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WEAK AND STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE SEMIGROUPS IN A BANACH SPACE SATISFYING OPIAL'S CONDITION

SACHIKO ATSUSHIBA AND WATARU TAKAHASHI

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Abstract. In this paper, we study the weak convergence of Mann's type iteration procedure and the existence of nonexpansive retractions for commutative semigroups in Banach spaces which satisfy Opial's condition. Further, we introduce an implicit iteration procedure for nonexpansive semigroups and then prove a strong convergence theorem for the nonexpansive semigroups in general Banach spaces.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself, that is,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. Mann [21] introduced the following iteration procedure for approximating fixed points of a nonexpansive self-mapping T on a nonempty, closed, convex subset C of a Hilbert space H:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad \text{for each } n = 1, 2, \dots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [25] studied this iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained that if T has a fixed point and $\{\alpha_n\}$ satisfies $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T. Shimizu and Takahashi [27, 28] introduced the first iteration procedure for finding common fixed points of families of nonexpansive mappings and obtained convergence theorems for the families. In [2], Atsushiba and Takahashi considered the following iteration procedure of Mann's type for approximating common fixed points of two nonexpansive mappings in a Banach space:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n^2} \sum_{i,j=0}^{n-1} S^i T^j x_n \quad \text{for each } n = 1, 2, \dots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and S, T are nonexpansive mappings from C into itself. Atsushiba and Takahashi [1] also studied an iteration procedure of Mann's type for approximating common fixed points for a family $\{T(t) : t \in S\}$ of nonexpansive mappings in a

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Hilbert space H:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n \quad \text{for each } n = 1, 2, \dots, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\{\mu_n\}$ is a sequence of means on the set $\mathbb{N}$ of positive integers and $T_{\mu_n} x_n$ is the unique point in C satisfying $\langle T_{\mu_n} x_n, y\rangle = (\mu_n)_s \langle T(s)x_n, y\rangle$ for all y in H (see also [5]). Recently, Suzuki [30] proved the weak convergence of the iteration procedure of Mann's type for approximating common fixed points for two commuting nonexpansive mappings in a Banach space which satisfies Opial's condition (see also [31]).

On the other hand, Xu and Ori [37] studied the following implicit iteration procedure for finite nonexpansive mappings $T_1, T_2, \dots, T_r$ in a Hilbert space: $x_0 = x \in C$ and

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n \tag{2}$$

for every $n = 1, 2, \dots$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $T_n = T_{n+r}$ for all $n \in \mathbb{N}$. Then, they established a weak convergence. Sun, He and Ni [29] studied the iterations defined by (2) and proved strong convergence of the iterations in a uniformly convex Banach space, requiring one mapping T_i in the family to be semi-compact.

In this paper, we study the weak convergence of Mann's type iteration procedure and the existence of nonexpansive retractions for commutative semigroups in Banach spaces which satisfy Opial's condition. Further, we introduce an implicit iteration procedure for nonexpansive semigroups and then prove a strong convergence theorem for the nonexpansive semigroups in general Banach spaces.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and and the set of all positive real numbers, respectively. Let E be a real Banach space. We denote by B_r the closed ball $\{x \in E : \|x\| \le r\}$. A Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for each $x, y \in B_1$ with $x \ne y$, and it is said to be *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x+y\|/2 \le 1 - \delta$ for each $x, y \in B_1$ with $\|x-y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex (see [36]).

Let C be a closed convex subset of a Banach space and let T be a mapping from C into itself. We denote by $F(T)$ the set $\{x \in C : x = Tx\}$. We also denote by I the identity mapping. We denote by $N(C)$ the set of all nonexpansive mappings from C into itself. We know from [9] that if C is a nonempty closed convex subset of a strictly convex Banach space, then $F(T)$ is convex for each $T \in N(C)$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to x. Similarly, $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) will symbolize weak convergence. For any element z and any set A, we denote the distance between z and A by $d(z, A) = \inf\{\|z-y\| : y \in A\}$.

