NORM INEQUALITIES FOR THE GEOMETRIC MEAN AND ITS REVERSE

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Received September 9, 2006; revised September 26, 2006

Abstract. If two positive operators A and B commute, then $A \sharp_\alpha B = A^{1-\alpha}B^\alpha$ for all $0 \le \alpha \le 1$. In this note, we prove a norm inequality for the geometric mean $A \sharp_\alpha B$ and its reverse inequality: Let A and B be positive operators on a Hilbert space such that $0 < m \le A, B \le M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then for each $0 \le \alpha \le 1$

$$K(h^2, \alpha)\|A^{1-\alpha}B^\alpha\| \le \|A \sharp_\alpha B\| \le \|A^{1-\alpha}B^\alpha\|,$$

where $K(h, \alpha)$ is a generalized Kantorovich constant.

1 Introduction. Let A and B be two positive operators on a Hilbert space. The arithmetic-geometric mean inequality says that

$$(1) \qquad (1-\alpha)A + \alpha B \ge A \sharp_\alpha B \qquad \text{for all } 0 \le \alpha \le 1,$$

where the α-geometric mean $A \sharp_\alpha B$ is defined as follows:

$$(2) \qquad A \sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} \qquad \text{for all } 0 \le \alpha \le 1.$$

On the other hand, Ando [1] proved the Matrix Young inequality: For positive semi-definite matrices A, B and $\frac{1}{p} + \frac{1}{q} = 1$

$$(3) \qquad \frac{1}{p}A^p + \frac{1}{q}B^q \ge U^*|AB|U$$

for some unitary matrix U. By (3), for positive semi-definite matrices A, B

$$(4) \qquad \|(1-\alpha)A + \alpha B\| \ge \|A^{1-\alpha}B^\alpha\| \qquad \text{for all } 0 \le \alpha \le 1$$

and by (1) we have

$$\|(1-\alpha)A + \alpha B\| \ge \|A \sharp_\alpha B\| \qquad \text{for } 0 \le \alpha \le 1 \text{ and } A, B \ge 0.$$

Here we remark that McIntosh [6] proved that (4) for $\alpha = 1/2$ holds for positive operators.

In this note, we prove a norm inequality and its reverse on the geometric mean. In other words, we estimate $\|A \sharp_\alpha B\|$ by $\|A^{1-\alpha}B^\alpha\|$ as mentioned in the abstract. Moreover we discuss it for the case $\alpha > 1$. Our main tools are Araki's inequality [2] and its reverse one [4].

2000 *Mathematics Subject Classification.* 47A63 and 47A64.

Key words and phrases. Positive operator, Araki's inequality, geometric mean, generalized Kantorovich constant, reverse inequality.

2 Norm inequalities. First of all, we cite Araki's inequality [2]:

Theorem A. *If A and B are positive operators, then*

$$\|BAB\|^p \le \|B^pA^pB^p\| \qquad \text{for all } p > 1 \tag{5}$$

or equivalently

$$\|B^pA^pB^p\| \le \|BAB\|^p \qquad \text{for all } 0 < p < 1. \tag{6}$$

As seen in [3], it is equivalent to the Cordes inequality

$$\|A^pB^p\| \le \|AB\|^p \qquad \text{for all } 0 < p < 1.$$

We show the following norm inequality, in which we use Theorem A twice:

Theorem 1. *Let A and B be positive operators. Then for each $0 \le \alpha \le 1$*

$$\|A \sharp_\alpha B\| \le \|A^{1-\alpha}B^\alpha\|. \tag{7}$$

Proof. It follows from (6) of Theorem A that

$$\|A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}\| \le \|A^{\frac{1}{2\alpha}}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{\frac{1}{2\alpha}}\|^\alpha = \|A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\|^\alpha$$

for $0 \le \alpha \le 1$.

Furtheremore, if $\alpha \ge 1/2$, then by (6) of Theorem A again

$$\|A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\|^\alpha \le \|A^{1-\alpha}B^{2\alpha}A^{1-\alpha}\|^{\frac{1}{2}} = \|A^{1-\alpha}B^\alpha\|.$$

Hence, if $1/2 \le \alpha \le 1$, then we have the desired inequality (7).

