K-THEORY OF THE PULLBACK AND PUSHOUT C^*-ALGEBRAS

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Received April 13, 2006

Abstract. We study K-theory of the pullback C^*-algebras and the pushout C^*-algebras such as amalgams of C^*-algebras and balanced tensor products of C^*-algebras, and obtain that their K-groups are isomorphic under the reasonable assumptions on their $*$-homomorphisms.

Introduction In the C^*-algebra theory, K-theory has played an important and useful role in some topics of C^*-algebras such as classification theory for amenable (or nuclear) C^*-algebras, extension theory and isomorphism problems such as the classification of irrational rotation C^*-algebras and the full or reduced C^*-algebras of free groups (see Rørdam [5], Davidson [2] and Wegge-Olsen [6]). On the other hand, some functorial methods of constructing examples of C^*-algebras such as the pullback construction of C^*-algebras and the pushout construction of C^*-algebras such as (universal) amalgamated free products (or amalgams) of C^*-algebras and (balanced) tensor products of C^*-algebras have been well studied (see Pedersen [3] (a survey) and [4]).

In this paper we study K-theory of the pullback C^*-algebras and the pushout C^*-algebras such as amalgams of C^*-algebras and balanced tensor products of C^*-algebras, and obtain that their K-groups are isomorphic under some reasonable assumptions on their $*$-homomorphisms. For this purpose, in Section 1 we first review about the pullback C^*-algebras and the pushout C^*-algebras and their successive construction from Pedersen [3] (and [4]). In Section 2 we include a formula for K-groups of (universal) amalgamated free products of C^*-algebras under an assumption for $*$-homomorphisms of common C^*-subalgebras to have (inverse) retractions (i.e., surjective $*$-homomorphisms) ¿from Blackadar [1] with our modified proof, while the case for full free products of C^*-algebras is first considered by J. Cuntz. Using this formula extensively we obtain a number of formulas for K-groups of successive amalgams and balanced tensor products of C^*-algebras through K-groups of their associated pullback C^*-algebras. To define the associated pullback C^*-algebras we need to assume that the $*$-homomorphisms from common C^*-subalgebras in the successive amalgams and balanced tensor products have (inverse) retractions.

See [1] and [6] for the details about K-theory of C^*-algebras, and see [3] for the details about the pullback and pushout constructions for C^*-algebras.

C^*-algebras of

1 The pullback and pushout C^*-algebras

Pullbacks For C^*-algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, suppose that there exist $*$-homomorphisms $\alpha_1 : \mathfrak{A} \to \mathfrak{C}$, $\alpha_2 : \mathfrak{B} \to \mathfrak{C}$. Then their pullback C^*-algebra denoted by $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ is defined by

$$\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} = \{(a, b) \in \mathfrak{A} \oplus \mathfrak{B} \mid \alpha_1(a) = \alpha_2(b)\}.$$

2000 *Mathematics Subject Classification.* Primary 46L05, 46L80.
Key words and phrases. C*-algebra, K-theory, Pullback, Amalgam, Tensor product.

We have the following diagram:

$$\begin{array}{ccc} \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} & \xrightarrow{p_2} & \mathfrak{B} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ \mathfrak{A} & \xrightarrow{\alpha_1} & \mathfrak{C} \end{array}$$

where p_1, p_2 are the canonical projections.

Now consider the commutative case. Let X, Y, Z be compact Hausdorff spaces and $C(X), C(Y), C(Z)$ the C^*-algebras of continuous functions on them respectively. Suppose that there exist continuous maps $f : Z \to X$, $g : Z \to Y$. Then the pullback C^*-algebra $C(X) \oplus_{C(Z)} C(Y)$ corresponds to the space $X \cup_Z Y$ obtained from the disjoint union $X \cup Y$ by identifying $f(Z)$ and $g(Z)$.

