FIXED POINTS OF UNIFORMLY LIPSCHITZIAN MAPPINGS IN METRIC TREES

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Abstract. In this paper we examine the basic structure of metric trees and prove fixed point theorems for uniformly Lipschitzian mappings in metric trees.

1 Introduction The study of injective envelopes of metric spaces, also known as metric trees ($\mathbb{R}$-trees or T-theory), has its motivation in many subdisciplines of mathematics as well as biology/medicine and computer science. Its relationship with biology and medicine stems from the construction of phylogenetic trees [25]. In computer science, concepts of "string matching" are closely related with the structure of metric trees [3]. In the definition of an ordinary tree all the edges are assumed to have the same length, and therefore the metric is not often stressed. A metric tree is a generalization of an ordinary tree which allows for different edge lengths, thereby emphasizing the behavior of free actions on metric trees. (for more details see [23],[24]). Metric trees were first introduced by J. Tits [26] in 1977. A metric tree is a metric space (M,d) such that for every x,y in M, there is a unique arc between x and y and this arc is isometric to an interval in $\mathbb{R}$. For example, a connected graph without loop is a metric tree. One basic property of metric trees is their one-dimensionality. Also in the late seventies, while studying t-RNA molecules of the *E. Coli* bacterium M. Eigen raised several questions which led A. Dress [8],[9] to construct metric trees (named as T -theory). Metric trees also arise naturally in the study of group isometries of hyperbolic spaces. For metric properties of trees we refer to [7] and the excellent book [5]. Topological characterization of metric trees were explored in [20] and [21] where it was proved that for a separable metric space (M,d) the following are equivalent:

1. M admits an equivalent metric ρ such that (M,ρ) is an metric tree.
2. M is locally arcwise connected and uniquely arcwise connected.

There is a close connection between hyperconvex spaces and metric trees. In [1], authors take A. Dress' definition of a metric tree and show that every complete metric tree is hyperconvex. On the other hand, in [16], by using J. Tits' definition of $\mathbb{R}$-tree, it is shown that a metric space M is a complete $\mathbb{R}$-tree if and only if M is hyperconvex with unique metric segments. For more about hyperconvex spaces and fixed point theorems in hyperconvex spaces we refer to [15] and [6]. Also in [13] Lipschitz quotients from metric trees and in [22], extension of Lipschitzian mappings on metric trees were considered. In the following, after proving basic properties of metric intervals, we will study convex subsets of metric trees and show that the collection of all convex subsets of a metric tree is uniformly normal. We will also prove under suitable conditions two fixed point theorems for uniformly Lipschizian mappings in metric trees. It should be mentioned that the maps, even though they are not intrinsically continuous, they are asymptotically continuous. It is unknown to us whether such theorems currently exist.

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2 Basic Properties and Results

Since a metric tree (M, d) is a space in which there is only one path between two points x and y, this would imply that if z is a point between x and y, by which we mean if $d(x,z)+d(z,y)=d(x,y)$ then we know that z is actually on the path between x and y. This will motivate the next concept of a metric interval.

Definition 2.1 Let (M, d) be a metric space and let $x, y \in M$. An *arc* from x to y is the image of a topological embedding $\alpha : [a, b] \to M$ of a closed interval $[a, b]$ of $\mathbb{R}$ such that $\alpha(a) = x$ and $\alpha(b) = y$. A geodesic segment from x to y is the image of an isometric embedding $\alpha : [a, b] \to M$ such that $\alpha(a) = x$ and $\alpha(b) = y$. The geodesic segment will be called metric segment and denoted by $[x, y]$ throughout this work.

Now we are ready to define a metric tree.

Definition 2.2 A *metric tree* is a nonempty metric space M satisfying:

(a) Any two points of $x, y \in M$, x and y are the endpoints of a metric segment $[x, y]$.
(b) If $x, y, z \in M$ then $[x, y] \cap [x, z] = [x, w]$ for some $w \in M$ (i.e., if we have two metric segments with a common endpoint, then their intersection is a metric segment.)
(c) If $x, y, z \in M$ and $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$ (i.e., if two metric segments intersect in a single point, then their union is a metric segment.)

