangle & \text{if } x = 1 \end{cases}$$

Let $|K|$ be the geometric realization of K with the weak topology or the metric topology. Then $\varphi : X \to 2^{|K|}$ is locally selectionable, but not lower semicontinuous.

3 Proof of Theorem 1.3. For a real-valued function f of a space X, set $\mathrm{Coz}(f) = \{x \in X \mid f(x) \neq 0\}$. A subset S of X is a *cozero-set* if $S = \mathrm{Coz}(f)$ for some real-valued continuous function f on X. A family $\{p_\lambda \mid \lambda \in \Lambda\}$ of continuous functions $p_\lambda : X \to [0, 1]$ is called a *partition of unity* on X if $\sum_{\lambda \in \Lambda} p_\lambda(x) = 1$ for each $x \in X$. A partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is said to be *locally finite* if $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ of X is locally finite.

For an open cover $\mathcal{U}$ of X, a partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is *subordinated to* $\mathcal{U}$ if $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ refines $\mathcal{U}$.

Let X be a space, K a simplicial complex and $f : X \to |K|$ a mapping. A mapping $g : X \to |K|$ is called a *K-approximation* ([2]) (or *K-modification*) of f if $g(x) \in \sigma$ for each simplex σ of K and each $x \in X$ with $f(x) \in \sigma$. For each $x \in X$ there exists the unique simplex $\sigma_{f(x)}$ of K such that $f(x) \in \operatorname{int}\sigma_{f(x)}$. Note that a mapping $g : X \to |K|$ is a K-approximation of f if and only if $g(x) \in \sigma_{f(x)}$ for each $x \in X$. Since the weak topology is finer than the metric topology, if the mapping $f : X \to |K|_w$ is continuous, then $f : X \to |K|_m$ is continuous. On the other hand, the following holds.

Proposition 3.1. *Let X be a space and K a simplicial complex. Then every continuous mapping $f : X \to |K|_m$ admits a continuous K-approximation $g : X \to |K|_w$.*

Proof. Let $f : X \to |K|_m$ be a continuous mapping. Since $\mathcal{U} = \{f^{-1}(\mathrm{St}(v, K)) \mid v \in K^{(0)}\}$ is a normal open cover of X, there exists a locally finite partition of unity $\mathcal{P} = \{p_v \mid v \in K^{(0)}\}$ on X such that $\mathrm{Coz}(p_v) \subset f^{-1}(\mathrm{St}(v, K))$ for each $v \in K^{(0)}$ ([1, Theorem 10.10]). Define a mapping $g : X \to |K|$ by $g(x) = \sum_{v \in K^{(0)}} p_v(x) \cdot v$ for each $x \in X$. Then g is a K-approximation of f. Since $\mathcal{P}$ is locally finite, each point $x \in X$ has a neighborhood U in X such that $g(U)$ is contained in some finite subcomplex L of K. Since the weak topology of $|L|$ coincides with the metric topology, the restriction $g|_U : U \to |L|_w$ is continuous. Hence the mapping $g : X \to |K|_w$ is continuous at x. □

Proposition 3.1 implies the following.

Proposition 3.2. *If a mapping $\varphi : X \to K$ from a space X to a simplicial complex K admits a continuous selection $f : X \to |K|_m$, then φ has a continuous selection $g : X \to |K|_w$.*

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. $(a) \Rightarrow (b)$. Assume that X is λ-PF-normal. Let K be a simplicial complex with $\mathrm{Card}\, K \le \lambda$ and $\varphi : X \to 2^{|K|_w}$ a locally selectionable simplex-valued mapping. By Proposition 2.1, $\mathcal{U} = \{\mathrm{Int}\,\varphi^{-1}[\{v\}] \mid v \in K^{(0)}\}$ is an open cover of X with $\mathrm{Card}\,\mathcal{U} \le \lambda$. Since X is λ-PF-normal and $\mathcal{U}$ is point-finite, the cover $\mathcal{U}$ is normal, and hence there exists a locally finite partition of unity $\mathcal{P} = \{p_v \mid v \in K^{(0)}\}$ on X such that $\mathrm{Coz}(p_v) \subset \mathrm{Int}\,\varphi^{-1}[\{v\}]$ for each $v \in K^{(0)}$ ([1, Theorem 10.10]). Then $v \in \bigcap\{\varphi(z) \mid z \in \mathrm{Coz}(p_v)\}$ for each $v \in K^{(0)}$. Define a mapping $f : X \to |K|$ by $f(x) = \sum_{v \in K^{(0)}} p_v(x) \cdot v$ for $x \in X$. Since each $\varphi(x)$ is convex, the mapping $f : X \to |K|$ is a selection of φ. By the same argument as in Proposition 3.1, the mapping $f : X \to |K|_w$ is continuous.

$(b) \Rightarrow (c)$. It immediately follows from the fact in section 2.

