ON SUNS, MOONS AND BEST APPROXIMATION IN M-SPACES

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Abstract. A metric space (X, d) in which for every $x, y \in X$ and for every t, $0 \leq t \leq 1$ there exists exactly one point $z \in X$ such that $d(x, z) = (1-t)d(x, y)$ and $d(z, y) = td(x, y)$ is called an M-space. In this paper we discuss suns and moons in M-spaces and characterize these via best approximation thereby extending corresponding known results in normed linear spaces to M-spaces.

Introduction The concept of a sun in Approximation Theory was first introduced in normed linear spaces by Klee [5] but the terminology 'sun' was proposed by Effimov and Steckin [3]. We recall that a set V is a sun iff whenever $v_0 \in V$ is a best approximation to some element $x \notin V$ then v_0 is a best approximation to every element on the ray from v_0 through x. Since every convex set in a normed linear space has this property, a sun may be regarded as a generalization of a convex set. Vlasov [10],who developed the concept further, showed that in a smooth Banach space every proximinal sun is convex. In view of Vlasov's result, the most famous unsolved problem in Approximation Theory viz. whether or not every Chebyshev set in a Hilbert space is convex, may be stated equivalently as "Is every Chebyshev set in a Hilbert space a sun?" The concept of a moon, which is a generalization of sun, was introduced by Amir and Deutsch [1] and their special interest was in determining those normed linear spaces in which every moon is a sun. Knowing such spaces is quite useful as it is much easier to verify that a given set is a moon than verify it is a sun.

Our purpose in this paper is to discuss these concepts in M-spaces [4] (also called strongly convex spaces [9]) and extend some of the results proved in [1] and [6] to M-spaces.

To start with, we give a few notations and recall a few definitions.

Let (X, d) be a metric space and $x, y, z \in X$. We say that z is <u>between</u> x and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points x, y of X, the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is called the <u>metric segment</u> and is denoted by $G[x, y]$.

A metric space (X, d) is said to be <u>convex</u> [9] if for every $x, y \in X$ and for every t, $0 \leq t \leq 1$ there exists at least one point $z \in X$ such that

$$d(x, z) = (1-t)d(x, y) \text{ and } d(z, y) = td(x, y)$$

The space (X, d) is said to be <u>strongly convex</u> [9] or an <u>M-space</u> [4] if such a z exists and is unique for each pair x, y of X .

Thus for strongly convex metric spaces each t, $0 \leq t \leq 1$, determines a unique point of the segment $G[x, y]$.

Let $G(x, y, -)$ denote the largest line segment containing $G[x, y]$ for which x is an extreme point i.e. the ray starting from x and passing through y, $G_1(x, y, -)$ denotes $G(x, y, -) \setminus G[x, y]$ and $K(v_0, x) \equiv \bigcup B(z, d(z, v_0))$, $z \in G_1(v_0, x, -)$ where $B(z, r)$ stands for an open ball with centre z and radius r.

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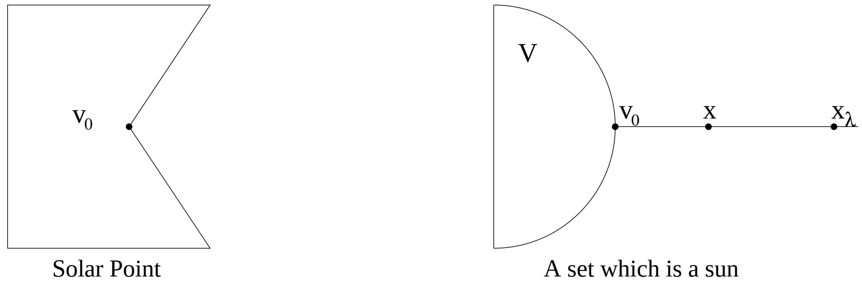

Figure 1: A non sun set and a set which is a sun

A subset V of an M-space (X,d) is said to be a cone with vertex v_0 if $G(v_0,y,-) \subseteq V$ whenever $y \in V$.

Let V be a non-empty subset of a metric space (X,d) and $x \in X$. An element $v_0 \in V$ is called a best approximation to x if $d(x,v_0) = dist(x,V)$. We denote by $P_V(x)$, the set of all best approximants to x in V. The set V is said to be proximinal if $P_V(x) \neq \emptyset$ for each $x \in X$ and is said to be Chebyshev if $P_V(x)$ is exactly singleton for each $x \in X$.