If $\alpha < 1/2$, then by using $A \sharp_\alpha B = B \sharp_{1-\alpha} A$, it reduces the proof to the case $\alpha \ge 1/2$ and so the proof is complete. □

We use also the notation $\natural$ to distingush from the operator mean $\sharp$;

$$A \natural_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} \qquad \text{for all } \alpha \notin [0,1]. \tag{8}$$

Theorem 2. *Let A and B be positive operators. If $3/2 \le \alpha \le 2$, then*

$$\|A \natural_\alpha B\| \le \|A^{1-\alpha}B^\alpha\|. \tag{9}$$

Proof. Put $\alpha = 1 + \beta$ and $1/2 \le \beta \le 1$. Then we have

$$\begin{aligned}
\|A \natural_\alpha B\| &= \|B \natural_{-\beta} A\| = \|B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-\beta} B^{\frac{1}{2}}\| \\
&= \|B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{\beta} B^{\frac{1}{2}}\| \\
&\le \|B^{\frac{1}{2\beta}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}B^{\frac{1}{2\beta}}\|^\beta \qquad \text{by (6) and } 1/2 \le \beta \le 1 \\
&= \|B^{\frac{1+\beta}{2\beta}}A^{-1}B^{\frac{1+\beta}{2\beta}}\|^\beta \\
&\le \|B^{1+\beta}A^{-2\beta}B^{1+\beta}\|^{\frac{1}{2}} \qquad \text{by (6) and } 0 < \tfrac{1}{2\beta} \le 1 \\
&= \|A^{-\beta}B^{1+\beta}\| = \|A^{1-\alpha}B^\alpha\|.
\end{aligned}$$

□

Remark 3. *In Theorem 2, the inequality* $\|A \natural_\alpha B\| \le \|A^{1-\alpha}B^\alpha\|$ *does not always hold for* $1 < \alpha < 3/2$. *In fact, Let* $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. *Then we have* $\|A \natural_{\frac{4}{3}} B\| = 3.38526 > \|A^{-\frac{1}{3}}B^{\frac{4}{3}}\| = 3.3759$. *Also,* $\|A \natural_{\frac{7}{5}} B\| = 3.49615 < \|A^{-\frac{2}{5}}B^{\frac{7}{5}}\| = 3.50464$.

3 Reverse type inequalities. In this section, we show reverse inequalities of the results obtained in the previous section. In order to prove our results, we need some preliminaries. For $h > 0$, a generalized Kantorovich constant $K(h,p)$ is defined by

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p \tag{10}$$

for any real numbers $p \in \mathbb{R}$. We state some properties of $K(h,p)$ (see [5, Theorem 2.54]):

Lemma 4. *Let* $h > 0$ *be given. Then a generalized Kantorovich constant* $K(h,p)$ *has the following properties.*
(i) $K(h,p) = K(h^{-1},p)$ *for all* $p \in \mathbb{R}$.
(ii) $K(h,p) = K(h,1-p)$ *for all* $p \in \mathbb{R}$.
(iii) $K(h,0) = K(h,1) = 1$ *and* $K(1,p) = 1$ *for all* $p \in \mathbb{R}$.
(iv) $K(h^r, \frac{p}{r})^{\frac{1}{p}} = K(h^p, \frac{r}{p})^{-\frac{1}{r}}$ *for* $pr \neq 0$.

The following theorem is reverse inequalities of Araki's inequality [4].

Theorem B. *If* A *and* B *are positive operators such that* $0 < m \le A \le M$ *for some scalars* $0 < m < M$, *then*

$$\|B^pA^pB^p\| \le K(h,p)\ \|BAB\|^p \qquad \text{for all } p > 1 \tag{11}$$

or equivalently

$$K(h,p)\ \|BAB\|^p \le \|B^pA^pB^p\| \qquad \text{for all } 0 < p < 1, \tag{12}$$

where a generalized Kantorovich constant $K(h,p)$ *is defined by (10) and* $h = \frac{M}{m}$ *is a generalized condition number of* A *in the sense of Turing [7].*

We show the following reverse inequality for Theorem 1:

Theorem 5. *If* A *and* B *are positive operators such that* $0 < m \le A, B \le M$ *for some scalars* $0 < m < M$ *and* $h = \frac{M}{m}$, *then for each* $0 \le \alpha \le 1$

$$K(h^2,\alpha)\|A^{1-\alpha}B^\alpha\| \le \|A \sharp_\alpha B\|. \tag{13}$$

Proof. Suppose that $0 \le \alpha \le \frac{1}{2}$. Since $\frac{m}{M} \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{M}{m}$, it follows that a generalized condition number of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is $\frac{M}{m}/\frac{m}{M} = h^2$ and we have

