SOME DECOMPOSITIONS OF IDEALS IN BF-ALGEBRAS

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Abstract. In this paper we study some properties of (normal, closed) ideals in BF-algebras, especially we show that any ideal of BF-algebra can be decomposed into the union of some sets, and obtain the greatest closed ideal I^0 of an ideal I of a BF-algebra X contained in I.

1. Introduction

The concept of B-algebras was introduced by J. Neggers and H. S. Kim ([1, 4, 5, 6]). They defined a B-*algebra* as an algebra $(X; *, 0)$ of type $(2, 0)$ (i.e., a non-empty set X with a binary operation $*$ and a constant 0) satisfying $(B1)$, $(B2)$ and (B) $(x*y)*z = x*[z*(0*y)]$, for any $x, y, z \in X$. In [2], Y. B. Jun, R. H. Roh and H. S. Kim introduced BH-algebras, which are generalization of $BCK/BCH/B$-algebras. An algebra $(X; *, 0)$ of type $(2, 0)$ is a BH-*algebra* if it satisfies $(B1)$, $(B2)$ and (BH) $x * y = 0 = y * x$ implies $x = y$. Recently, C. B. Kim and H. S. Kim ([3]) defined a BG-*algebra* as an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying $(B1)$, $(B2)$ and (BG) $x = (x * y) * (0 * y)$, for any $x, y \in Z$. A. Walendziak ([9]) introduced the notion of BF-algebras, which is a generalization of B-algebras, and investigated some properties of (normal) ideals in BF-algebras. For another generalization of B-algebras we refer to [7, 8]. S. W. Wei and Y. B. Jun ([10]) studied ideals in BCI-algebras and decomposed some ideals into the union of some sets. We apply this concept to BF-algebras. In this paper we study some properties of (normal, closed) ideals in BF-algebras, especially we show that any ideal of BF-algebra can be decomposed into the union of some sets, and obtain the greatest closed ideal I^0 of an ideal I of a BF-algebra X contained in I.

2. Decompositions of ideals in BF-algebras

Let us review some definitions and results. By a BF-*algebra* ([9]) we mean a non-empty set X with a binary operation " $*$ " and a constant 0 satisfying the following conditions:

$(B1)$ $x * x = 0$,

$(B2)$ $x * 0 = x$,

(BF) $0 * (x * y) = y * x$

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for any $x, y, z \in X$.

A non-empty subset I of a BF-algebra X is said to be a *subalgebra* if $x \in I$ and $y \in I$ imply $x * y \in I$.
An *ideal* of a BF-algebra X is a subset I containing 0 such that if $x * y \in I$ and $y \in I$ then $x \in I$.
An ideal I of a BF-algebra X is said to be *normal* if for any $x, y, z \in X$, $x * y \in I$ implies $(z * x) * (z * y) \in I$.

Lemma 2.1. ([9]) *If I is a normal ideal of a BF-algebra X, then*

(a) $x \in I \Rightarrow 0 * x \in I$,

(b) $x * y \in I \Rightarrow y * x \in I$,

for any $x, y \in X$.

An ideal I of X is said to be *closed* if $x \in I$ then $0 * x \in I$. By Lemma 2.1-(a), it is known that every normal ideal of a BF-algebra X is a closed ideal of X. Note that a closed ideal need not be a subalgebra. See the following example.

Example 2.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	3	2	1
1	1	0	2	2
2	2	2	0	2
3	3	2	2	0

Then $(X; *, 0)$ is a BF-algebra, and $I := \{0, 1, 3\}$ is a closed ideal of X, but not a subalgebra of X, since $1 * 3 = 2 \notin I$. Moreover, $J := \{0, 1\}$ is an ideal of X, but not closed, since $0 * 1 = 3 \notin J$. The set $K := \{0, 2\}$ is a subalgebra of X, but not an ideal of X, since $3 * 2 = 2 \in K$,$2 \in K$, $3 \notin K$.

For any BF-algebra X and $x, y \in X$, we denote

$$A(x, y) = \{z \in X | (z * x) * y = 0\}.$$

Theorem 2.3. *If I is an ideal of a BF-algebra X, then*

$$I = \bigcup_{x,y \in I} A(x, y).$$

Proof. Let I be an ideal of a BF-algebra X. If $z \in I$, then $(z * 0) * z = z * z = 0$. Hence $z \in A(0, z)$. It follows that

$$I \subseteq \bigcup_{z \in I} A(0, z) \subseteq \bigcup_{x,y \in I} A(x, y).$$

Let $z \in \bigcup_{x,y\in I} A(x,y)$. Then there exist $a, b \in I$ such that $z \in A(a,b)$, so that $(z*a)*b = 0$. Since I is an ideal, it follows that $z \in I$. Thus $\bigcup_{x,y\in I} A(x,y) \subseteq I$, and consequently, $I = \bigcup_{x,y\in I} A(x,y)$. □

Corollary 2.4. *If I is an ideal of a BF-algebra X, then*

$$I = \bigcup_{x\in I} A(0,x).$$

Proof. By Theorem 2.3. we have that $\bigcup_{x\in I} A(0,x) \subseteq \bigcup_{x,y\in I} A(x,y) = I$. If $x \in I$, then $x \in \bigcup_{x\in I} A(0,x)$, since $(x*0)*x = 0$. Hence $I \subseteq \bigcup_{x\in I} A(0,x)$. This completes the proof. □

We give an example satisfying Theorem 2.3 and Corollary 2.4. See the following example.