We say that a Banach space E satisfies *Opial's condition* [23] if for each sequence $\{x_n\}$ in E with $x_n \rightharpoonup x$,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\|$$

for each $y \in E$ with $y \ne x$. We know that if the duality mapping $x \mapsto \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ from E into E^* is single-valued and weakly sequentially continuous, then E satisfies Opial's condition. In particular, each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [17, 23]). Although an L^p-space with $p \ne 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [12, 23]).

Let S be a commutative semigroup with identity. In this case, $(S, \leq)$ is a directed system when the binary relation $\leq$ on S is defined by $a \leq b$ if and only if there is $c \in S$ with $a+c = b$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define an element $r_s f$ in $B(S)$ by $(r_s f)(t) = f(s+t)$ for each $t \in S$. Let X be a subspace of $B(S)$ with $1 \in X$. An element μ in X^* is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$; see also [36]. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be r_s-invariant, i.e., $r_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be *invariant* if $\mu(r_s f) = \mu(f)$ for each $s \in S$ and $f \in X$. We know that if S is a commutative semigroup and μ is an invariant mean on X, then

$$\varliminf_{s \in S} f(s) \leq \mu(f) \leq \varlimsup_{s \in S} f(s)$$

for each $f \in X$; see [33, 36] for more details. A sequence $\{\mu_n\}$ of means on X is said to be asymptotically invariant if $\mu_n - r_s^* \mu_n \to 0$ for each $s \in S$, in the sense of the weak-star topology, where r_s^* is the adjoint operator of r_s [14, 20]. Let E be a Banach space, let X be a subspace of $B(S)$ with $1 \in X$ and let μ be a mean on X. Let f be a mapping from S into E such that $\{f(t) : t \in S\}$ is contained in a weakly compact convex subset of E and the mapping $t \mapsto \langle f(t), x^* \rangle$ is in X for each $x^* \in E^*$. We know from [14, 32] that there exists a unique element $x_0 \in E$ such that $\langle x_0, x^* \rangle = \mu_t \langle f(t), x^* \rangle$ for all $x^* \in E^*$. Following [14], we denote such x_0 by $\int f(t)\, d\mu(t)$ or f_μ. We know that f_μ is contained in $\overline{\mathrm{co}}\{f(t) : t \in S\}$ (for example see [15, 16, 32]). Let C be a nonempty closed convex subset of a Banach space E. A family $\mathcal{S} = \{T(t) : t \in S\}$ is said to be a *nonexpansive semigroup* on C if it satisfies the following:

(1) For each $t \in S$, $T(t)$ is a nonexpansive mapping from C into itself;
(2) $T(t+s) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that for each $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact convex subset of C. Let X be a subspace of $B(S)$ with $1 \in X$ such that the mapping $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$, and let μ be a mean on X. Following [26], we also write $T_\mu x$ instead of $\int T(t)x\, d\mu(t)$ for $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$; for more details, see [36].

For a nonempty subset D of S, we define the characteristic function I_D by

$$I_D(t) = \begin{cases} 1, & t \in D, \\ 0, & t \notin D. \end{cases} \tag{3}$$

The following lemma is used in the proof of Proposition 3.4.

Lemma 2.1. *Let S be a commutative semigroup with identity. Let $k \in \mathbb{N}$ and let $A_1, A_2, \ldots, A_k$ be subsets of S. Let X be a subspace of $B(S)$ with $1 \in X$ such that $I_{A_1}, I_{A_2}, \cdots, I_{A_k}$ are contained in X. Put $D = \bigcap_{j=1}^k A_j$ and put*

$$\alpha = \sum_{j=1}^{k} \mu_t(I_{A_j}(t)) - (k-1).$$

where μ is an invariant mean on X. Suppose $\alpha > 0$. Then,

$$\mu(I_D) \geq \alpha$$

holds and

$$\{s \in S : s \geq p\} \cap D \neq \emptyset$$

for each $p \in S$.

