Scientiae Mathematicae Japonicae Online, **e-2006**, 1261–1266

MARGINAL MEASURE PROBLEMS ON THE RANKED SPACE S′

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Received August 24, 2006

Abstract. Let S be the set of all rapidly decreasing C^∞-functions defined in R^k and S' its topological dual for the usual topology. S' is definable as a ranked space. The family of preneighbourhoods defines a topology for it. We denote the space equipped with the topology by S'_R. Our aim in this note is to study Strassen's marginal measure problems for S'_R. Our results are that if we read continuous functions and open sets as Borel measurable functions and Borel sets, respectively, in [8], Theorem 7 and Theorem 11, then the similalities of Strassen's results hold still.

§1. Introduction

Let $S,\ S'$ and S'_R be the same as in the above. S'_R satisfies the second axiom of countability and is metrizable for the topology (see [6], Paragraph II). Our aim in this note is to study Strassen's marginal measure problems (see [8], Theorem 7 and Theorem 11) for S'_R. S'_R is not a Polish space. Hence our results differ a little from Strassen's one, that is, we have to read continuous functions and open sets as Borel measurable functions and Borel sets, respectively, in [8], Theorem 7 and Theorem 11. However our conditions are equivalent to Strassen's one for Polish spaces.

To obtain our results the following facts play important roles;

(1) the Borel σ-algebra of S'_R coincides with it of the spaces S'_w equipped with the week topology $\sigma(S',\ S)$ and it of the space S'_r topologized by the family of all r-open sets of the ranked space S' (see [6], p. 810, Theorem 11),

(2) the projection of any Borel set of the product space $S'_R \times S'_R$ is universally measurable with respect to $(S'_R, \beta(S'_R))$.

For it the notion of r-convergence plays an important role.

§2. The ranked space S' and measurability

Let S and S' be the same as in §1 and $\mathbf{N}$ the set of all non-negative integers. For each m in $\mathbf{N}$, let $(\ ,\ )_m$ be an inner product on S defined by

$$(\phi, \psi)_m = \sum_q \int_{R^k} D^q\phi(x)\overline{D^q\psi(x)}dx\ (D^q = \frac{\partial^{q_1+\cdots+q_k}}{\partial x_1^{q_1}\cdots\partial x_k^{q_k}}),$$

where q runs through all multi-indices $q = (q_1, \cdots, q_k)$ with $0 \le q_i \le 3m$ for each $i = 1,\ 2, \cdots, k$. Putting

$$\|\phi\|_m = \sqrt{(\phi,\ \phi)_m},$$

we have

$$\|\phi\|_0 \le \|\phi\|_1 \le \cdots\ (\phi \in S).$$

2000 *Mathematics Subject Classification.* 46A19, 60B05.

Key words and phrases. Marginal measures, ranked spaces.

Put, for any f in S',

$$\|f\|_{-m} = \sup\{|f(\phi)| : \ \phi \in S \text{ and } \|\phi\|_m \leq 1\},$$

and

$$p(f) = \min\{m : \ \|f\|_{-m} < \ +\infty\}.$$

Let $S'_m = \{f \in S' : \ p(f) \leq m\}$ for each m in $\mathbf{N}$.

Proposition 2.1 ([6], p. 807). *The followings hold.*
(1) $S' = \cup_{m=0}^{\infty} S'_m$.
(2) $m \leq m'$ *iff* $S'_m \subset S'_{m'}$.
(3) *For each* m *in* $\mathbf{N}$, $\| \ \|_{-m}$ *is a norm on* S'_m *and each normed space* $(S'_m, \ \| \ \|_{-m})$ *is complete.*

For each f in S' put

$$V(f; \ m, \ j) = f + \{g \in S'_m : \ \|g\|_{-m} < \frac{1}{2^j}\} \ (j, \ m \in \mathbf{N}),$$

$$\mathcal{U}(f) = \{V(f; \ m, \ j); \ j = 0, \ 1, \ \cdots, \ \max(1, \ p(f)) \leq m, \ m \in \mathbf{N}\}$$

and

$$\mathcal{U}_j = \{V(f; \ m, \ j) : \ f \in S', \ \max(1, \ p(f)) \leq m, \ m \in \mathbf{N}\}.$$

Then $(S', \mathcal{U}(f), \mathcal{U}_j)$ becomes a ranked space (see [5], §2). Since the family $\mathcal{U}(f)$ of preneighbourhoods for each f in S' satisfies the axioms (B) and (C) of Hausdorff, $\mathcal{U}(f)$ becomes a fundamental system of neighbourhoods of f (see [5], §1, 1.6). $\mathcal{O}(S')$ denotes this topology. Furthermore the ranked space $(S', \ \mathcal{U}(f), \ \mathcal{U}_j)$ satisfies $(r - T_2)$ (see [6], P. 807, Theorem 7). Hence the topological space $(S', \ \mathcal{O}(S'))$ is a Hausdorff space. S. Nakanishi showed the following theorem in [6], §7 and §8.