Example 2.5. Let $X := \{0, 1, 2, 3, \}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	0
3	3	1	0	0

Then $(X; *, 0)$ is a BF-algebra and $I := \{0, 2, 3\}$ is an ideal of X. Moreover, it is easy to check that $I = A(2,0) \cup A(3,2)$ and $I = A(0,0) \cup A(0,2)$.

Theorem 2.6. *Let I be a subset of a BF-algebra X such that $0 \in I$ and*

$$I = \bigcup_{x,y\in I} A(x,y).$$

Then I is an ideal of X.

Proof. Let $x*y, y \in I = \bigcup_{x,y\in I} A(x,y)$. Since $(x*y)*(x*y) = 0$, it follows that $x \in A(y, x*y) \subseteq I$. Hence I is an ideal of X. □

Combining Theorems 2.3 and 2.6, we have the following corollary.

Corollary 2.7. *Let X be a BF-algebra and I be a subset of X containing 0. Then I is an ideal of X if and only if*

$$I = \bigcup_{x,y\in I} A(x,y).$$

Now, we give a characterization of normal and closed ideal in BF-algebras.

Proposition 2.8. *Let I be a normal ideal of a BF-algebra X. If $x*z \in I$, $y*z \in I$ and $z \in I$, then $x*y \in I$.*

Proof. Let I be a normal ideal of X. Assume that $x * z \in I$, $y * z \in I$ and $z \in I$. Since I is an ideal of X, we obtain $x, y \in I$. By Lemma 2.1-(a), $0 * y \in I$ and by definition of normal, $(x * 0) * (x * y) \in I$, i.e., $x * (x * y) \in I$. Also, by Lemma 2.1-(b), we have $(x * y) * x \in I$. Since I is an ideal of X and $x \in I$, we obtain $x * y \in I$. □

The converse of Proposition 2.8 need not be true in general. See the following example.

Example 2.9. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

Then $(X; *, 0)$ is a BF-algebra and $I := \{0\}$ is an ideal of X. Although I satisfies the condition: $x * z \in I$, $y * z \in I$ and $z \in I$ imply $x * y \in I$, I is not a normal ideal of X, since $1 * 3 = 0 \in I$, $(2 * 1) * (2 * 3) = 2 \notin I$. □

Corollary 2.10. *If I is a subset of a BF-algebra X with satisfying the conditions:*

(1) $0 \in I$,

(2) $x * z \in I$, $y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$

for any $x, y, z \in X$, then I is a subalgebra of X.

Proof. Given $x, y \in I$, by $(B2)$, we have $x = x * 0, y = y * 0$. It follows from (2) that $x * y \in I$. □

Proposition 2.11. *Let I be a subset of a BF-algebra X with the following conditions:*

(1) $0 \in I$,

(2) $x * z \in I$, $y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$

Then I is a closed ideal of X.

Proof. Assume that I satisfies (1) and (2). We claim that I is a closed ideal of X. Let $x * y, y \in I$. Since $0 * 0, y * 0, 0 \in I$, by (2), we have $0 * y \in I$, which proves that I is closed. Since $x * y, 0 * y, y \in I$, by applying (2) again, we obtain that $x = x * 0 \in I$, so that I is an ideal of X. □

Lemma 2.12. ([9]) *If $(X; *, 0)$ is a BF-algebra, then $0 * (0 * x) = x$ for any $x \in X$.*

Theorem 2.13. *Let I be an ideal of a BF-algebra X. Then the set*

$$I^0 := \{x \in I | 0 * x \in I\}$$

is the greatest closed ideal of X which is contained in I.

Proof. First, we show that I^0 is an ideal of X. Clearly, $0 \in I^0$. If $x * y, y \in I^0$, then $x * y, y \in I$, since $I^0 \subseteq I$. Since I is an ideal of X, $x \in I$. By applying Lemma 2.12, we have $0 * (0 * x) = x \in I$. This means that $0 * x \in I^0$. Since $I^0 \subseteq I$, $0 * x \in I$ and hence $x \in I^0$. Hence I^0 is an ideal of X.

If $x \in I^0$, by definition of I^0, we have $0 * x \in I$ and $x \in I$. By Lemma 2.12 we have $0*(0*x) = x \in I$, it follows $0*x \in I^0$. Hence $0*x \in I^0 \subseteq I$, which proves that I^0 is closed.

Now, assume that A is a closed ideal of X which is contained in I. If $x \in A$, then $0 * x \in A$. Since A is contained in I, we have $x, 0 * x \in I$, and so $x \in I^0$. Thus $A \subseteq I^0$. Therefore, I^0 is the greatest closed ideal of X which is contained in I. □

Example 2.14. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	2	1	3
1	1	0	1	2
2	2	2	0	2
3	3	1	1	0

Then $(X; *, 0)$ is a BF-algebra and $I := \{0, 1, 3\}$ is an ideal of X. Let $I_1 := \{0, 1\}$, $I_2 := \{0, 3\}$ and $I_3 := \{0, 1, 3\}$ be subsets of I. We can see that I_1 is not an ideal, since $3*1 = 1 \in I_1, 1 \in I_1$, but $3 \notin I_1$, I_2 is a closed ideal, but I_3 is not closed, since $0*1 = 2 \notin I_3$. Hence I_2 is the greatest closed ideal of X which is contained in I, i.e., $I^0 = I_2$.

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