Proof. Let μ be an invariant mean on $B(S)$. ¿From $D = \bigcap_{j=1}^{k} A_j$, we have

$$I_D(t) \geq \sum_{j=1}^{k} I_{A_j}(t) - (k-1)$$

for all $t \in S$ and hence

$$\begin{aligned} \mu_t(I_D(t)) &\geq \mu_t\left(\sum_{j=1}^{k} I_{A_j}(t) - (k-1)\right) \\ &= \sum_{j=1}^{k} \mu(I_{A_j}) - (k-1) = \alpha > 0. \end{aligned} \tag{4}$$

Since μ is an invariant mean, we have $\mu_t(I_D(t)) = \mu_t(I_D(t+s))$ for any $s \in S$. Fix $p \in S$. Then, it follows from (4) that

$$\mu_t(I_D(t+p)) \geq \alpha > 0.$$

Hence, we obtain

$$\{t + p : t \in S\} \cap D \neq \emptyset.$$

Since $p \in S$ is arbitrary,

$$\{t \in S : t \geq p\} \cap D \neq \emptyset$$

holds for each $p \in S$. □

The following theorem was proved by Edelstein and O'Brien [13].

Theorem 2.2 ([13]). *Let E be a Banach space which satisfies Opial's condition and let C be a nonempty weakly compact convex subset of E. Let T be a nonexpansive mapping of C into itself. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_{n+1} = \alpha x_n + (1-\alpha)Tx_n \quad \text{for each } n \in \mathbb{N},$$

where α is a constant number in $(0,1)$. Then $\{x_n\}$ converges weakly to a fixed point of T.

3. Lemmas

Let C be a nonempty closed convex subset of a Banach space E. Throughout the rest of this paper, we assume that S is a commutative semigroup with identity, $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on C, and X is a subspace of $B(S)$ with $1 \in X$ such that it is r_s-invariant for each $s \in S$, and the functions $t \mapsto \langle T(t)x, x^* \rangle$ and $t \mapsto \|T(t)x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$. We will call such a subspace X of $B(S)$ $\mathcal{S}$-stable. We know the following lemma (for example see [18, 22]). For the sake of completeness, we provide a proof.

Lemma 3.1. *Let C be a nonempty bounded closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-stable. For an invariant mean μ on X and $z \in C$ with $T_\mu z = z$, put $L_0 = \mu_t\|T(t)z - z\|$. Then, $\|T(t)z - z\| \leq L_0$ holds for each $t \in S$.*

Proof. Since μ is an invariant mean on X, for each $t \in S$, we have

$$\begin{aligned}\|T(t)z - z\| = \|T(t)z - T_\mu z\| &= \sup_{x^* \in S(E^*)} |\langle T(t)z - T_\mu z, x^*\rangle| \\ &= \sup_{x^* \in S(E^*)} |\mu_s \langle T(t)z - T(s)z, x^*\rangle| \\ &\leq \sup_{x^* \in S(E^*)} \mu_s(\|T(t)z - T(s)z\| \|x^*\|) \\ &= \mu_s \|T(t)z - T(s)z\|.\end{aligned}$$

Putting $g(s) = \|T(t)z - T(s)z\|$ for each $s \in S$, we have $(r_t g)(s) = \|T(t)z - T(s+t)z\|$ and hence

$$\begin{aligned}\mu_s \|T(t)z - T(s)z\| &= \mu(g) = \mu(r_t g) \\ &= \mu_s(\|T(t)z - T(s+t)z\|) \leq \mu_s(\|z - T(s)z\|) = L_0.\end{aligned}$$

So, we have $\|T(t)z - z\| \leq L_0$ holds for each $t \in S$. □

Let C be a nonempty bounded closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. For $z \in C$, $f \in E^*$ and $a, b \in \mathbb{R}$, let us define two subsets of S as follows

$$A(z, f, a) = \{s \in S : \langle T(s)z - z, f\rangle \leq a\}$$

and

$$B(z, b) = \{s \in S : \|T(s)z - z\| \geq b\}.$$

We will call a subspace X of $B(S)$ $\mathcal{S}$-admissible if X is $\mathcal{S}$-stable and contains $I_{A(z,f,a)}$ and $I_{B(z,b)}$ for all $z \in C$, $f \in E^*$ and $a, b \in \mathbb{R}$. Occasionally, we use $I_{A(f,a)}$ and $I_{B(b)}$ instead of $I_{A(z,f,a)}$ and $I_{B(z,b)}$, respectively.