Amalgams Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. Assume that there exists a common C^*-subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ and $\mathfrak{B}$ with embeddings $\mu_1 : \mathfrak{C} \to \mathfrak{A}$, $\mu_2 : \mathfrak{C} \to \mathfrak{B}$. Then as their pushout C^*-algebra we define the (universal) amalgamated free product (or amalgam) of $\mathfrak{A}, \mathfrak{B}$ over $\mathfrak{C}$, denoted by $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$, to be the quotient C^*-algebra of the (universal) free product C^*-algebra $\mathfrak{A} * \mathfrak{B}$ of $\mathfrak{A}, \mathfrak{B}$ by the closed ideal generated by the set $\{\mu_1(c) - \mu_2(c) \,|\, c \in \mathfrak{C}\}$. We have the following diagram:

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\mu_2} & \mathfrak{B} \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{B}}} \\ \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} \end{array}$$

where $\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{B}}$ are the canonical inclusions.

Balanced tensor products Let $\mathfrak{A}, \mathfrak{B}$ be unital C^*-algebras. Assume that there exists a common C^*-subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ and $\mathfrak{B}$ with embeddings $\mu_1 : \mathfrak{C} \to \mathfrak{A}$, $\mu_2 : \mathfrak{C} \to \mathfrak{B}$. Then as another version of their pushout C^*-algebra we define the balanced tensor product C^*-algebra of $\mathfrak{A}, \mathfrak{B}$ over $\mathfrak{C}$, denoted by $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$, to be the quotient C^*-algebra of the (maximal) tensor product C^*-algebra $\mathfrak{A} \otimes \mathfrak{B}$ of $\mathfrak{A}, \mathfrak{B}$ by the closed ideal generated by the set $\{\mu_1(c) - \mu_2(c) \,|\, c \in \mathfrak{C}\}$. We have the following diagram:

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\mu_2} & \mathfrak{B} \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{B}}} \\ \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B} \end{array}$$

where $\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{B}}$ are the canonical inclusions. We may take nonunital $\mathfrak{A}$, $\mathfrak{B}$ if not use this diagram.

If we have continuous maps $f : X \to Z$, $g : Y \to Z$, then the space $X \times_Z Y$ defined by

$$X \times_Z Y = \{(x, y) \in X \times Y \,|\, f(x) = g(y)\}$$

corresponds to $C(X) \otimes_{C(Z)} C(Y)$ (or $C(X) *_{C(Z)} C(Y)$).

Successive construction Let $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be a pullback C^*-algebra and $\mathfrak{D}$, E be C^*-algebras. Suppose that there exist $*$-homomorphisms $\beta_1 : \mathfrak{C} \to E$, $\beta_2 : \mathfrak{D} \to E$. Then we can define the extension of β_1 by the same symbol $\beta_1 : \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \to E$. Thus, we can define the pullback C^*-algebra $(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}$ such that

$$\begin{array}{ccc} (\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D} & \xrightarrow{p_2} & \mathfrak{D} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} & \xrightarrow{\beta_1} & E \end{array}$$

where p_1, p_2 are the canonical projections. Moreover,

$$(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D} \cong (\mathfrak{A} \oplus_E \mathfrak{D}) \oplus_{\mathfrak{C} \oplus_E \mathfrak{D}} (\mathfrak{B} \oplus_E \mathfrak{D}).$$

Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be an amalgam C^*-algebra and $\mathfrak{D}$, E be C^*-algebras. Suppose that there exist $*$-homomorphisms $\nu_1 : E \to \mathfrak{C}$, $\nu_2 : E \to \mathfrak{D}$. Then we can define the extension of ν_1 by the same symbol $\nu_1 : E \to \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$. Thus, we can define the amalgam C^*-algebra $(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D}$ such that

$$\begin{array}{ccc} E & \xrightarrow{\nu_2} & \mathfrak{D} \\ \nu_1 \downarrow & & \downarrow \mathrm{id}_{\mathfrak{D}} \\ \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} & \xrightarrow{\mathrm{id}} & (\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D} \end{array}$$

where id, $\mathrm{id}_{\mathfrak{D}}$ are the canonical inclusions. Moreover,

$$(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D} \cong (\mathfrak{A} *_E \mathfrak{D}) *_{\mathfrak{C} *_E \mathfrak{D}} (\mathfrak{B} *_E \mathfrak{D}).$$