Next we give some basic properties of metric segments.

Lemma 2.1 *Let (M, d) be a metric space and $x, y \in M$, with $x \neq y$.*

1. If $z \in [x, y]$, then $[x, z] \subset [x, y]$.

2. If M is a metric tree, then for any $z \in M$, there is a unique $w \in [x, y]$ such that

$$[x, z] \cap [y, z] = [w, z].$$

The proof is classical and may be found in [4] on page 33.

Definition 2.3 Let M be a metric tree and $C \subset M$. We say C is *convex*, if for all $x, y \in C$ we have $[x, y] \subset C$.

Clearly, a metric tree M and the $\emptyset$ are convex. Also any closed ball $B(a, r) = \{z \in M : d(a, z) \leq r\}$ in a metric tree is also convex. To see this, take two arbitrary elements x, y of $B(a, r)$ and let $z \in [x, y]$. From the above lemma, there exists a unique $w \in [x, y]$ such that

$$[x, a] \cap [y, a] = [w, a].$$

Since $[x, y] = [x, w] \cup [w, y]$, we have $z \in [x, w]$ or $z \in [w, y]$. Without loss of generality, assume $z \in [x, w]$, then

$$d(a, z) \leq d(a, w) + d(w, z) \leq d(a, w) + d(w, z) + d(x, z) = d(a, w) + d(x, w) = d(a, x) \leq r\ ,$$

which implies $z \in B(a, r)$.

Throughout this paper we will make abundant use of the following property which is closely related to uniform convexity in Banach spaces.

Theorem 2.1 *Let M be a metric tree. Let $x, y \in M$ and let z be the middle-point of $[x, y]$. For $a \in M$, we have*

$$d(a,z) \leq \max\Big(d(a,x); d(a,y)\Big) - \frac{d(x,y)}{2}\ .$$

We will refer to this conclusion as property (UC).

Proof: Let $x, y, a, z \in M$ as in the theorem 2.3. Since M is a metric tree, there exists $w \in [x, y]$ such that

$$[a,x] \cap [a,y] = [a,w]\ .$$

Since $z \in [x, w] \cup [w, y]$, then without loss of generality we may assume $z \in [x, w]$. So

$$d(a,z) + d(z,x) = d(a,x) \leq \max\Big(d(a,x); d(a,y)\Big)\ .$$

Hence

$$d(a,z) \leq \max\Big(d(a,x); d(a,y)\Big) - d(z,x) = \max\Big(d(a,x); d(a,y)\Big) - \frac{d(x,y)}{2}$$

which completes the proof of our theorem.

The following notations will be needed throughout this paper. Let (M, d) be a metric space and let A be a nonempty bounded subset of M. Set

$$co(A) = \cap\{B : \text{ B is a closed ball and } A \subset B\}.$$

The subset A is called *admissible* if $co(A) = A$, (i.e., A is an intersection of closed balls.) Let $\mathcal{A}(M)$ denotes the collection of admissible subsets in M. $\mathcal{A}(M)$ is said to be *uniformly normal* if for each $C \in \mathcal{A}(M)$ for which $diam(C) > 0$ there exists $\alpha < 1$ such that $R(C) \leq \alpha\, diam(C)$, where

$$\begin{cases} r(x) & = & \sup\{d(x,c); c \in C\} \\ R(C) & = & \inf\{r(x); x \in C\} \\ diam(C) & = & \sup\{d(x,y) : x, y \in C\}. \end{cases}$$

Letting $\mathcal{C}(M)$ denote the collection of all closed *convex* subsets of the metric tree M, we set:

$$conv(A) = \cap\{B : \ B \text{ is a convex subset of } M \text{ such that } \ A \subseteq B\}.$$