Lemma 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let μ be an invariant mean on X. Suppose that $T_\mu z = z$ for some $z \in C$. Put $L_0 = \mu_t \|T(t)z - z\|$. Fix $p \in S$ and $f \in E^*$ such that $\|f\| = 1$ and $\langle T(p)z - z, f\rangle = \|T(p)z - z\|$. Let δ be a positive real number satisfying $\|T(p)z - z\| \geq L_0 - \delta$. Then,*

$$\mu(I_{A(f,\varepsilon)}) \geq \frac{\varepsilon}{\varepsilon + \delta}$$

holds for all $\varepsilon > 0$.

Proof. Let $s \in S$ with $s \geq p$. There exists p_1 such that $s = p + p_1$. By Lemma 3.1, we have

$$\|T(s)z - T(p)z\| \leq \|T(p_1)z - z\| \leq L_0$$

and hence

$$\begin{aligned}\langle T(s)z - z, f\rangle &= \langle T(s)z - T(p)z, f\rangle + \langle T(p)z - z, f\rangle \\ &= \langle T(s)z - T(p)z, f\rangle + \|T(p)z - z\| \\ &\geq -|\langle T(s)z - T(p)z, f\rangle| + \|T(p)z - z\| \\ &\geq -\|f\| \|T(s)z - T(p)z\| + \|T(p)z - z\| \\ &\geq -L_0 + L_0 - \delta = -\delta.\end{aligned}$$

We also have

$$\begin{aligned}\langle T_\mu z - z, f\rangle &= \langle T_\mu z, f\rangle - \langle z, f\rangle \\ &= (\mu)_t \langle T(t)z, f\rangle - \langle z, f\rangle = (\mu)_t \langle T(t)z - z, f\rangle. \qquad (5)\end{aligned}$$

On the other hand, for $s \in S$ with $s \geq p$, we obtain

$$
\begin{aligned}
\langle T(s)z - z, f\rangle &= I_{S\setminus A(f,\varepsilon)}(s)\langle T(s)z - z, f\rangle + I_{A(f,\varepsilon)}(s)\langle T(s)z - z, f\rangle \\
&\geq \varepsilon \cdot I_{S\setminus A(f,\varepsilon)}(s) - \delta \cdot I_{A(f,\varepsilon)}(s) \\
&\geq \varepsilon \cdot (I_S(s) - I_{A(f,\varepsilon)}(s)) - \delta \cdot I_{A(f,\varepsilon)}(s) \\
&= \varepsilon \cdot I_S(s) - (\varepsilon + \delta)\, I_{A(f,\varepsilon)}(s) = \varepsilon - (\varepsilon + \delta)\, I_{A(f,\varepsilon)}(s). \qquad (6)
\end{aligned}
$$

Then, it follows from (5) and (6) that

$$
\begin{aligned}
\langle T_\mu z - z, f\rangle &= (\mu)_t\langle T(t)z - z, f\rangle \\
&= \mu_t(\langle T(t+p)z - z, f\rangle) \\
&\geq \mu(\varepsilon - (\varepsilon+\delta)I_{A(f,\varepsilon)}) = \varepsilon - (\varepsilon + \delta)\cdot\mu(I_{A(f,\varepsilon)}).
\end{aligned}
$$

¿From $T_\mu z = z$, we have

$$\mu(I_{A(f,\varepsilon)}) \geq \frac{\varepsilon}{\varepsilon + \delta}.$$

□

Lemma 3.3. *Let C be a nonempty bounded closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let μ be an invariant mean on X. Suppose that $T_\mu z = z$ for some $z \in C$. Then,*

$$\mu(I_{B(L_0-\varepsilon)}) = 1$$

holds for all $\varepsilon > 0$.

