ABSOLUTE CONTINUITY OF FORMS AND ABSOLUTE J-CONTRACTIONS

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Abstract. In this note, we study absolute continuity of a pair of forms $\langle Ax, y\rangle, \langle Bx, y\rangle$. As an application, we have a characterization of absolutely J-contractive operators A on a Krein space:

$$|[Ax, Ay]| \leqq |[x, y]|$$

for all x, y.

Let $\mathcal{H}$ be a Hilbert space whose inner product is denoted as $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. A linear operator V on $\mathcal{H}$ is said to be isometric if

$$\langle Vx, Vy\rangle = \langle x, y\rangle$$

for all $x, y \in \mathcal{H}$. A linear operator A is called absolutely contractive if

$$|\langle Ax, Ay\rangle| \leqq |\langle x, y\rangle|$$

for all $x, y \in \mathcal{H}$. Then it is natural to try to find a relation between absolutely contractive operators A and isometric operators V. Indeed, A is characterized as αV for some real number α $(0 \leqq \alpha \leqq 1)$ and an isometric operator V, which is shown as Corollary 4 below.

In this note, we would like to extend this to linear operators on a Krein space with selfadjoint involution $J : J = J^*,\ J^2 = I$. We refer the reader to [1] for Krein spaces. The J-inner product $[\cdot, \cdot]$ on $\mathcal{H}$ is defined by

$$[x, y] := \langle Jx, y\rangle \quad (x, y \in \mathcal{H}).$$

A linear operator V is said to be J-isometric if

$$[Vx, Vy] = [x, y]$$

for all $x, y \in \mathcal{H}$, and is called absolutely J-isometric if

$$|[Vx, Vy]| = |[x, y]|$$

for all $x, y \in \mathcal{H}$. A linear operator A is called absolutely J-contractive if

$$|[Ax, Ay]| \leqq |[x, y]|$$

for all $x, y \in \mathcal{H}$.

Let us consider an example. Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $[A_1x, A_1y] = -[x, y]$ $(x, y \in \mathbb{C}^2)$; hence, $|[A_1x, A_1y]| = |[x, y]|$ for all $x, y \in \mathbb{C}^2$. Suppose that there were a complex number α and a J-isometry V such that $A_1 = \alpha V$. Then $[A_1x, A_1y] = |\alpha|^2[x, y]$, and $|\alpha|^2 = -1$; this is a contradiction.

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We have another example; let

$$A_2 = \begin{pmatrix} a & b \\ e^{i\theta}a & e^{i\theta}b \end{pmatrix}$$

for $a, b \in \mathbb{C}, \theta \in \mathbb{R}$. Then for the same J, $[A_2x, A_2y] = 0$ $(x, y \in \mathbb{C}^2)$ and $A_2 \neq O$ in general.

These examples lead us to the following:

Theorem 1. *Let A be a bounded linear operator on $\mathcal{H}$, J a selfadjoint involution on $\mathcal{H}$, and $[\cdot,\cdot]$ the J-inner product. Then A is absolutely J-contractive if and only if $A^*JA = O$ or $A = \alpha V$ for a real number α $(0 < \alpha \leqq 1)$ and an absolutely J-isometric operator V.*

To prove this, we prepare the following, which seems of importance in itself:

Theorem 2. *Let A, B be linear operators on $\mathcal{H}$. The form $\langle Ax, y\rangle$ is absolutely continuous for $\langle Bx, y\rangle$, i.e., $\langle Bx, y\rangle = 0$ implies $\langle Ax, y\rangle = 0$, if and only if there is a derivative $\alpha \in \mathbb{C}$ with $A = \alpha B$.*

This follows from more general one, which might be well-known, but we have a proof for the reader's convenience:

Proposition 3. *Let X, Y be complex vector spaces and let $\varphi_i : X \times Y \longrightarrow \mathbb{C}$ be bilinear $(i = 1, 2)$. Then*

$$\varphi_1(x, y) = 0 \Longrightarrow \varphi_2(x, y) = 0$$

if and only if

$$\varphi_2(x, y) = \alpha\varphi_1(x, y) \quad (x \in X, y \in Y)$$

for a complex number α.

Proof. Since the sufficiency is clear, we show the necessity.
Case 1: either X or Y is of dimension 1. For instance, assume that $\dim Y = 1 : Y = \mathbb{C}y_0$ for some $y_0 \in Y$. Then by assumption,

$$\varphi_1(x, y_0) = 0 \Longrightarrow \varphi_2(x, y_0) = 0.$$

By the standard fact on linear functionals ([3, Proposition 1.1.1] or [2, Appendix A]), there is a complex number α such that

$$\varphi_2(x, y_0) = \alpha\varphi_1(x, y_0) \quad (x \in X),$$

and the assertion follows.
Case 2: $\dim X, \dim Y \geqq 2$. For each $x \in X$, since the linear functionals $\varphi_1(x, \cdot), \varphi_2(x, \cdot)$ on Y satisfy the assumption, we have a complex number $\alpha(x) \in \mathbb{C}$ such that

$$\varphi_2(x, \cdot) = \alpha(x)\varphi_1(x, \cdot).$$

Similarly, for each $y \in Y$ we have a complex number $\beta(y)$ such that

$$\varphi_2(\cdot, y) = \beta(y)\varphi_1(\cdot, y).$$

If $\varphi_1(x, \cdot) = 0$ for all $x \in X$, then $\varphi_2(x, \cdot) = 0$ for all $x \in X$ by assumption, and the conclusion follows for any α. Hence, we assume that there is a vector $x_0 \in X$ with $\varphi_1(x_0, \cdot) \neq 0$ and take any $x \in X$. Our claim is that $\alpha(\cdot)$ can be taken to be identical.