$$\begin{aligned}
\|A \sharp_\alpha B\| &= \|(A^{\frac{1}{2\alpha}})^\alpha \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha (A^{\frac{1}{2\alpha}})^\alpha\| \\
&\ge K(h^2,\alpha)\|A^{\frac{1}{2\alpha}}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{\frac{1}{2\alpha}}\|^\alpha \qquad \text{by (12) and } 0 \le \alpha \le \tfrac{1}{2} \\
&= K(h^2,\alpha)\|A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\|^\alpha \\
&\ge K(h^2,\alpha)\|A^{1-\alpha}B^{2\alpha}A^{1-\alpha}\|^{\frac{1}{2}} \qquad \text{by (5) and } \tfrac{1}{2\alpha} \ge 1 \\
&= K(h^2,\alpha)\|A^{1-\alpha}B^\alpha\|.
\end{aligned}$$

Suppose that $\frac{1}{2} \leq \alpha \leq 1$. Since $0 \leq 1-\alpha \leq \frac{1}{2}$, we have

$$\begin{aligned} \|A \sharp_\alpha B\| &= \|B \sharp_{1-\alpha} A\| \\ &\geq K(h^2, 1-\alpha)\|B^{1-(1-\alpha)}A^{1-\alpha}\| \\ &= K(h^2,\alpha)\|A^{1-\alpha}B^\alpha\| \qquad \text{by (ii) of Lemma 4} \end{aligned}$$

and so the proof is complete. □

We show the following reverse inequality for Theorem 2:

Theorem 6. *If A and B are positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$, then for each $\frac{3}{2} \leq \alpha \leq 2$*

$$K(h^2, \alpha-1)K(h, 2(\alpha-1))^{-\frac{1}{2}}\|A^{1-\alpha}B^\alpha\| \leq \|A \natural_\alpha B\|.$$

Proof. Put $\alpha = 1+\beta$ and $1/2 \leq \beta \leq 1$. Then we have

$$\begin{aligned} &\|A \natural_\alpha B\| = \|B \natural_{-\beta} A\| \\ &= \|B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^\beta B^{\frac{1}{2}}\| \\ &\geq K(h^2,\beta)\|B^{\frac{1}{2\beta}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}B^{\frac{1}{2\beta}}\|^\beta \qquad \text{by (12) of Theorem B and } 1/2 \leq \beta \leq 1 \\ &= K(h^2,\beta)\|B^{\frac{1+\beta}{2\beta}}A^{-\frac{2\beta}{2\beta}}B^{\frac{1+\beta}{2\beta}}\|^\beta \\ &\geq K(h^2,\beta)\left(K(h^{-2\beta},\frac{1}{2\beta})\|B^{1+\beta}A^{-2\beta}B^{1+\beta}\|^{\frac{1}{2\beta}}\right)^\beta \qquad \text{by (12) and } 0 < \tfrac{1}{2\beta} \leq 1 \\ &= K(h^2,\beta)K(h^{-2\beta},\frac{1}{2\beta})^\beta\|A^{-\beta}B^{1+\beta}\| \\ &= K(h^2,\beta)K(h,2\beta)^{-\frac{1}{2}}\|A^{1-\alpha}B^\alpha\|. \end{aligned}$$

The last equality follows from

$$K(h^{-2\beta},\frac{1}{2\beta})^\beta = K(h^{2\beta},\frac{1}{2\beta})^\beta = K(h,2\beta)^{-\frac{\beta}{2\beta}} = K(h,2\beta)^{-\frac{1}{2}}$$

by (i) and (iv) of Lemma 4. □

As mentioned in Remark 3, we have no relation between $\|A \natural_\alpha B\|$ and $\|A^{1-\alpha}B^\alpha\|$ for $1 \leq \alpha \leq \frac{3}{2}$. We have the following result:

Theorem 7. *If A and B are positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$, then for each $1 \leq \alpha \leq \frac{3}{2}$*

$$K(h^2,\alpha-1)\|A^{1-\alpha}B^\alpha\| \leq \|A \natural_\alpha B\| \leq K(h, 2(\alpha-1))^{-\frac{1}{2}}\|A^{1-\alpha}B^\alpha\|.$$