Proof. We fix $\varepsilon > 0$ and $\eta \in \mathbb{R}^+$ with $\frac{1}{2} < \eta < 1$ and put

$$\delta = \frac{\varepsilon(1-\eta)}{2\eta}.$$

We note that $0 < \delta < \frac{\varepsilon}{2}$. Put $L_0 = \mu_p\|T(p)z - z\|$ and $d = \overline{\lim}_p \|T(p)z - z\|$. Then, we have $d \geq L_0$. By the definition of d, there exists $p \in S$ such that $\|T(p)z - z\| \geq d - \delta$. So, it follows that

$$\|T(p)z - z\| \geq d - \delta \geq L_0 - \delta.$$

Fix $f \in E^*$ with

$$\|f\| = 1 \quad \text{and} \quad \langle T(p)z - z, f\rangle = \|T(p)z - z\|.$$

Let μ be an invariant mean on X. So, by Lemma 3.2, we have

$$\mu(I_{A(f,\varepsilon/2)}) \geq \frac{\varepsilon/2}{\varepsilon/2 + \delta} = \eta. \qquad (7)$$

If $u + p \in A(f, \varepsilon/2)$, then we have

$$
\begin{aligned}
\|T(u)z - z\| &\geq \|T(u+p)z - T(p)z\| \\
&\geq \langle T(p)z - T(u+p)z, f\rangle \\
&= \langle T(p)z - z, f\rangle + \langle z - T(u+p)z, f\rangle \\
&= \|T(p)z - z\| + \langle z - T(u+p)z, f\rangle = \|T(p)z - z\| - \langle T(u+p)z - z, f\rangle \\
&\geq L_0 - \delta - \frac{\varepsilon}{2} \geq L_0 - \varepsilon
\end{aligned}
$$

and hence $u \in B(L_0 - \varepsilon)$. Therefore, $I_{B(L_0-\varepsilon)}(u) \geq I_{A(f,\varepsilon/2)}(u + p)$ for all $u \in S$. So, by (7), we obtain

$$\mu_u\Big(I_{B(L_0-\varepsilon)}(u)\Big) \geq \mu_u\Big(I_{A(f,\varepsilon/2)}(u+p)\Big) = \mu_u\Big(I_{A(f,\varepsilon/2)}(u)\Big) \geq \eta.$$

Since η is arbitrary, we have the desired result. □

Proposition 3.4. *Let C be a nonempty bounded closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let μ be an invariant mean on X. Suppose that $T_\mu z = z$ for some $z \in C$. Let $L_0 = \mu_p\|T(p)z - z\|$ and let $t \in S$. Then, there exists sequences $\{p_n\}$ in S and $\{f_n\}$ in E^* such that*

$$p_{n+1} \geq p_n + t,$$
$$\|T(p_n)z - z\| \geq L_0 - \frac{1}{3^{n+1}},$$
$$\langle T(p_n)z - z, f_\ell\rangle \leq \frac{2^{\ell+1}}{3^{\ell+1}} \quad \text{for all} \quad \ell = 1, 2, \dots, n-1$$

and

$$\|f_n\| = 1 \quad \text{and} \quad \langle T(p_n)z - z, f_n\rangle = \|T(p_n)z - z\| \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. ¿From $L_0 = \mu_t\|T(t)z - z\|$, there exists $p_1 \in S$ such that

$$\|T(p_1)z - z\| \geq L_0 - \frac{1}{3^2}.$$

Take $f_1 \in E^*$ with $\|f_1\| = 1$ and $\langle T(p_1)z - z, f_1\rangle = \|T(p_1)z - z\|$. By Lemma 3.2, we have

$$\mu(I_{A(f_1,(\frac{2}{3})^2)}) \geq \left(\frac{2}{3}\right)^2 \Big/ \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2^2}{2^2+1}.$$

Putting $A_1 = B(L_0 - \frac{1}{3^{2+1}})$ and $A_2 = A\big(f_1, \big(\frac{2}{3}\big)^2\big)$ in Lemma 2.1, we have that

$$\begin{aligned}\alpha &= \mu(I_{A_1}) + \mu(I_{A_2}) - 1 = \mu(I_{B(L_0-(\frac{1}{3})^3)}) + \mu(I_{A(f_1,(\frac{2}{3})^2)}) - 1\\ &\geq 1 + \frac{2^2}{2^2+1} - 1 > 0.\end{aligned}$$