For $v_0 \in V$, ${P_V}^{-1}(v_0) = \{x \in X : v_0 \in P_V(x)\}$. It is easy to prove (see [8]) that if $x \in {P_V}^{-1}(v_0)$ then $x_\lambda \in {P_V}^{-1}(v_0)$ for every $x_\lambda \in G[v_0,x]$ i.e.$v_0 \in P_V(x_\lambda)$. On the other hand, v_0 may not be in $P_V(x_\lambda)$ for $x_\lambda \in G_1(v_0,x,-)$. This motivates the following definition introduced in normed linear spaces by Effimov and Steckin [3]: If V is a proximinal subset of an M-space (X,d) , a point $v_0 \in V$ is called a solar point (see Fig. 1 left diagram) of V if $x \in {P_V}^{-1}(v_0)$ implies $x_\lambda \in {P_V}^{-1}(v_0)$ for every $x_\lambda \in G_1(v_0,x,-)$. The set V is called a sun (see Fig. 1 right diagram) if for each $x \in X \setminus V$, every $v_0 \in P_V(x)$ is a solar point of V i.e. for all $v_0 \in P_V(x)$, $v_0 \in P_V(z)$ for all $z \in G_1(v_0,x,-)$.

Let V be a subset of an M-space (X,d) . A point $v_0 \in V$ is called a lunar point if $x \in X$ and $K(v_0,x) \cap V \neq \emptyset$ imply $v_0 \in \overline{K(v_0,x) \cap V}$. The set V is called a moon if each of its point is lunar.

The set $V = \{(x,y) \in R^2 : x^2 + 4y^2 \geq 1\}$ in Euclidean 2-space R^2 is a moon (see [2], p.38). We shall see that each sun in an M-space is a moon. However, the converse is not true (see [1]).

For proximinal subsets of an M-space, we have:

Theorem 1 *A proximinal subset V of an M-space (X,d) is a sun if and only if for any $v_0 \in V$, the set ${P_V}^{-1}(v_0)$ is a cone with vertex v_0 .*

Proof Suppose V is a sun and $x \in {P_V}^{-1}(v_0)$ i.e. $v_0 \in P_V(x)$. We want to show that $G(v_0,x,-) \subseteq {P_V}^{-1}(v_0)$. Since $v_0 \in P_V(x)$ and V is a sun, $v_0 \in P_V(z)$ for all $z \in G_1(v_0,x,-)$ and consequently for all $z \in G(v_0,x,-)$ i.e. $z \in {P_V}^{-1}(v_0)$ for all $z \in G(v_0,x,-)$ i.e. ${P_V}^{-1}(v_0)$ is a cone with vertex v_0.

Conversely, let $x \in X \setminus V$ and $y \in P_V(x)$ i.e. $x \in {P_V}^{-1}(y)$ where $y \in V$. Since ${P_V}^{-1}(y)$ is a cone with vertex y, $G(y,x,-) \subseteq {P_V}^{-1}(y)$ i.e. $y \in P_V(z)$ for all $z \in G(y,x,-)$. Hence V is a sun.

Theorem 2 *A proximinal subset V of an M-space (X,d) is a sun if and only if for any $v_0 \in V$ and $x \in {P_V}^{-1}(v_0)$, $K(v_0,x) \cap V = \emptyset$.*

Proof Suppose V is a sun. Let $v_0 \in V$ and $x \in {P_V}^{-1}(v_0)$. Since $v_0 \in P_V(x)$ and V is a sun, $v_0 \in P_V(z)$ for all $z \in G(v_0,x,-)$. To show $K(v_0,x) \cap V = \emptyset$. Suppose $u \in K(v_0,x) \cap V$

i.e. $u \in B(z, d(z, v_0))$ for some $z \in G_1(v_0, x, -)$ i.e. $d(z, u) \leq d(z, v_0)$ and so $v_0 \notin P_V(z)$ as $u \in V$, a contradiction. Therefore $K(v_0, x) \cap V = \emptyset$.

For the converse part, suppose V is not a sun. Then there exists $x \in X \setminus V$ and $v_0 \in P_V(x)$ such that $v_0 \notin P_V(z)$ for some $z \in G(v_0, x, -)$. Then $d(z, v_1) \leq d(z, v_0)$ where $v_1 \in P_V(z)$ i.e. $v_1 \in B(z, d(z, v_0))$ for some $z \in G(v_0, x, -)$. i.e $v_1 \in K(v_0, x)$. Also $v_1 \in V$ and therefore $K(v_0, x) \cap V \neq \emptyset$, a contradiction. Hence V is a sun.

Note In normed linear spaces, Theorem 2 was proved by Amir and Deutsch [1] (see also [6], p. 467).

Lemma 3 *$K(v_0, x) = K(v_0, y)$ for all $y \in G[v_0, x]$, where $x \in X$, $V \subset X$ and $v_0 \in P_V(x)$.*

Proof $K(v_0, x) \equiv \bigcup B(z_1, d(z_1, v_0)), z_1 \in G_1(v_0, x, -)$, $K(v_0, y) \equiv \bigcup B(z_2, d(z_2, v_0)), z_2 \in G_1(v_0, y, -)$.