Proof. Put $\alpha = 1+\beta$ and $0 \le \beta \le \frac{1}{2}$. Since a generalized condition number of $A^{-2\beta}$ is $h^{-2\beta}$, it follows that

$$\begin{aligned}
\|A \natural_\alpha B\| &= \|B \natural_{-\beta} A\| = \|B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-\beta}B^{\frac{1}{2}}\| \\
&= \|B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{\beta}B^{\frac{1}{2}}\| \\
&\le \|B^{\frac{1}{2\beta}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}B^{\frac{1}{2\beta}}\|^{\beta} \qquad \text{by (6) and } 0 \le \beta \le 1 \\
&= \|B^{\frac{1+\beta}{2\beta}}A^{-1}B^{\frac{1+\beta}{2\beta}}\|^{\beta} \\
&\le \left(K(h^{-2\beta},\frac{1}{2\beta})\|B^{1+\beta}A^{-2\beta}B^{1+\beta}\|^{\frac{1}{2\beta}}\right)^{\beta} \qquad \text{by (11) and } 1 \le \tfrac{1}{2\beta} \\
&= K(h,2(\alpha-1))^{-\frac{1}{2}}\|A^{1-\alpha}B^{\alpha}\| \qquad \text{by (i) and (iv) of Lemma 4.}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|A \natural_\alpha B\| &= \|B \natural_{-\beta} A\| = \|B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-\beta}B^{\frac{1}{2}}\| \\
&= \|B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{\beta}B^{\frac{1}{2}}\| \\
&\ge K(h^2,\beta)\|B^{\frac{1}{2\beta}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}B^{\frac{1}{2\beta}}\|^{\beta} \qquad \text{by (12) and } 0 \le \beta \le 1 \\
&= K(h^2,\beta)\|B^{\frac{1+\beta}{2\beta}}A^{-1}B^{\frac{1+\beta}{2\beta}}\|^{\beta} \\
&\ge K(h^2,\beta)\|B^{1+\beta}A^{-2\beta}B^{1+\beta}\|^{\frac{1}{2}} \qquad \text{by (5) and } \tfrac{1}{2\beta} \ge 1 \\
&= K(h^2,\alpha-1)\|A^{1-\alpha}B^{\alpha}\|
\end{aligned}$$

and so the proof is complete. □

Finally, we consider the case of $\alpha \ge 2$:

Theorem 8. *If A and B are positive operators such that $0 < m \le A, B \le M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$, then for each $\alpha \ge 2$*

$$K(h,2(\alpha-1))^{-\frac{1}{2}}\|A^{1-\alpha}B^{\alpha}\| \le \|A \natural_\alpha B\| \le K(h^2,\alpha-1)\|A^{1-\alpha}B^{\alpha}\|.$$

Proof. Put $\alpha = 1+\beta$ and $\beta \ge 1$. Then we have

$$\begin{aligned}
\|A \natural_\alpha B\| &= \|B \natural_{-\beta} A\| = \|B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-\beta}B^{\frac{1}{2}}\| \\
&= \|B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{\beta}B^{\frac{1}{2}}\| \\
&\le K(h^2,\beta)\|B^{\frac{1}{2\beta}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}B^{\frac{1}{2\beta}}\|^{\beta} \qquad \text{by (11) and } \beta \ge 1 \\
&= K(h^2,\beta)\|B^{\frac{1+\beta}{2\beta}}A^{-1}B^{\frac{1+\beta}{2\beta}}\|^{\beta} \\
&\le K(h^2,\alpha-1)\|A^{1-\alpha}B^{\alpha}\| \qquad \text{by (6) and } 0 < \tfrac{1}{2\beta} \le 1.
\end{aligned}$$

Also, it follows that

$$\begin{aligned}\|A \natural_\alpha B\| &= \|B \natural_{-\beta} A\| = \|B^{\frac{1}{2}} \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right)^{-\beta} B^{\frac{1}{2}}\| \\ &= \|B^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)^{\beta} B^{\frac{1}{2}}\| \\ &\geq \|B^{\frac{1}{2\beta}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} B^{\frac{1}{2\beta}}\|^{\beta} \qquad \text{by (5) and } \beta \geq 1 \\ &= \|B^{\frac{1+\beta}{2\beta}} A^{-1} B^{\frac{1+\beta}{2\beta}}\|^{\beta} \\ &\geq K(h^{-2\beta}, \frac{1}{2\beta})^{\beta} \|B^{1+\beta} A^{-2\beta} B^{1+\beta}\|^{\frac{1}{2}} \qquad \text{by (11) and } 0 < \tfrac{1}{2\beta} \leq 1 \\ &= K(h, 2(\alpha-1))^{-\frac{1}{2}} \|A^{1-\alpha} B^{\alpha}\| \qquad \text{by (i) and (iv) of Lemma 4.}\end{aligned}$$

□

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