Let $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$ be a balanced tensor product C^*-algebra and $\mathfrak{D}$, E be C^*-algebras. Suppose that there exist $*$-homomorphisms $\nu_1 : E \to \mathfrak{C}$, $\nu_2 : E \to \mathfrak{D}$. Then we can define the extension of ν_1 by the same symbol $\nu_1 : E \to \mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$. Thus, we can define the balanced tensor product C^*-algebra $(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \otimes_E \mathfrak{D}$. Moreover,

$$(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \otimes_E \mathfrak{D} \cong (\mathfrak{A} \otimes_E \mathfrak{D}) \otimes_{\mathfrak{C} \otimes_E \mathfrak{D}} (\mathfrak{B} \otimes_E \mathfrak{D}).$$

Furthermore, under the successive assumptions on $*$-homomorphisms involved we can construct an n-successive pullback C^*-algebra as follows:

$$(\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{B}_1} \mathfrak{A}_2) \oplus_{\mathfrak{B}_2} \mathfrak{A}_3) \cdots) \oplus_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are C^*-algebras, and we assume that there exist $*$-homomorphisms: $\alpha_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, $\alpha_j : \mathfrak{A}_j \to \mathfrak{B}_{j-1}$ $(2 \le j \le n)$, $\beta_j : \mathfrak{B}_j \to \mathfrak{B}_{j+1}$ $(1 \le j \le n-2)$.

Also, we can construct an n-successive amalgam C^*-algebra:

$$(\cdots((\mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2) *_{\mathfrak{B}_2} \mathfrak{A}_3) \cdots) *_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are C^*-algebras, and we assume that there exist $*$-homomorphisms: $\mu_1 : \mathfrak{B}_1 \to \mathfrak{A}_1$, $\mu_j : \mathfrak{B}_j \to \mathfrak{A}_{j+1}$ $(2 \le j \le n-1)$, $\nu_j : \mathfrak{B}_{j+1} \to \mathfrak{B}_j$ $(1 \le j \le n-2)$.

Similarly, we can construct an n-successive balanced tensor product C^*-algebra:

$$(\cdots((\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \otimes_{\mathfrak{B}_2} \mathfrak{A}_3) \cdots) \otimes_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are C^*-algebras, and we assume that there exist $*$-homomorphisms: $\mu_1 : \mathfrak{B}_1 \to \mathfrak{A}_1$, $\mu_j : \mathfrak{B}_j \to \mathfrak{A}_{j+1}$ $(2 \le j \le n-1)$, $\nu_j : \mathfrak{B}_{j+1} \to \mathfrak{B}_j$ $(1 \le j \le n-2)$.

2 K-theory

Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras. Let $\mathfrak{C}$ be a common C^*-subalgebra of $\mathfrak{A}$ and $\mathfrak{B}$ with embeddings $\mu_1 : \mathfrak{C} \to \mathfrak{A}$, $\mu_2 : \mathfrak{C} \to \mathfrak{B}$. Let $\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}$ be the amalgam of $\mathfrak{A}$, $\mathfrak{B}$ over $\mathfrak{C}$. Let $\iota_1 : \mathfrak{A} \to \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$, $\iota_2 : \mathfrak{B} \to \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be the natural injective $*$-homomorphisms. Suppose that there exist retractions (i.e., surjective $*$-homomorphisms) $r_1 : \mathfrak{A} \to \mathfrak{C}$ and $r_2 : \mathfrak{B} \to \mathfrak{C}$ satisfying $r_1 \circ \mu_1 = \mathrm{id}_{\mathfrak{C}}$ and $r_2 \circ \mu_2 = \mathrm{id}_{\mathfrak{C}}$. Let $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the pullback C^*-algebra associated with r_1, r_2 defined by

$$\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} = \{(a, b) \in \mathfrak{A} \oplus \mathfrak{B} \mid r_1(a) = r_2(b)\}.$$

Theorem 2.1 (Blackadar [1, 10.11.11]) *Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be the amalgamated free product of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$ with retractions r_1, r_2 to $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the associated pull back C^*-algebra. Then*

$$K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \cong K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \quad (j = 0, 1).$$