So, it follows from Lemma 2.1 that

$$\{s \in S : s \geq p_1 + t\} \cap B\left(L_0 - \frac{1}{3^3}\right) \cap A\left(f_1, \left(\frac{2}{3}\right)^2\right) \neq \emptyset.$$

This implies that there exists $p_2 \in S$ with $p_2 \geq p_1 + t$ such that

$$\|T(p_2)z - z\| \geq L_0 - \frac{1}{3^3} \quad \text{and} \quad \langle T(p_2)z - z, f_1\rangle \leq \frac{2^2}{3^2}.$$

Let us prove Proposition 3.4 by induction. Suppose $p_k \in S$ and $f_k \in E^*$ are known. By Lemmas 3.2 and 3.3, we have

$$\begin{aligned}&\mu(I_{B(L_0-(\frac{1}{3})^{k+2})}) + \sum_{\ell=1}^{k} \mu(I_{A(f_\ell,(\frac{2}{3})^{\ell+1})}) - k\\ &\geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}}{2^{\ell+1}+1} - k\\ &\geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}-1}{2^{\ell+1}} - k\\ &= 1 + \sum_{\ell=1}^{k} \frac{-1}{2^{\ell+1}} > \frac{1}{2} > 0.\end{aligned}$$

So it follows from Lemma 2.1 that

$$\{s \in S : s \geq p_k + t\} \cap B\left(L_0 - \frac{1}{3^{k+2}}\right) \cap \bigcap_{\ell=1}^{k} A\left(f_\ell, \left(\frac{2}{3}\right)^{\ell+1}\right) \neq \emptyset,$$

i.e, there exists $p_{k+1} \geq p_k + t$ such that $\|T(p_{k+1})z - z\| \geq L_0 - \frac{1}{3^{k+2}}$ and

$$\langle T(p_{k+1})z - z, f_\ell \rangle \leq \frac{2^{\ell+1}}{3^{\ell+1}}$$

for all $l = 1, 2, \dots, k$. Take $f_{k+1} \in E^*$ with

$$\|f_{k+1}\| = 1 \text{ and } \langle T(p_{k+1})z - z, f_{k+1} \rangle = \|T(p_{k+1})z - z\|.$$

It follows from Lemma 3.2 that

$$\mu(I_{A(f_{k+1}, (\frac{2}{3})^{k+2})}) \geq \frac{2^{k+2}}{2^{k+2} + 1}.$$

This completes the proof. □

4. Weak Convergence Theorems

In this section, we first show that for a weakly compact convex subset C of a Banach space with Opial's condition, the fixed point set of a commutative semigroup of nonexpansive mappings in C is precisely the fixed point set of the nonexpansive mapping determined by an invariant mean.

Theorem 4.1. *Let E be a Banach space which satisfies Opial's condition and let C be a nonempty weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. If $z \in C$, then the following are equivalent:*

(i) z is a common fixed point of $\mathcal{S} = \{T(t) : t \in S\}$;
(ii) $T_\mu z = z$ for some invariant mean μ on X;
(iii) $\{T_{\mu_\alpha} z\}$ converges weakly to z.

Proof. It is clear that (i) implies (ii) and that (i) implies (iii).

We prove that (ii) implies (i). Put $L_0 = \mu_s \|T(s)z - z\|$ and let $t \in S$. By Proposition 3.4, there exists sequences $\{p_n\}$ in S and $\{f_n\} \subset E^*$ such that

$$p_{n+1} \geq p_n + t,$$
$$\|T(p_n)z - z\| \geq L_0 - \frac{1}{3^{n+1}},$$
$$\langle T(p_n)z - z, f_\ell \rangle \leq \frac{2^{\ell+1}}{3^{\ell+1}} \text{ for all } \ell = 1, 2, \dots, n-1$$

and

$$\|f_n\| = 1 \text{ and } \langle T(p_n)z - z, f_n \rangle = \|T(p_n)z - z\| \text{ for all } n \in \mathbb{N}.$$