Since closed balls are convex, we have $\mathcal{A}(M) \subset \mathcal{C}(M)$. Moreover, this is a proper inclusion. To see this, we need to make a couple of observations. First, if one has $x \neq y$ with $[x, y] \subset B(a, r)$ and m is the midpoint of $[x, y]$, then using the (UC) property we get

$$B\left(m, \frac{d(x,y)}{2}\right) \subset B(a,r),$$

which implies $B\left(m, \frac{d(x,y)}{2}\right) \subset co([x, y])$. Moreover closed unit balls in a metric tree may not be compact. Indeed, take closed unit ball in $\mathbb{R}^2$ with radial metric. Looking at $x, y \in B(0, 1)$ with $x \neq y$, then the radial distance between x and y is 2. Also any metric segment in $B(0, 1)$ is contained in $co([x, y])$, which implies that $co([x, y])$ is not compact for this distance. But $[x, y]$ is compact and convex. This example was suggested to us by Kirk. [17]

In order to prove our first fact about $\mathcal{C}(M)$, we need the following result of Baillon [2].

Theorem 2.2 [2] *Let M be a bounded metric space and let $\{H_\beta\}_{\beta\in\Gamma}$ be a decreasing family of nonempty hyperconvex subsets of of M. Then $\bigcap_{\beta\in\Gamma} H_\beta \neq \emptyset$ and is hyperconvex.*

Since convex subsets of a metric tree are hyperconvex [1] and [16] and the intersection of convex subsets is also convex, we get the following amazing fact.

Theorem 2.3 *Let M be a bounded metric tree. Then $\mathcal{C}(M)$ is compact, (i.e., for any family $\{C_\beta\}_{\beta\in\Gamma}$ in $\mathcal{C}(M)$ such that $\bigcap_{\beta\in\Gamma_f} C_\beta \neq \emptyset$, where Γ_f is any finite susbet of Γ, we have*

$$\bigcap_{\beta\in\Gamma} C_\beta \neq \emptyset$$

and is in $\mathcal{C}(M)$.) Moreover $\mathcal{C}(M)$ is uniformly normal.

The following remark will help shed more light on the properties of the convex sets and metric segments.

Remark 2.1 In [14] (see also [10]) a natural isometric embedding of any metric space M into the Banach space $l_\infty(M)$ is given. So if M is a metric tree, it is also hyperconvex. Then, there exists a nonexpansive retract $R : l_\infty(M) \to M$. For any $x, y \in M$, we write

$$tx \oplus (1-t)y = R(tx + (1-t)y)$$

for any $t \in [0,1]$. Here we are using the linear convexity of $l_\infty(M)$. It is not hard to check that $tx \oplus (1-t)y \in [x,y]$ and that for any $x, y, z, w \in M$, we have

$$d(z, tx \oplus (1-t)y) \leq td(z,x) + (1-t)d(z,y)$$

and

$$d(tx \oplus (1-t)y, tz \oplus (1-t)w) \leq td(x,z) + (1-t)d(y,w)$$

for any $t \in [0,1]$. Recall that R is nonexpansive if

$$d(R(x), R(y)) \leq d(x,y)$$

for any x, y.

3 Main Results Up till now the metric fixed point theory in metric trees is based on what is currently known about hyperconvex spaces. But because metric trees enjoy the property (UC) which fails in the classical hyperconvex spaces, metric trees inherit some known results in the uniformly convex Banach spaces. In fact, in this paper, we give the proofs in a larger class of maps. It is worth to mention the recent works [11] and [18] where the authors deal with similar questions.

Definition 3.1 Let M be a metric tree. A mapping $T : C \to C$ of a subset C of M is said to be *Lipschitzian* if there exists a non-negative number k such that $d(Tx, Ty) \leq kd(x,y)$ for all x and y in C. The smallest such k is called *Lipschitz constant* and will be denoted by $Lip(T)$. Same mapping is called *uniformly Lipschitzian* (respectively *eventually uniformly Lipschitzian*) if $\sup_{n\geq 1} Lip(T^n) < \infty$ (respectively $\sup_{n\geq n_0} Lip(T^n) < \infty$ for some $n_0 \geq 1$).