Proof. Define the map $r : \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \to \mathfrak{C}$ by $r(a, b) = r_1(a) = r_2(b) \in \mathfrak{C}$ and let $i : \mathfrak{C} \to \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be the canonical inclusion. Define the map g by the following composition:

$$g : \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \xrightarrow{\ r\ } \mathfrak{C} \xrightarrow{\ i\ } \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}.$$

Let $k : \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} \to \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the map induced by setting

$$k(a) = (a, r_1(a)) \text{ for } a \in \mathfrak{A} \text{ and } k(b) = (r_2(b), b) \text{ for } b \in \mathfrak{B}$$

and using the universal property of $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$. Define $f : \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \to M_2(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ (the 2×2 matrix algebra over $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$) by $f(a, b) = a \oplus b$ the diagonal sum. Then we have the following composition:

$$(1 \otimes k) \circ f : \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \xrightarrow{\ f\ } M_2(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \xrightarrow{\ 1\otimes k\ } M_2(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}),$$

$$(1 \otimes k) \circ f(a, b) = \begin{pmatrix} (a, r_1(a)) & 0 \\ 0 & (r_2(b), b) \end{pmatrix} \equiv (a, r_1(a)) \oplus (r_2(b), b),$$

and this homomorphism is homotopic to $1_{\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}} \oplus (k \circ g)$ by conjugation by the unitaries $(1_{M_2(\mathfrak{A})} \oplus u_t)$, where

$$u_t = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Indeed, $(k \circ g)(a, b) = k(r_1(a)) = (r_1(a), r_1(r_1(a))) = (r_1(a), r_2(b))$ and

$$\begin{aligned} &(a, b) \oplus (k \circ g)(a, b) \\ &= (a, b) \oplus (r_1(a), r_2(b)) = (a \oplus r_1(a)) \oplus (b \oplus r_2(b)) \\ &= \begin{pmatrix} a & 0 \\ 0 & r_1(a) \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & r_2(b) \end{pmatrix} \in M_2(\mathfrak{A}) \oplus M_2(\mathfrak{B}), \\ &(1_{M_2(\mathfrak{A})} \oplus u_1)((a \oplus r_1(a), b \oplus r_2(b))(1_{M_2(\mathfrak{A})} \oplus u_1^*) \\ &= (a \oplus r_1(a)) \oplus u_1(b \oplus r_2(b))u_1^* \\ &= (a \oplus r_1(a)) \oplus (r_2(b) \oplus b) \in M_2(\mathfrak{A}) \oplus M_2(\mathfrak{B}). \end{aligned}$$

Hence, it follows that $k_* \circ f_* - k_* \circ g_*$ is the identity map on the K-groups $K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B})$ of $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ $(j = 0, 1)$. Also we have the following composition:

$$h_1 = f \circ k : \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} \xrightarrow{\ k\ } \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} \xrightarrow{\ f\ } M_2(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}),$$

which is homotopic to $h_0 = 1_{\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}} \oplus (g \circ k)$ via the path of homomorphisms h_t defined by $h_t(a) = a \oplus r_1(a) = (f \circ k)(a)$, $h_t(b) = u_t((b \oplus r_2(b))u_t^*$. Indeed, $(g \circ k)(a) = g(a, r_1(a)) = r_1(a) = r_2(r_1(a))$ and $(g \circ k)(b) = g(r_2(b), b) = r_2(b) = r_1(r_2(b))$ and

$$\begin{aligned} &h_0(a) = a \oplus r_1(a), \quad h_0(b) = b \oplus r_2((b), \\ &u_1(b \oplus r_2(b))u_1^* = r_2(b) \oplus b = (f \circ k)(b) \end{aligned}$$

Thus, it follows that $f_* \circ k_* - g_* \circ k_*$ is the identity map on the K-groups $K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ of $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ $(j = 0, 1)$.