Since C is weakly compact, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{T(p_{n_k})z\}$ converges weakly to some point $u \in C$. Fix $\ell \in \mathbb{N}$. If $n_k > \ell$, then

$$\langle T(p_{n_k})z - z, f_\ell \rangle \leq \frac{2^{\ell+1}}{3^{\ell+1}}. \tag{8}$$

So, it follows from (8) that

$$\langle u - z, f_\ell \rangle \leq \frac{2^{\ell+1}}{3^{\ell+1}}$$

for all $l \in \mathbb{N}$. Since

$$\begin{aligned}\|T(p_\ell)z - u\| &= \|f_\ell\|\|T(p_\ell)z - u\| \\ &\geq \langle T(p_\ell)z - u, f_\ell\rangle \\ &= \langle T(p_\ell)z - z, f_\ell\rangle + \langle z - u, f_\ell\rangle \\ &= \|T(p_\ell)z - z\| + \langle z - u, f_\ell\rangle = \|T(p_\ell)z - z\| - \langle u - z, f_\ell\rangle \\ &\geq L_0 - \frac{1}{3^{\ell+1}} - \frac{2^{\ell+1}}{3^{\ell+1}}\end{aligned}$$

for all $\ell \in \mathbb{N}$, we have

$$\underline{\lim}_k \|T(p_{n_k})z - u\| \geq L_0. \tag{9}$$

Suppose $z \neq u$. Since E satisfies Opial's condition, by Lemma 3.1 and (9), we have

$$\begin{aligned}\underline{\lim}_k \|T(p_{n_k})z - z\| &\leq L_0 \\ &\leq \underline{\lim}_k \|T(p_{n_k})z - u\| < \underline{\lim}_k \|T(p_{n_k})z - z\|.\end{aligned}$$

This is a contradiction. So, we have $z = u$. For each $\ell \geq 2$, we obtain $p_\ell \geq p_1 + t \geq t$. So, there exists $t_k \in S$ such that $p_{n_k} = t_k + t$. Then, it follows from Lemma 3.1 that

$$\begin{aligned}\|T(p_{n_k})z - T(t)z\| &= \|T(t_k + t)z - T(t)z\| \\ &\leq \|T(t_k)z - z\| \leq L_0\end{aligned}$$

and hence

$$\underline{\lim}_k \|T(p_{n_k})z - T(t)z\| \leq L_0. \tag{10}$$

Suppose $T(t)z \neq u$. Since E satisfies Opial's condition, by (9) and (10), we have

$$\begin{aligned}\underline{\lim}_k \|T(p_{n_k})z - T(t)z\| &\leq L_0 \\ &\leq \underline{\lim}_k \|T(p_{n_k})z - u\| \\ &< \underline{\lim}_k \|T(p_{n_k})z - T(t)z\|.\end{aligned}$$

This is a contradiction. Hence, $T(t)z = u$. We remark that $t \in S$ is arbitrary. Hence, we have $z = T(t)z = u$ for all $t \in S$. Therefore, (i) and (ii) are equivalent.

We prove that (iii) implies (ii). Let μ be a cluster point of $\{\mu_\alpha\}$ in the weak* topology. Then, we know that [36] that μ is an invariant mean on X. Without loss of generality, we may assume that $\{\mu_\alpha\}$ converges to μ in the weak* topology. So, we have

$$\langle T_{\mu_\alpha} z, x^*\rangle = \mu_\alpha \langle T(t)z, x^*\rangle \to \mu\langle T(t)z, x^*\rangle = \langle T_\mu z, x^*\rangle \tag{11}$$

for each x^* in E^*. We obtain from (iii) that

$$\langle T_{\mu_\alpha} z, x^*\rangle \to \langle z, x^*\rangle$$

for each x^* in E^*. Then, it follows that $T_\mu z = z$. □

For a similar result, see Lau, Miyake and Takahashi [19]. Now, we prove a weak convergence theorem for a commutative semigroup in a Banach space which satisfies Opial's condition.