If the linear functionals $\varphi_1(x_0, \cdot), \varphi_1(x, \cdot)$ are linearly independent, then there are vectors $y_1, y_2 \in Y$ such that $\varphi_1(x_0, y_1) = \varphi_1(x, y_2) = 1$ and $\varphi_1(x_0, y_2) = \varphi_1(x, y_1) = 0$. Put $y_3 := y_1 + y_2 \in Y$, then we have $\varphi_1(x_0, y_3) = \varphi_1(x, y_3) = 1$. Hence, it follows that

$$\begin{aligned} \alpha(x) &= \alpha(x)\varphi_1(x, y_3) = \varphi_2(x, y_3) = \beta(y_3)\varphi_1(x, y_3) = \beta(y_3) \\ &= \beta(y_3)\varphi_1(x_0, y_3) = \varphi_2(x_0, y_3) = \alpha(x_0)\varphi_1(x_0, y_3) = \alpha(x_0). \end{aligned}$$

If $\varphi_1(x_0,\cdot)$ and $\varphi_1(x,\cdot)$ are linearly dependent and $\varphi_1(x,\cdot) \neq 0$, then there is a complex number $\lambda(\neq 0)$ such that $\varphi_1(x,\cdot) = \lambda\varphi_1(x_0,\cdot)$. Taking a vector $y_4 \in Y$ with $\varphi_1(x_0,y_4) = 1$, we have

$$\begin{aligned} \alpha(x) &= \alpha(x)\varphi_1(x_0,y_4) = \frac{1}{\lambda}\alpha(x)\lambda\varphi_1(x_0,y_4) = \frac{1}{\lambda}\alpha(x)\varphi_1(x,y_4) = \frac{1}{\lambda}\varphi_2(x,y_4) \\ &= \frac{1}{\lambda}\beta(y_4)\varphi_1(x,y_4) = \beta(y_4)\varphi_1(x_0,y_4) = \varphi_2(x_0,y_4) = \alpha(x_0)\varphi_1(x_0,y_4) = \alpha(x_0). \end{aligned}$$

When $\varphi_1(x,\cdot) = 0$, we can take any number as $\alpha(x)$.

Therefore, the proof is complete. □

Proof of Theorem 1. It suffices to show the necessity. Applying Theorem 2, we have a complex number β ($|\beta| \leqq 1$) such that

$$A^*JA = \beta J.$$

When $\beta = |\beta|e^{i\theta} \neq 0$, then

$$[\frac{A}{\sqrt{|\beta|}}x, \frac{A}{\sqrt{|\beta|}}y] = e^{i\theta}[x,y].$$

Therefore, $V := \dfrac{A}{\sqrt{|\beta|}}$ is absolutely J-isometric. For $\alpha := \sqrt{|\beta|}$, the proof is complete. □

We remark that since A^*JA and J are selfadjoint β should be real and that the existence of V with $[Vx,Vy] = -[x,y]$ $(\forall x,y)$ depends on each Krein space.

Let $J = I$, the identity operator on $\mathcal{H}$. Then in the above proof, the scalar β with $A^*A = \beta I$ should be non-negative, since A^*A is positive. When $\beta = 0$, that is, $A^*A = O$, then $A = O = 0 \cdot I$. When $\beta > 0$, $\dfrac{A}{\sqrt{\beta}}$ is isometric. Therefore, we have:

Corollary 4. *Let A be a linear operator on $\mathcal{H}$. Then A is absolutely contractive if and only if $A = \alpha V$ for a real number α $(0 \leqq \alpha \leqq 1)$ and an isometric operator V.*

Finally, we have a comment on the condition that $A^*JA = O$. Assume that

$$P_+ := \frac{1}{2}(I+J) \neq O,\ P_- := \frac{1}{2}(I-J) \neq O,$$

that is, J is indefinite. Denote the corresponding subspaces by $\mathcal{H}_+, \mathcal{H}_-$ and consider the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In correspondence to this decomposition, $J, A \in B(\mathcal{H})$ are represented as

$$J = \begin{pmatrix} I_+ & O \\ O & -I_- \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

In this case, $A^*JA = O$ if and only if there is a partial isometry W from $\mathcal{H}_+$ to $\mathcal{H}_-$ such that

$$WA_{11} = A_{21}, \quad WA_{12} = A_{22}.$$

In fact, $A^*JA = O$ means that

$$\|A_{11}x + A_{12}y\| = \|A_{21}x + A_{22}y\| \ \ (x \in \mathcal{H}_+, y \in \mathcal{H}_-).$$

Hence, $W_0 : A_{11}x + A_{12}y \mapsto A_{21}x + A_{22}y$ $(x \in \mathcal{H}_+, y \in \mathcal{H}_-)$ is well-defined and we extend this to a desired partial isometry $W : \mathcal{H}_+ \to \mathcal{H}_-$. Therefore, we conclude:

Proposition 5. *Let $J \in B(\mathcal{H})$ be an indefinite selfadjoint involution, $A \in B(\mathcal{H})$, and let us represent them as*

$$J = \begin{pmatrix} I_+ & O \\ O & -I_- \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then $A^*JA = O$ *if and only if there is a partial isometry* W *from* $\mathcal{H}_+$ *to* $\mathcal{H}_-$ *such that*

$$WA_{11} = A_{21}, \quad WA_{12} = A_{22}.$$

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References

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