Therefore, we conclude that $k_* : K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \to K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B})$ is an isomorphism with its inverse $f_* - g_*$ $(j = 0, 1)$. □

Remark. If $\mathfrak{C} = \{0\}$, then we can take the retractions r_1, r_2 as zero ones, and $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} \cong \mathfrak{A} * \mathfrak{B}$ the free product C^*-algebra of $\mathfrak{A}$, $\mathfrak{B}$. Moreover, for $j = 0, 1$,

$$K_j(\mathfrak{A} * \mathfrak{B}) \cong K_j(\mathfrak{A} \oplus \mathfrak{B}).$$

Furthermore,

Theorem 2.2 *We have the following splitting exact sequence*:

$$0 \longrightarrow K_j(\mathfrak{C}) \xrightarrow{(\mu_{1*}, \mu_{2*})} K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}) \xrightarrow{\iota_{1*} - \iota_{2*}} K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \longrightarrow 0.$$

Proof. By Mayer-Vietoris sequence for K-theory, the following sequence:

$$0 \longrightarrow K_j(\mathfrak{C}) \xrightarrow{(\mu_{1*}, \mu_{2*})} K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}) \xrightarrow{\iota_{1*} - \iota_{2*}} K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \longrightarrow 0$$

is exact and splitting ([1, 10.11.11]). □

Corollary 2.3 *We have*

$$K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \cong (K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C}) \quad (j = 0, 1).$$

Exactly by the same way as Theorem 2.1, under an additional assumption on commutativity we obtain

Theorem 2.4 *Let $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$ be the balanced tensor product C^*-algebra of unital C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ over a common nonzero unital C^*-subalgebra $\mathfrak{C}$ with retractions r_1, r_2 to $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the associated pull back C^*-algebra defined as above. Assume that $\mathfrak{C}$ commutes with $\mathfrak{A}$ and $\mathfrak{B}$ and has the same unit with them. Then*

$$K_j(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \cong K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \quad (j = 0, 1).$$

Proof. Since $\mathfrak{A}, \mathfrak{B}$ are unital, they are assumed to be C^*-subalgebras of $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$ via $a = a \otimes 1$ and $b = 1 \otimes b$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Since $a \otimes b = (a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1)$ and we need to have that the following elements:

$$(a, r_1(a))(r_2(b), b) = (ar_2(b), r_1(a)b), \quad (r_2(b), b) = (a, r_1(a)) = (r_2(b)a, br_1(a))$$

are equal to define the map k' corresponding to the map k in the proof of Theorem 2.1, from which we need to assume that $\mathfrak{C}$ commutes with $\mathfrak{A}$ and $\mathfrak{B}$. Also, $\mathfrak{C}$ can not be zero since if $\mathfrak{C}$ is zero, $k'(1 \otimes 1) = (1, 0)$ and $k'(1 \otimes 1) = (0, 1)$. Thus, k' is not well-defined. If $\mathfrak{C}$ is unital and nonzero, $k'(1 \otimes 1) = (1, r_1(1))$ and $k'(1 \otimes 1) = (r_2(1), 1)$, Thus, to have $(1, r_1(1)) = (r_2(1), 1)$ we need to assume that $\mathfrak{C}$ has the same unit with $\mathfrak{A}$, $\mathfrak{B}$. □

Corollary 2.5 *Under the same assumption as above we have*

$$K_j(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \cong (K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C}) \quad (j = 0, 1).$$

Theorem 2.6 *Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be the amalgam of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$ with retractions r_1, r_2 to $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the associated pullback C^*-algebra defined as above. Let $(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D}$ be the successive amalgam defined above for C^*-algebras $\mathfrak{D}, \mathfrak{E}$ with retractions $s_1 : \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B} \to E$, $s_2 : \mathfrak{D} \to E$, and $(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}$ be the associated pullback C^*-algebra. Then for $j = 0, 1$,*

$$\begin{aligned} K_j((\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D}) &\cong [((K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C})) \oplus K_j(\mathfrak{D})]/K_j(E) \\ &\cong K_j((\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}) \end{aligned}$$

where $(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}$ is the successive pullback C^*-algebra associated with r_1, r_2 and s_1, s_2.