$(c) \Rightarrow (a)$. Assume (c) and let $\mathcal{U}$ be a point-finite open cover of X with $\mathrm{Card}\,\mathcal{U} \le \lambda$. Note that the cardinal of the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is at most λ. Define a mapping $\varphi : X \to N(\mathcal{U})$ by $\varphi(x) = \langle U \in \mathcal{U} \mid x \in U\rangle$ for each $x \in X$. Since $\mathcal{U}$ is point-finite, φ is actually defined. Since $\varphi^{-1}[\{U\}] = U$ for each $U \in \mathcal{U}$, φ is lower semicontinuous by Proposition 2.2, and hence it admits a continuous selection $f : X \to |N(\mathcal{U})|_m$. Then $f^{-1}(\mathrm{St}(U, N(\mathcal{U})))$ is contained in U for each $U \in \mathcal{U}$. Since the open cover $\{\mathrm{St}(U, N(\mathcal{U})) \mid U \in \mathcal{U}\}$ of $|N(\mathcal{U})|_m$ is normal, so is $\mathcal{U}$. □

4 Characterizations of dimension-like properties. For a cover $\mathcal{U}$ of a space X and a point $x \in X$, let $\mathrm{ord}_x\,\mathcal{U} = \mathrm{Card}\,\{U \in \mathcal{U} \mid x \in U\}$ and $\mathrm{ord}\,\mathcal{U} = \sup\{\mathrm{ord}_x\,\mathcal{U} \mid x \in X\}$. The *covering dimension* $\dim X$ of a space X is the least number n such that any finite open cover of X is refined by a finite open cover $\mathcal{U}$ of X of $\operatorname{ord}\mathcal{U} \leq n+1$. A space X is called *finitistic* if for every open cover of X is refined by an open cover $\mathcal{U}$ of X such that $\operatorname{ord}\mathcal{U}$ is finite. Finitistic spaces were introduced by R. Swan [15] for work in fixed point theory. Some of their properties were investigated from the dimensional viewpoints ([2], [5] and [6]). A space X is said to be *pseudofinitistic* ([18]) if every normal open cover of X has a refinement which is normal and whose order is finite. V. Matijević [10] defined this concept under the name finitistic and gave characterizations of pseudofinitistic spaces by means of approximate resolutions. It is clear that every finitistic normal space and every finite-dimensional normal space is pseudofinitistic and that every pseudofinitistic paracompact Hausdorff space is finitistic. Note that a normal space X is pseudofinitistic if and only if every countable locally finite open cover of X is refined by a normal open cover $\mathcal{V}$ of X satisfying $\sup\{\operatorname{ord}_x \mathcal{V} \mid x \in D\} < \infty$ for each closed discrete subset D of X ([18]).

A mapping $g : X \to |K|$ is said to be *n-dimensional* (respectively, *finite-dimensional*) ([2]) if $g(X) \subset |K^{(n)}|$ (respectively, $g(X) \subset |K^{(m)}|$ for some m). J. Dydak, S. N. Mishra and R. A. Shukla [2, Theorem 2.2] established the following.

Theorem 4.1 (J. Dydak, S. N. Mishra and R. A. Shukla [2]). *For a normal space X, $\dim X \leq n$ if and only if for every simplicial complex K, every continuous mapping $f : X \to |K|_m$ has an n-dimensional continuous K-approximation $g : X \to |K|_m$.*

Proposition 4.2. *A T_1-space X is λ-PF-normal and $\dim X \leq n$ if and only if for every simplicial complex K with $\operatorname{Card} K \leq \lambda$, every locally selectionable simplex-valued mapping $\varphi : X \to 2^{|K|_w}$ admits an n-dimensional continuous selection $f : X \to |K|_w$.*

Proof. The "only if" part is immediate from Theorems 1.3, 4.1 and Proposition 3.1. Let us show the "if" part. By Theorem 1.3, it suffices to show $\dim X \leq n$. We shall apply Theorem 4.1. Let K be a simplicial complex and $f : X \to |K|_m$ a continuous mapping. For each $x \in X$, let $\sigma_{f(x)}$ be the unique simplex of K such that $f(x) \in \operatorname{int} \sigma_{f(x)}$. Define a mapping $\varphi : X \to K$ by $\varphi(x) = \sigma_{f(x)}$ for each $x \in X$. Since $f^{-1}(\operatorname{St}(v,K)) \subset \varphi^{-1}[\operatorname{St}(v,K)] = \varphi^{-1}[\{v\}]$ for each vertex v of K, φ is locally selectionable by Proposition 2.1. Then φ admits an n-dimensional continuous selection $g : X \to |K|_w$, which is a K-approximation of f. Hence we have $\dim X \leq n$. □

The following is a slight modification of [2, Theorem 2.1]. Let $\mathcal{D}(X)$ denote the set of all closed discrete subsets of a space X.