Let $z \in K(v_0, x)$ then $z \in B(z_1, d(z_1, v_0))$ for at least one $z_1 \in G_1(v_0, x, -)$. Now any $z_1 \in G_1(v_0, x, -)$ is also a point on $G_1(v_0, y, -)$ i.e. $z_1 = z_2$ for some $z_2 \in G_1(v_0, y, -)$ i.e. $z \in \bigcup B(z_2, d(z_2, v_0)), z_2 \in G_1(v_0, y, -)$. Therefore,

$$K(v_0, x) \subseteq K(v_0, y) \tag{1}$$

Let $z \in K(v_0, y)$ i.e $z \in B(z_2, d(z_2, v_0)$ for at least one $z_2 \in G_1(v_0, y, -)$. If $z_2 \in G_1(v_0, x, -)$ then $z \in K(v_0, x)$ and so $K(v_0, y) \subseteq K(v_0, x)$. If $z_2 \in G[y, x]$, consider $z\prime \in G_1(v_0, x, -)$. Then

$$\begin{aligned} d(z, z\prime) &\leq d(z, z_2) + d(z_2, z\prime) \\ &\leq d(z_2, v_0) + d(z_2, z\prime) \\ &= d(z\prime, v_0). \end{aligned}$$

Therefore $z \in B(z\prime, d(z\prime, v_0))$ and so $z \in K(v_0, x)$. Consequently

$$K(v_0, y) \subseteq K(v_0, x). \tag{2}$$

(1) and (2) imply $K(v_0, x) = K(v_0, y)$.

The following theorem shows that we may assume in the definition of lunar point that x has v_0 as a best approximation from V.

Theorem 4 *Let V be a subset of an M-space (X, d) and $v_0 \in V$. Then the following are equivalent:*
(i) v_0 is a lunar point
(ii) whenever v_0 is a best approximation to x with $K(v_0, x) \cap V \neq \emptyset$ then $v_0 \in \overline{K(v_0, x) \cap V}$.

Proof $\overline{(i) \Rightarrow (ii)}$ is trivial. $(ii) \Rightarrow (i)$. Let $x \in X$ and $K(v_0, x) \cap \overline{V \neq \emptyset}$. To show $v_0 \in \overline{K(v_0, x) \cap V}$. If v_0 is a best approximation to x then by (ii), $v_0 \in \overline{K(v_0, x) \cap V}$. If v_0 is not a best approximation to x then two cases arise:
(a) v_0 is not a local best approximation to x,
(b) v_0 is a local best approximation to x.
Case (a): If v_0 is not a local best approximation to x i.e. for all $\epsilon \geq 0$ there exists $v_\epsilon \in V$ such that $d(v_\epsilon, v_0) \leq \epsilon$ and $d(v_\epsilon, x) \leq d(v_0, x)$. Then $v_\epsilon \in B(x, d(v_0, x)) \subset K(v_0, x)$. Therefore every neighbourhood of v_0 contains an $\overline{\text{element } v_\epsilon \text{ of}}$ $K(v_0, x) \cap V$ other than v_0 i.e. v_0 is a limit point of $K(v_0, x) \cap V$ and so $v_0 \in \overline{K(v_0, x) \cap V}$. Hence v_0 is a lunar point.
Case (b): If v_0 is a local best approximation to x i.e. v_0 is a best approximation to x from $V \cap B(v_0, \epsilon)$ for some $\epsilon \geq 0$. Let $z \in G[v_0, x]$ such that $d(z, v_0) \leq \frac{\epsilon}{2}$ then by Lemma 3 $K(\overline{v_0, z) = K(}v_0, \overline{x) \text{ and } v_0 \text{ is}}$ a best approximation to z from V [7]. So (ii) implies $v_0 \in \overline{K(v_0, z) \cap V} = \overline{K(v_0, x) \cap V}$ and therefore v_0 is a lunar point.

Remark For normed linear spaces, above theorem was proved by Amir and Deutsch [1]-Lemma 2.7.

Corollary 5 *Every sun in an M-space is a moon.*

Proof Let V be a sun. Suppose V is not a moon i.e. there exists $v_0 \in V$ which is not a lunar point i.e. v_0 is a best approximation to $x \in X$ with $K(v_0, x) \cap V \neq \emptyset$ but $v_0 \notin \overline{K(v_0, x) \cap V}$.

As V is a sun, Theorem 2 implies $K(v_0, x) \cap V = \emptyset$ whenever v_0 is a best approximation to $x \in X$.

Since these two statements are contradictory, the result follows.

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