Proof. Using Theorem 2.1 and Corollary 2.3 we compute

$$\begin{aligned} K_j((\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) *_E \mathfrak{D}) &\cong K_j((\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}) \\ &\cong [K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \oplus K_j(\mathfrak{D})]/K_j(E) \\ &\cong [K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus K_j(\mathfrak{D})]/K_j(E) \\ &\cong [((K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C})) \oplus K_j(\mathfrak{D})]/K_j(E). \end{aligned}$$

On the other hand, using Mayer-Vietoris sequence repeatedly we obtain

$$\begin{aligned} K_j((\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}) &\cong [K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus K_j(\mathfrak{D})]/K_j(E) \\ &\cong [((K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C})) \oplus K_j(\mathfrak{D})]/K_j(E). \end{aligned}$$

□

Similarly, using Theorem 2.4 and Corollary 2.5 we obtain

Theorem 2.7 *Let $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$ be the balanced tensor product C^*-algebra of unital C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ over a common nonzero unital C^*-subalgebra $\mathfrak{C}$ with retractions r_1, r_2 to $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the associated pullback C^*-algebra defined as above. Let $(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \otimes_E \mathfrak{D}$ be the successive balanced tensor product C^*-algebra defined in Section 1 for unital C^*-algebras $\mathfrak{D}, E$ with retractions $s_1 : \mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B} \to E$, $s_2 : \mathfrak{D} \to E$, and $(\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}$ be the associated pullback C^*-algebra. Assume that $\mathfrak{C}$ commutes with $\mathfrak{A}$ and $\mathfrak{B}$, and E commutes with $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$ and $\mathfrak{D}$. Then for $j = 0, 1$,*

$$\begin{aligned} K_j((\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}) \otimes_E \mathfrak{D}) &\cong [((K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathfrak{C})) \oplus K_j(\mathfrak{D})]/K_j(E) \\ &\cong K_j((\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}) \end{aligned}$$

where $(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \oplus_E \mathfrak{D}$ is the successive pullback C^*-algebra associated with r_1, r_2 and s_1, s_2.

Theorem 2.8 *Let $\mathfrak{A}$ be the n-successive pullback C^*-algebra as follows*:

$$\mathfrak{A} = (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{B}_1} \mathfrak{A}_2) \oplus_{\mathfrak{B}_2} \mathfrak{A}_3) \cdots) \oplus_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are C^*-algebras, and we assume that there exist $*$-homomorphisms: $\alpha_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, $\alpha_j : \mathfrak{A}_j \to \mathfrak{B}_{j-1}$ $(2 \le j \le n)$, $\beta_j : \mathfrak{B}_j \to \mathfrak{B}_{j+1}$ $(1 \le j \le n-2)$. Then for $j = 0, 1$,

$$K_j(\mathfrak{A}) \cong ((\cdots((((K_j(\mathfrak{A}_1) \oplus K_j(\mathfrak{A}_2))/K_j(\mathfrak{B}_1)) \oplus K_j(\mathfrak{A}_3))/K_j(\mathfrak{B}_2)) \cdots) \oplus K_j(\mathfrak{A}_n))/K_j(\mathfrak{B}_{n-1}).$$

Proof. We use the Mayer-Vietoris sequence for K-theory repeatedly. □

Theorem 2.9 *Let $\mathfrak{A}$ be the n-successive amalgam C^*-algebra as follows*:

$$\mathfrak{A} = (\cdots((\mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2) *_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) *_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are C^*-algebras, and we assume that there exist $*$-homomorphisms: $\mu_1 : \mathfrak{B}_1 \to \mathfrak{A}_1$, $\mu_j : \mathfrak{B}_j \to \mathfrak{A}_{j+1}$ $(2 \le j \le n-1)$, $\nu_j : \mathfrak{B}_{j+1} \to \mathfrak{B}_j$ $(1 \le j \le n-2)$. Suppose that there exist retractions $r_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, $r_j : \mathfrak{A}_j \to \mathfrak{B}_{j-1}$ $(2 \le j \le n)$ and $s_j : \mathfrak{B}_j \to \mathfrak{B}_{j+1}$ $(1 \le j \le n-2)$. Let P be the associated n-bullback C^*-algebra as follows: $P = (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{B}_1} \mathfrak{A}_2) \oplus_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) \oplus_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$. *Then for $j = 0, 1$,*

$$\begin{aligned} &K_j(\mathfrak{A}) \cong K_j(P) \\ &\cong ((\cdots((((K_j(\mathfrak{A}_1) \oplus K_j(\mathfrak{A}_2))/K_j(\mathfrak{B}_1)) \oplus K_j(\mathfrak{A}_3))/K_j(\mathfrak{B}_2))\cdots) \oplus K_j(\mathfrak{A}_n))/K_j(\mathfrak{B}_{n-1}). \end{aligned}$$