Note that eventually uniformly Lipschitzian mappings need not be continuous. If $Lip(T) \leq 1$, then T is called nonexpansive and eventually nonexpansive if $\sup\limits_{n \geq n_0} Lip(T^n) \leq 1$ for some $n_0 \geq 1$.

It is well-known fact that if a map is uniformly Lipschitzian, then one may find an equivalent distance for which the map is nonexpanive. Indeed, let $T : C \to C$ be uniformly Lipschitzian. Setting

$$\rho(x, y) = sup\{d(T^n x, T^n y) : n = 0, 1, 2...\}$$

for $x, y \in C$, one can obtain a metric ρ on C which is equivalent to the metric d and relative to which T is nonexpansive. In this context, it is natural to ask the question: if a set C has the fixed point property (fpp) for nonexpansive mappings with respect to the metric d, then does C also have (fpp) for mappings which are nonexpansive relative to an equivalent metric? This is known as the stability of (fpp). The first result in this direction is due to Goebel and Kirk [12]. Motivated by such questions,the following fixed point theorems of uniformly Lipschitzian mappings in metric trees are given:

Theorem 3.1 *Let M be a metric tree and K be a nonempty, closed, convex, and bounded subset of M with $diam(K) > 0$. Let $T : K \to K$ be eventually uniformly Lipschitzian such that*

$$\sigma(T) = \limsup_{n \to \infty} Lip(T^n) < \frac{3}{2} \, .$$

Then T has a fixed point.

Proof: Let $k > 0$ such that $\sigma(T) < k < 3/2$. By the definition of $\sigma(T)$, there exists $n_0 \geq 1$ such that $Lip(T^n) \leq k$ for $n \geq n_0$. Next let $x \in K$ and set

$$d(x) = \limsup_{n \to \infty} d(x, T^n(x))$$

and

$$r(x) = \inf \left\{ \rho > 0 : \exists \; n \geq 1 \;\text{ such that } K \cap \Big(\bigcap_{i \geq n} B(T^i x, \rho) \Big) \neq \emptyset \right\}.$$

Observe that since the diameter of the set K is finite, then $r(x) \leq \text{diam}(K)$ is finite. Next, for each $\varepsilon > 0$ we define

$$C_\varepsilon(x) = \bigcup_{n \geq 1} \Big(\bigcap_{i \geq n} B(T^i, r(x) + \varepsilon) \Big).$$

Then for each $\varepsilon > 0$ the set $C_\varepsilon(x)$ is nonempty ($C_\varepsilon(x) \cap K \neq \emptyset$) and convex. Compactness of $\mathcal{C}(M)$ implies that

$$C(x) = \bigcap_{\varepsilon > 0} \overline{C_\varepsilon} \cap K \neq \emptyset.$$

Let $z \in C(x)$, then z and $r(x)$ have the properties:

1. For any $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for any $n \geq n_0$ we have $T^n(x) \in B(z, r(x) + \varepsilon)$.
2. For any $z \in K$ and $0 < r < r(x)$, the set $\{i : d(T^i(x), z) > r\}$ is infinite.

Observe that if $r(x) = 0$ or if $d(z) = 0$, then $\lim\limits_{n \to \infty} T^n(x) = z$. Let us prove that in this case we have $Tz = z$. Indeed, let $n_0 \geq 1$ such that T^n is Lipschitzian for any $n \geq n_0$. In particular, T^n will be continuous (for $n \geq n_0$). So for $N \geq 1$, we have $\lim\limits_{n\to\infty} T^{n+N}(x) = T^N(z)$. But

$$\lim_{n\to\infty} T^{n+N}(x) = \lim_{n\to\infty} T^n(x) = z.$$

So for $N \geq n_0$, we have $T^N(z) = z$. This clearly implies $T^{N+1}(z) = z$ as well. Combining the two, we get $T(z) = z$. Of course, if we knew that T was continuous the proof would have been easier. Assume now that $r(x) > 0$ and $d(z) > 0$. Let $\varepsilon > 0, \varepsilon \leq d(z)$, and select j big enough so that $d(z, T^j z) \geq d(z) - \varepsilon$. By property (1.) above, there exists $n_0 \geq 1$ such that if $i \geq n_0$, then