Theorem 4.3. *For a normal space X, the following are equivalent.*

(*a*) *X is pseudofinitistic.*

(*b*) *For every simplicial complex K, every continuous mapping $f : X \to |K|_m$ has a finite-dimensional continuous K-approximation $g : X \to |K|_w$.*

(*c*) *For every simplicial complex K and every continuous mapping $f : X \to |K|_m$, there exist a mapping $m : \mathcal{D}(X) \to \mathbf{N}$ and a continuous K-approximation $g : X \to |K|_w$ of f such that $g(D) \subset |K^{(m(D))}|$ for each $D \in \mathcal{D}(X)$.*

By the same argument as in Theorem 4.2, we have the following.

Proposition 4.4. *For a T_1-space X, the following are equivalent.*

(*a*) *X is λ-PF-normal and pseudofinitistic.*

(*b*) *For every simplicial complex K with* $\operatorname{Card} K \leq \lambda$*, every locally selectionable simplex-valued mapping* $\varphi : X \to 2^{|K|_w}$ *admits a finite-dimensional continuous selection* $f : X \to |K|_w$.

(*c*) *For every simplicial complex K with* $\operatorname{Card} K \leq \lambda$ *and every locally selectionable simplex-valued mapping* $\varphi : X \to 2^{|K|_w}$*, there exist a mapping* $m : \mathcal{D}(X) \to \mathbf{N}$ *and a continuous selection* $f : X \to |K|_w$ *of* φ *such that* $f(D) \subset |K^{(m(D))}|$ *for each* $D \in \mathcal{D}(X)$.

A normal space is said to be *strongly countable-dimensional* if it can be represented as the union of countably many finite-dimensional closed subspaces. The following theorem is due to Y. Hattori [7, Corollary].

Theorem 4.5 (Y. Hattori [7]). *A normal space X is strongly countable-dimensional if and only if there exists a mapping* $m : X \to \mathbf{N}$ *such that for every simplicial complex K, every continuous mapping* $f : X \to |K|_m$ *has a continuous K-approximation* $g : X \to |K|_m$ *such that* $g(x) \in |K^{(m(x))}|$ *for each* $x \in X$.

Theorems 1.3, 4.5 and Proposition 3.1 yield

Proposition 4.6. *A T_1-space X is λ-PF-normal and strongly countable-dimensional if and only if there exists a mapping* $m : X \to \mathbf{N}$ *such that for every simplicial complex K with* $\operatorname{Card} K \leq \lambda$*, every locally selectionable simplex-valued mapping* $\varphi : X \to 2^{|K|_w}$ *admits a continuous selection* $f : X \to |K|_w$ *such that* $f(x) \in |K^{(m(x))}|$ *for each* $x \in X$.

Proposition 3.1, Theorem 1.3 and [7, Theorem] also provide the following characterization of strong large transfinite dimension (for the definition of strong large transfinite dimension, see [7]).

Proposition 4.7. *A metrizable space X has strong large transfinite dimension if and only if there exists a mapping* $m : \mathcal{D}(X) \to \mathbf{N}$ *such that for every simplicial complex K, every locally selectionable simplex-valued mapping* $\varphi : X \to 2^{|K|_w}$ *admits a continuous selection* $f : X \to |K|_w$ *such that* $f(D) \subset |K^{(m(D))}|$ *for each* $D \in \mathcal{D}(X)$.

A normal space is called *locally finite-dimensional* if every point has a neighborhood U such that $\dim \operatorname{Cl} U < \infty$. A mapping $m : X \to \mathbf{N}$ is said to be *lower semicontinuous* if the set $m^{-1}(\{k \in \mathbf{N} \mid k < n\})$ is open in X for each $n \in \mathbf{N}$.

Proposition 4.8. *A normal weakly paracompact space X is locally finite-dimensional if and only if there exists a lower semicontinuous mapping* $m : X \to \mathbf{N}$ *such that for every simplicial complex K, every continuous mapping* $f : X \to |K|_m$ *has a continuous K-approximation* $g : X \to |K|_w$ *such that* $g(x) \in |K^{(m(x))}|$ *for each* $x \in X$.

Proof. Due to [4, Theorem 5.5.12], a normal weakly paracompact space X is locally finite-dimensional if and only if there is a lower semicontinuous mapping $m : X \to \mathbf{N}$ such that every finite open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that $\operatorname{ord}_x \mathcal{V} \leq m(x)$ for each $x \in X$. Note that the phrase "finite open cover" can be replaced with that "locally finite open cover". Thus the assertion follows from an argument analogous to Proposition 3.1. □

Theorem 1.3 and Proposition 4.8 yield

Proposition 4.9. *A weakly paracompact T_1-space X is λ-PF-normal and locally finite-dimensional if and only if there exists a lower semicontinuous mapping* $m : X \to \mathbf{N}$ *such that for every simplicial complex K with* $\operatorname{Card} K \leq \lambda$*, every locally selectionable simplex-valued mapping* $\varphi : X \to 2^{|K|_w}$ *admits a continuous selection* $f : X \to |K|_w$ *such that* $f(x) \in |K^{(m(x))}|$ *for each* $x \in X$.

Remark 4.10. The phrase "locally selectionable" in Propositions 4.2, 4.4, 4.6 and 4.9 can be replaced by "lower semicontinuous".

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