Theorem 2.2. *The topological space* $(S', \ \mathcal{O}(S'))$ *satisfies the second axiom of countability and is metrizable.*

We denote the topological space $(S', \ \mathcal{O}(S'))$ and S' equipped with the weak topology $\sigma(S', \ S)$ by S'_R and S'_w, respectively. We, also, denote the topological space S' topologized by the family of all r-open sets of the ranked space $(S', \ \mathcal{U}(f), \ \mathcal{U}_j)$ by S'_r. $\beta(S'_R), \ \beta(S'_r)$ and $\beta(S'_w)$ denote the Borel σ-algebras of $S'_R, \ S'_r$ and S'_w, respectively.

Theorem 2.3 ([6], p. 810, Theorem 11). $\beta(S'_R) = \beta(S'_r) = \beta(S'_w)$ *hold.*

Proposition 2.4. *For each* m *in* $\mathbf{N}$ *a normed space* $(S'_m, \ \| \ \|_{-m})$ *is a separable Banach spaces.*

Proof. Since the topological space $(S', \ \mathcal{O}(S'))$ satisfies the second axiom of countability, the topological subspace $(S'_m, \ \mathcal{O}(S'_m))$ of $(S', \ \mathcal{O}(S'))$ satisfies the second axiom of countability. The topology of $(S'_m, \ \mathcal{O}(S'_m))$ is stronger than it of $(S'_m, \| \ \|_{-m})$. Therefore the normed space $(S'_m,\ \|\ \|_{-m})$ is separable. By Proposition 2.1 $(S'_m,\ \|\ \|_{-m})$ is a separable Banach space.

Theorem 2.5. *Let $\beta((S'_m,\ \|\ \|_{-m}))$ denote the Borel σ-algebra of the normed space $(S'_m,\ \|\ \|_{-m})$ for each m in $\mathbf{N}$. Then S'_m belongs to $\beta(S'_r)$ and $\beta(S'_r) \cap S'_m = \beta((S'_m,\ \|\ \|_{-m}))$ holds.*

Proof. For any positive integer n we have

$$\{f \in S'_m :\ \|f\|_{-m} \leq n\} = \{f \in S' :\ \|f\|_{-m} \leq n\}$$

By [6], p. 809, Lemma 18 the right hand is r-closed (see [4], p. 180 for the definition of r-closed sets). Hence S'_m belongs to $\beta(S'_r)$. Next we shall show the second assertion. Since the topology of the topological subspace $(S_m,\ \mathcal{O}(S_m))$ is stronger than it of the normed space $(S_m,\ \|\ \|_{-m})$, it is obvious that $\beta(S'_r) \cap S'_m \supset \beta((S'_m,\ \|\ \|_{-m}))$ holds. We shall show the converse inclusion. Let F be an r-closed subset of S'_r. Put, for each m in $\mathbf{N}$, $F_m = F \cap S'_m$. Then F_m is a closed subset of the normed space $(S'_m,\ \|\ \|_{-m})$. Indeed, let $\{f_n\}$ be a sequence of F_m converging to an f in S' with respect to the norm $\|\ \|_{-m}$. Since S'_m is a Banach space, f belong to S'_m. This convergence, by [6], p. 807, Proposition 17, implies that $\{f_n\}$ r-converges to the f in S'. Since F is r-closed, the f belongs to F. Since it has been proved that F_m is closed in $(S'_m,\ \|\ \|_{-m})$, F_m belongs to $\beta((S'_m,\ \|\ \|_{-m}))$. Accordingly it is easily seen that $\beta(S'_r) \cap S'_m = \beta((S'_m,\ \|\ \|_{-m}))$ holds.