$$d(z, T^i x) \leq r(x) + \varepsilon \leq k(r(x) + \varepsilon)$$

where n_0 is chosen so that $Lip(T^n) \leq k$ for $n \geq n_0$. Thus if $i - j \geq n_0$, we have

$$d(T^j(x), T^i(x)) \leq k\; d(z, T^{i-j}(x)) \leq k\, \Big(r(x) + \varepsilon\Big).$$

Considering the midpoint m of the interval $[z, T^j(z)]$, and using the property (UC), we have

$$d(m, T^i(x)) + \frac{1}{2} d(z, T^j(z)) \leq k\, \Big(r(x) + \varepsilon\Big),$$

equivalently,

$$d(m, T^i(x)) \leq k\, \Big(r(x) + \varepsilon\Big) - \frac{1}{2} d(z, T^j(z)) \leq k\, \Big(r(x) + \varepsilon\Big) - \frac{1}{2}\Big(d(z) - \varepsilon\Big).$$

Therefore, we have

$$r(x) \leq k\, \Big(r(x) + \varepsilon\Big) - \frac{1}{2}\Big(d(z) - \varepsilon\Big)$$

from the definition of $r(x)$. Since ε was arbitrary we get

$$r(x) \leq k\; r(x) - \frac{1}{2} d(z)$$

or equivalently

$$d(z) \leq 2(k-1) r(x).$$

Set $a = 2(k-1) < 1$, then

$$d(z) \leq a\; r(x) \leq a\; d(x)$$

and

$$d(z, x) \leq d(x) + r(x) \leq 2\; d(x).$$

To complete the proof, fix $x \in K$ and construct by induction a sequence $\{x_n\}$, with $x_0 = x$, such that

$$d(x_{n+1}) \leq a\; d(x_n) \;\text{ and }\; d(x_{n+1}, x_n) \leq 2\; d(x_n)$$

for $n = 1, ...,$. If we have $d(x_n) = 0$ for some n, then the above argument yields a fixed point of T. Otherwise, we have

$$d(x_{n+1}, x_n) \leq 2\; d(x_n) \leq 2\; a^n\; d(x)$$

which implies that the sequence $\{x_n\}$ is Cauchy, therefore $\lim\limits_{n\to\infty} x_n = z \in K$ exists. Also

$$d(z, T^i(z)) \leq d(z, x_n) + d(x_n, T^i(x_n)) + d(T^i(x_n), T^i(z))$$

for any $i \geq 1$. If we chose i large enough to assume that $Lip(T^i) \leq k$, then

$$d(z, T^i(z)) \leq (k+1)d(z, x_n) + d(x_n, T^i(x_n)).$$

This implies

$$d(z) \leq (k+1)d(z, x_n) + d(x_n),$$

hence $d(z) = 0$ which implies $Tz = z$.

In order to improve the constant $\frac{3}{2}$, one needs to use Lifschitz's [19] ideas.

Theorem 3.2 *Let M be a metric tree. Then for any $4/3 < b < 2$, there exists $a > 1$ such that for any $x, y \in M$ and $r > 0$ with $d(x, y) > r$ it implies that there exists $z \in M$ such that*

$$B(x, br) \cap B(y, ar) \subset B(z, r).$$

Proof: Let $4/3 \leq b < 2$. Choose $c > 0$ such that

$$\max\left\{\frac{1}{b}, 2 - \frac{2}{b}, \frac{4}{b} - 2\right\} < c < 1.$$

Then choose $a > 0$ such that

$$1 < a \leq 2 - \frac{bc}{2}.$$

It is easy to check that $a < b$. Let $x, y \in M$ such that $d(x, y) > r$. Let $w \in B(x, br) \cap B(y, ar)$. We have three cases.