Corollary 2.10 *Let $\mathfrak{A}$ be the n-successive amalgam C^*-algebra as follows*:

$$\begin{aligned} \mathfrak{A} &= (\cdots((\mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2) *_{\mathbb{C}} \mathfrak{A}_3)\cdots) *_{\mathbb{C}} \mathfrak{A}_n \\ &\cong \mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2 *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathfrak{A}_n \quad (n\text{-fold unital free product}) \end{aligned}$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$ are unital C^*-algebras. Suppose that there exist retractions $r_j : \mathfrak{A}_j \to \mathbb{C}$ $(1 \le j \le n)$. Let P be the associated n-bullback C^*-algebra as follows:

$$P = (\cdots((\mathfrak{A}_1 \oplus_{\mathbb{C}} \mathfrak{A}_2) \oplus_{\mathbb{C}} \mathfrak{A}_3)\cdots) \oplus_{\mathbb{C}} \mathfrak{A}_n.$$

Then

$$\begin{aligned} &K_0(\mathfrak{A}) \cong K_0(P) \\ &\cong ((\cdots((((K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2))/\mathbb{Z}) \oplus K_0(\mathfrak{A}_3))/\mathbb{Z})\cdots) \oplus K_0(\mathfrak{A}_n))/\mathbb{Z}, \quad \text{and} \\ &K_1(\mathfrak{A}) \cong K_1(P) \cong K_1(\mathfrak{A}_1) \oplus K_1(\mathfrak{A}_2) \oplus K_1(\mathfrak{A}_3) \oplus \cdots \oplus K_1(\mathfrak{A}_n). \end{aligned}$$

Proof. Note that $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) \cong 0$. □

Remark. In the theorem above, if $\mathfrak{B}_j = 0$ $(1 \le j \le n-1)$, then

$$\begin{aligned} \mathfrak{A} &\cong \mathfrak{A}_1 * \mathfrak{A}_2 * \cdots * \mathfrak{A}_n \quad (n\text{-fold free product}), \\ P &\cong \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_n \quad (n\text{-direct sum}), \end{aligned}$$

and $K_j(\mathfrak{A}) \cong K_j(P) \cong K_j(\mathfrak{A}_1) \oplus K_j(\mathfrak{A}_2) \oplus \cdots \oplus K_j(\mathfrak{A}_n)$ for $j = 0, 1$.

Theorem 2.11 *Let $\mathfrak{A}$ be the n-successive balanced tensor product C^*-algebra as follows*:

$$\mathfrak{A} = (\cdots((\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \otimes_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) \otimes_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$$

where $\mathfrak{A}_j$ $(1 \le j \le n)$, $\mathfrak{B}_j$ $(1 \le j \le n-1)$ are nonzero unital C^*-algebras, and we assume that there exist $*$-homomorphisms: $\mu_1 : \mathfrak{B}_1 \to \mathfrak{A}_1$, $\mu_j : \mathfrak{B}_j \to \mathfrak{A}_{j+1}$ $(2 \le j \le n-1)$, $\nu_j : \mathfrak{B}_{j+1} \to \mathfrak{B}_j$ $(1 \le j \le n-2)$. Suppose that there exist retractions $r_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, $r_j : \mathfrak{A}_j \to \mathfrak{B}_{j-1}$ $(2 \le j \le n)$ and $s_j : \mathfrak{B}_j \to \mathfrak{B}_{j+1}$ $(1 \le j \le n-2)$. Let P be the associated n-bullback C^*-algebra as follows: $P = (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{B}_1} \mathfrak{A}_2) \oplus_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) \oplus_{\mathfrak{B}_{n-1}} \mathfrak{A}_n$. *Assume that $\mathfrak{B}_j$ $(1 \le j \le n-1)$ commute with $\mathfrak{A}_{j+1}$ and*