Remark. *By Theorem 2.3 it is seen that, for each m in $\mathbf{N}$, S'_m belongs to $\beta(S'_R)$ and $\beta(S'_w)$, and $\beta(S'_R) \cap S'_m = \beta(S'_w) \cap S'_m = \beta((S'_m,\ \|\ \|_{-m}))$ hold.*

Definition 2.6 ([2], p. 280). *Let $(X,\ \beta)$ be an abstract measurable space. A subset of X is universally measurable with respect to $(X,\ \beta)$ iff it is μ-measurable for every finite measure μ on $(X,\ \beta)$.*

Theorem 2.7. ([3], p. 388). *Let X and Y be Polish spaces. If B is a Borel subset of X and f is a continuous map from B to Y, then $f(B)$ is an analytic set in Y.*

Theorem 2.8 ([2], p. 281). *Evry analytic subset of a Polish space is universally measurable.*

p_{r_1} and p_{r_2} denote the projections from the product space $S'_R \times S'_R$ onto the first component space and the second component space, respectively.

Theorem 2.9. *Let F be a Borel subset of $S'_R \times S'_R$. Then for any Borel subset B of S'_R $p_{r_1}(F \cap (S'_R \times B))$ is universally measurable with respect to $(S'_R,\ \beta(S'_R))$.*

Proof. Since S'_R is a metric space satisfying the second axiom of countability, we have

$$\beta(S'_R) \otimes \beta(S'_R) = \beta(S'_R \times S'_R),$$

where $\beta(S'_R) \otimes \beta(S'_R)$ denotes the product σ-algebra of two Borel σ-algebra $\beta(S'_R)$'s. Putting $B_m = S'_m \cap B$ for each m in $\mathbf{N}$, by Theorem 2.5 and Theorem 2.3 $F \cap (S'_m \times B_m)$ belongs to $\beta((S'_m,\ \|\ \|_{-m}) \times (S'_m,\ \|\ \|_{-m}))$. Put $E_m = p_{r_1}(F \cap (S'_m \times B_m))$ for every m in $\mathbf{N}$. Since $(S'_m,\ \|\ \|_{-m})$ is a separable Banach space, by Theorem 2.8 E_m is universally measurable in the normed space $(S'_m,\ \|\ \|_{-m})$. Let μ be a finite measure on S'_R. Since, by Remark, $\beta((S'_m,\ \|\ \|_{-m})) = \beta(S'_R) \cap S'_m$ and S'_m belong to $\beta(S'_R)$, we can restrict μ to the measurable space $(S'_m,\ \beta((S'_m,\ \|\ \|_m)))$. We denote it by μ_m. Since E_m is universally measurable in

$(S'_m, \| \|_m)$, there exist Borel sets A_m and C_m in $\beta((S'_m, \| \|_{-m}))$ such that $A_m \subset E_m \subset C_m$ and $\mu_m(A_m) = \mu_m(C_m)$. Since

$$F \cap (S'_R \times B) = F \cap (\cup_{m=0}^{\infty}(S'_m \times B_m)) = \cup_{m=0}^{\infty}(F \cap (S'_m \times B_m)),$$

we have

$$p_{r_1}(F \cap (S'_R \times B)) = \cup_{m=0}^{\infty} E_m.$$

Putting $A = \cup_{m=0}^{\infty} A_m$ and $C = \cup_{m=0}^{\infty} C_m$, A and B belong to $\beta(S'_R)$ and we have

$$A \subset p_{r_1}(F \cap (S'_R \times B)) \subset C \text{ and } \mu(C - A) \leq \sum_{m=0}^{\infty} \mu_m(C_m - A_m) = 0.$$

Hence $p_{r_1}(F \cap (S'_R \times B))$ is μ-measurable. Thus Theorem 2.9 has been proved.

§3. Strassen's Theorem

Let X be a topological space and let $\beta(X)$ denote the Borel σ-algebra of X. $B^b(X)$ (*resp.* $C^b(X)$) denotes the set of all bounded real valued Borel measurable (resp. continuous) functions on X. $P(X)$ denotes the set of all probability Borel measures on X. Put $Z = S'_R \times S'_R$. S'_w is a Lusin space (see [7], p. 115, (C)). Let $\mathcal{K}(S'_w)$ be the set of all compact subsets of S'_w and $\mathcal{K}_0(S'_w \times S'_w)$ the set of all finite disjoint unions of compact rectangles in $S'_w \times S'_w$. $\beta_0(Z)$ denotes the set of all finite disjoint unions of Borel measurable rectangles of Z. Put

$$B_0^b(Z) = \{f \circ p_{r_1} + g \circ p_{r_2} : \ f, \ g \in B^b(S'_R)\}.$$

Theorem 3.1. *Let μ and ν be probability Borel measures on S'_R and Λ a non-void weakly closed convex subset of $P(Z)$. Then the following conditions are mutually equivalent:*

(1) There exists a θ in Λ having μ and ν as marginals.