Case 1. If $d(x, y) \geq br$, let z be the midpoint of $[x, y]$. Then, we have

$$d(z, w) + \frac{d(x, y)}{2} \leq br$$

which implies

$$d(z, w) \leq br - \frac{br}{2} = \frac{br}{2} < r$$

(i.e., $w \in B(z, r)$.)

Case 2. If $ar \leq d(x, y) < br$, let $z \in [x, y]$ such that

$$d(x, z) = \frac{br}{2}\ .$$

Since M is a metric tree, there exists $w_{xy} \in [x, y]$ such that

$$[w, x] \cap [w, y] = [w, w_{xy}].$$

So either $z \in [x, w_{xy}]$ or $z \in [w_{xy}, y]$. Assume $z \in [x, w_{xy}]$. Then

$$d(z, w) = d(x, w) - d(x, z) = d(x, w) - \frac{br}{2} \leq br - \frac{br}{2} < r.$$

Otherwise, assume $z \in [w_{xy}, y]$. Then

$$d(z,w) = d(y,w) - d(y,z) = d(y,w) - \left(d(x,y) - \frac{br}{2}\right) \leq ar - d(x,y) + \frac{br}{2} \leq \frac{br}{2} < r$$

since $ar - d(x,y) \leq 0$.

Case 3. If $d(x,y) < ar$, let $z \in [x,y]$ such that

$$d(x,z) = \frac{bcr}{2} .$$

Let w_{xy} be as in the previous case. Then again either $z \in [x, w_{xy}]$ or $z \in [w_{xy}, y]$. Assume $z \in [x, w_{xy}]$. Then

$$d(z,w) = d(x,w) - d(x,z) = d(x,w) - \frac{bcr}{2} \leq br - \frac{bcr}{2} = br\left(1 - \frac{c}{2}\right) < r$$

since $b - \dfrac{bc}{2} < 1$. Otherwise, assume $z \in [w_{xy}, y]$. Then

$$d(z,w) = d(y,w) - d(y,z) = d(y,w) - \left(d(x,y) - \frac{bcr}{2}\right) \leq ar - d(x,y) + \frac{bcr}{2} \leq ar - r + \frac{bcr}{2} \leq r$$

since $a - 1 + \dfrac{bc}{2} \leq 1$.

This completes the proof of our theorem.

Using this theorem, we are ready to state the main result of this paper.

Theorem 3.3 *Let M be a metric tree and C be a nonempty, closed, convex subset of M with diam$(C) > 0$. Let $T : C \to C$ be eventually uniformly Lipschitzian such that*

$$\sigma(T) = \limsup_{n \to \infty} Lip(T^n) < 2 .$$

Then T has a fixed point provided that T has bounded orbits.

Proof: Let $k > 0$ such that $\sigma(T) < k < 2$. By definition of $\sigma(T)$, there exists $n_0 \geq 1$ such that $Lip(T^n) \leq k$ for any $n \geq n_0$. Let $y \in C$ and set

$$R(y) = \inf\Big\{d > 0 : \ \exists\, x \in C \ \text{ such that for any } n \geq 1 \ \ d(T^n(x), y) \leq d\Big\}.$$

Since the orbit of y is bounded, we get $R(y) < \infty$. Assume that $R(y) = 0$. Then for all $\varepsilon > 0$, there exists $x_\varepsilon \in C$ such that $d(T^n(x_\varepsilon), y) < \varepsilon$ for any $n \geq 1$. If we choose $i \geq n_0$, then we get

$$d(T^{n+i}(x_\varepsilon), T^i(y)) < k\, \varepsilon$$

which implies $d(y, T^i(y)) < \varepsilon\,(1+k)$, for any $i \geq n_0$. Since ε was arbitrary, we get $T^i(y) = y$, for any $i \geq n_0$. So $T^{n_0}(y) = T^{n_0+1}(y) = y$ which obviously implies $T(y) = y$. Now assume that $R(y) > 0$. Since $k < 2$, let $k < b < 2$ and $a > 1$ such that $\forall x, y \in C$, $\forall r > 0$ with $d(x,y) > r$, there exists $z \in [x,y]$ such that

$$\begin{cases} d(x,w) & \leq b\ r \\ d(y,w) & \leq a\ r \end{cases} \Longrightarrow d(w,z) \leq r.$$