$$(\cdots((\mathfrak{A}_1 \otimes_{\mathfrak{B}_1} \mathfrak{A}_2) \otimes_{\mathfrak{B}_2} \mathfrak{A}_3)\cdots) \otimes_{\mathfrak{B}_{j-1}} \mathfrak{A}_j$$

and have the same units with them. Then for $j = 0, 1$,

$$\begin{aligned} &K_j(\mathfrak{A}) \cong K_j(P) \\ &\cong ((\cdots((((K_j(\mathfrak{A}_1) \oplus K_j(\mathfrak{A}_2))/K_j(\mathfrak{B}_1)) \oplus K_j(\mathfrak{A}_3))/K_j(\mathfrak{B}_2))\cdots) \oplus K_j(\mathfrak{A}_n))/K_j(\mathfrak{B}_{n-1}). \end{aligned}$$

Corollary 2.12 *Let $\mathfrak{A}$ be the n-successive balanced tensor product C^*-algebra as follows*:

$$\begin{aligned}\mathfrak{A} &= (\cdots((\mathfrak{A}_1 \otimes_{\mathbb{C}} \mathfrak{A}_2) \otimes_{\mathbb{C}} \mathfrak{A}_3) \cdots) \otimes_{\mathbb{C}} \mathfrak{A}_n \\ &\cong \mathfrak{A}_1 \otimes_{\mathbb{C}} \mathfrak{A}_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathfrak{A}_n \quad (n\text{-fold unital tensor product})\end{aligned}$$

where $\mathfrak{A}_j$ $(1 \leq j \leq n)$ are unital C^*-algebras. Suppose that there exist retractions $r_j : \mathfrak{A}_j \to \mathbb{C}$ $(1 \leq j \leq n)$. Let P be the associated n-bullback C^*-algebra as follows:

$$P = (\cdots((\mathfrak{A}_1 \oplus_{\mathbb{C}} \mathfrak{A}_2) \oplus_{\mathbb{C}} \mathfrak{A}_3) \cdots) \oplus_{\mathbb{C}} \mathfrak{A}_n.$$

Then

$$\begin{aligned}&K_0(\mathfrak{A}) \cong K_0(P) \\ &\cong ((\cdots((((K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2))/\mathbb{Z}) \oplus K_0(\mathfrak{A}_3))/\mathbb{Z}) \cdots) \oplus K_0(\mathfrak{A}_n))/\mathbb{Z}, \quad \text{and} \\ &K_1(\mathfrak{A}) \cong K_1(P) \cong K_1(\mathfrak{A}_1) \oplus K_1(\mathfrak{A}_2) \oplus K_1(\mathfrak{A}_3) \oplus \cdots \oplus K_1(\mathfrak{A}_n).\end{aligned}$$

Remark. In the theorem above, if $\mathfrak{B}_j = 0$ $(1 \leq j \leq n-1)$, then

$$\begin{aligned}\mathfrak{A} &\cong \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \cdots \otimes \mathfrak{A}_n \quad (n\text{-fold tensor product}), \\ P &\cong \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_n \quad (n\text{-direct sum}),\end{aligned}$$

but $K_j(\mathfrak{A}) \not\cong K_j(P) \cong K_j(\mathfrak{A}_1) \oplus K_j(\mathfrak{A}_2) \oplus \cdots \oplus K_j(\mathfrak{A}_n)$ for $j = 0, 1$ in general. For instance, if $\mathfrak{A}_j = \mathbb{C}$ $(1 \leq j \leq n)$, then $\mathfrak{A} \cong \mathbb{C}$ and $K_0(\mathfrak{A}) \cong \mathbb{Z}$ but $K_0(P) \cong \oplus_{j=1}^{n} \mathbb{Z}$. See [1] for Künneth Theorem for K-groups of tensor products of C^*-algebras.

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