(2) For any functions f and g in $B^b(S'_R)$ one has

$$\int_{S'_R} f d\mu + \int_{S'_R} g d\nu \leq \sup\{\int_Z (f \circ p_{r_1} + g \circ p_{r_2}) d\theta : \ \theta \in \Lambda\}.$$

Proof. The implication (1) $\to$ (2) is easy. We shall prove the converse assertion. Put, for any f and g in $B^b(S'_R)$,

$$W_0(f \circ p_{r_1} + g \circ p_{r_2}) = \int_{S'_R} f d\mu + \int_{S'_R} g d\nu.$$

W_0 is a linear functional on $B_0^b(Z)$. Put, for any h in $B^b(Z)$,

$$P(h) = \sup\{\int_Z h d\theta : \ \theta \in \Lambda\}.$$

P is positively homogeneous and subadditive on $B^b(Z)$ and satisfies $W_0 \leq P$ on $B_0^b(Z)$. By the Hahn-Banach theorem W_0 is extended to a linear functional W on $B^b(Z)$ with $W \leq P$. Put, for any set E in $\beta_0(Z)$, $\theta_0(E) = W(\chi_E)$, where χ_E denotes the characteristic function of E. θ_0 is a finitely additive probability measure on $\beta_0(Z)$ having μ and ν as marginals. Since S'_w is a Lusin space and $\beta(S'_w) = \beta(S'_R)$, for any positive number ϵ and any set E in $\beta_0(Z)$ there exists a compact set K in $\mathcal{K}_0(S'_w \times S'_w)$ such that $K \subset E$ and $\theta_0(E - K) < \epsilon$.

Therefore by [7], p. 51, Theorem 16 θ_0 is extended to probability Radon measure θ on $S'_w \times S'_w$ having μ and ν as marginals. Since

$$\beta(S'_w \times S'_w) = \beta(S'_w) \otimes \beta(S'_w) = \beta(S'_R) \otimes \beta(S'_R) = \beta(Z),$$

θ is a probability Borel measure on Z. Since Λ is a weakly closed convex subset of $P(Z)$, in order to show that θ belongs to Λ it is sufficient to show that $\int_Z hd\theta \leq P(h)$ for any h in $C^b(Z)$. By adding positive constants we may assume that h is positive. Since h is a bounded continuous function, by [1], Chapitre 9, §2, $n°6$, Lemma 3, for any positive number ϵ there exists a linear combination h_0 of the characteristic functions of open subsets of Z with positive coefficients such that $0 \leq h(z) - h_0(z) < \epsilon$ for all z in Z. Let

$$h_0(z) = \sum_{i=1}^{n} \alpha_i \chi_{O_i}(z),$$

where each α_i is positive and each O_i is open. Since Z satisfies the second axiom of countability, for each i there exists an open set U_i in $\beta_0(Z)$ such that $U_i \subset O_i$ and $\theta(O_i - U_i) < \dfrac{\epsilon}{Mn}$, where $M = \max\{\alpha_i : \ i = 1, 2, \cdots, n\}$. Putting $l(z) = \sum_{i=1}^{n} \alpha_i \chi_{U_i}(z)$, we have

$$\int_Z hd\theta - 2\epsilon < \int_Z h_0 d\theta - \epsilon < \int_Z ld\theta = \int_Z ld\theta_0 \leq P(l) \leq P(h).$$

Since ϵ is arbitrary, we have

$$\int_Z hd\theta \leq P(h).$$

Thus Theorem 3.1 has been proved.

Corollary 3.2. *Let F be a non-void closed subset of Z and $\epsilon \geq 0$. Given the probability Borel measures μ and ν on S'_R, there exists a probability Borel measure λ on Z with the marginals μ and ν such that $\lambda(F) \geq 1 - \epsilon$ iff, for any Borel set B of S'_R, one has*

$$\nu(B) \leq \mu(p_{r_1}(F \cap (S'_R \times B)) + \epsilon.$$

Proof. By Theorem 2.9 $p_{r_1}(F \cap (S'_R \times B))$ is universally measurable with respect to $(S'_R,\ \beta(S'_R))$. Accordingly reading open sets (resp. continuous functions) as Borel sets (resp. Borel measurable functions) in [8], Theorem 11, Corollary 3.2 is proved similarly to the proof of [8], Theorem 11.

Acknowledgment: The author wishes to express his hearty thanks to Professor S. Nakanishi for her usefull advice.

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