Letting $\lambda < 1$ be such that $\gamma = \min\{a\lambda, b\lambda/2\}$, we will construct by induction a sequence $\{y_n\} \in C$ such that

$$R(y_{n+1}) \leq \lambda R(y_n) \quad \text{and} \quad d(y_n, y_{n+1}) \leq (\lambda+\gamma) R(y_n).$$

Let $y_1 \in C$ and assume $y_1, ...y_n$ are known. Again if $R(y_n) = 0$ we are done. Assuming that $R(y_n) > 0$, then there exists $j \geq n_0$ such that $\lambda R(y_n) \leq d(T^j(y_n), y_n)$ and $x \in C$ with $d(T^m(x), y_n) \leq \gamma R(y_n)$ for all $m \geq 1$. Let $x^* = T^j(x)$. Then for $i \geq 1$ we have

$$T^i(x^*) = T^{i+j}(x) \in B(y_n, \gamma R(y_n)) \subset B(y_n, a\gamma R(y_n))$$

which implies

$$d(T^i(x^*), T^j(y_n)) = d(T^{i+j}(x), T^j(y_n)) \leq k\ d(T^i x, y_n) \leq k\ \gamma\ R(y_n) \leq b\ \lambda\ r(y_n),$$

hence

$$T^i(x^*) \in B(y_n, a\ \lambda\ R(y_n)) \cap B(T^j(y_n), b\ \lambda\ R(y_n)) = D.$$

Since $b < 2$, there exists $w \in [y_n, T^j(y_n)] \subset C$ such that $D \subset B(w, \lambda\ R(y_n))$ yielding $T^i(x^*) \in B(w, \lambda\ R(y_n))$ for all $i \geq 1$. Thus, $R(w) \leq \lambda R(y_n)$. Set $y_{n+1} = w$, then $R(y_{n+1}) \leq \lambda R(y_n)$ and

$$d(y_{n+1}, y_n) \leq d(y_{n+1}, T^i(x^*)) + d(T^i(x^*), y_n) \leq \lambda R(y_n) + \gamma R(y_n) \leq (\lambda+\gamma) R(y_n)\ .$$

By induction, the sequence $\{y_n\}$ is built. It is easy to see that $\{y_n\}$ is a Cauchy sequence. Therefore $\lim\limits_{n\to\infty} y_n = z \in C$ exists. Let $\varepsilon > 0$,so that there exists $n_1 \geq n_0$ such that for all $n \geq n_1$, $d(z, y_n) < \varepsilon$. By fixing $n \geq n_1$, there exists $x \in C$ such that $R(y_n)+\varepsilon \geq d(y_n, T^i(x))$ and hence

$$d(T^i(x), z) \leq d(z, y_n) + R(y_n) + \varepsilon\ .$$

Thus $R(z) \leq d(z, y_n) + R(y_n) + \varepsilon$ yielding $R(z) = 0$. This will imply $T(z) = z$.

As a direct consequence of the above result, we have the following stability result.

Theorem 3.4 *Let (M,d) be a metric tree. Let d^* be an equivalent distance such that $d \leq d^* \leq b\ d$, with $b < 2$. Then (M,d^*) has the fixed point property for eventually nonexpanisve mappings, (i.e., for any nonempty closed convex subset C of M and any eventually nonexpansive map $T : C \to C$ with bounded orbits, T has a fixed point.)*

Proof: Let T be as in the statement of the theorem. Then for any $n \geq 1$, we have

$$d(T^n(x), T^n(y)) \leq d^*(T^n(x), T^n(y)) \leq d^*(x,y) \leq b\ d(x,y)\ .$$

So T is eventually Lipschitzian for the distance d with $\sigma(T) \leq b < 2$. The previous results assure us of the existence of a fixed point of T in C.

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