Theorem 4.2. *Let E be a Banach space which satisfies Opial's condition and let C be a nonempty weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let μ be an invariant mean on X. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_{n+1} = \alpha x_n + (1-\alpha)T_\mu x_n \quad \text{for each } n \in \mathbb{N},$$

where α is a constant number in $(0,1)$. Then $\{x_n\}$ converges weakly to a point of $F(\mathcal{S})$.

Proof. By Theorem 2.2, $\{x_n\}$ converges weakly to a fixed point z_0 of T_μ. Then, it follows from Theorem 4.1 that $z_0 \in F(\mathcal{S})$. This is a completes the proof. □

Remark 4.3. In Theorems 4.1 and 4.2, we may replace "E satisfies Opial's condition" with the following condition: for each weakly convergent sequences $\{x_n\}$ in C which converges weakly to x,

$$\underline{\lim}_{n\to\infty} \|x_n - x\| < \underline{\lim}_{n\to\infty} \|x_n - y\|$$

for each $y \in C$ with $y \neq x$. We note that the above condition is satisfied in the case that C is compact (see [23, 31, 30]). So, we have the following theorem (It was obtained in [22], see also [6]).

Theorem 4.4 ([22])**.** *Let C be a nonempty compact convex subset of a Banach space E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. If $z \in C$, then the following are equivalent:*

(i) z is a common fixed point of $\mathcal{S} = \{T(t) : t \in S\}$;
(ii) $T_\mu z = z$ for some invariant mean μ on X;
(iii) there exists a subnet $\{T_{\mu_{\alpha_\beta}} z\}$ of $\{T_{\mu_\alpha} z\}$ converging strongly to z;
(iv) $\underline{\lim}_{\alpha} \|T_{\mu_\alpha} z - z\| = 0$ holds.

Next, we prove the existence of nonexpansive retractions for a commutative semigroup in a Banach space which satisfies Opial's condition.

Theorem 4.5. *Let E be a Banach space which satisfies Opial's condition and let C be a nonempty weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let μ be an invariant means on X. Then, there exists a nonexpansive retraction Q from C onto $F(\mathcal{S})$ such that $Q = QT(t) = T(t)Q$ for all $t \in S$.*

Proof. We shall first define a mapping Q of C into C. Let μ be an invariant mean on X and let $x \in C$. Define a sequence $\{x_n\}$ by $x_1 = T_\mu x$ and

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}T_\mu x_n$$

for $n \in \mathbb{N}$. By Theorem 4.2, $\{x_n\}$ converges weakly to a common fixed point z_0 of $\mathcal{S}$. We note

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{2}T_\mu x_{n-1} = (\frac{1}{2}I + \frac{1}{2}T_\mu)x_{n-1} = \cdots = (\frac{1}{2}I + \frac{1}{2}T_\mu)^{n-1}x_1.$$

We put

$$Qx = \text{w-}\lim_{n\to\infty}(\frac{1}{2}T_\mu + \frac{1}{2}I)^{n-1}T_\mu x = z_0$$

where I is the identity mapping on C. For $x, y \in C$, we have

$$\left\| \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu x - \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu y \right\| \leq \|x - y\|$$

and hence

$$\begin{aligned}\|Qx - Qy\| &\le \varliminf_{n\to\infty} \left\| \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu x - \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu y \right\| \\ &\le \|x - y\|.\end{aligned}$$

So, Q is nonexpansive. For $x \in C$ and $t \in S$, we also have

$$\begin{aligned}\|T_\mu T(t)x - T_\mu x\| &= \sup_{x^* \in S(E^*)} |(\mu)_s \langle T(s)T(t)x, x^*\rangle - (\mu)_s \langle T(s)x, x^*\rangle| \\ &= \sup_{x^* \in S(E^*)} |(\mu)_s \langle T(s+t)x, x^*\rangle - (\mu)_s \langle T(s)x, x^*\rangle| \\ &= \sup_{x^* \in S(E^*)} |(\mu)_s \langle T(s)x, x^*\rangle - (\mu)_s \langle T(s)x, x^*\rangle| \\ &= 0\end{aligned}$$

and hence $T_\mu T(t)x = T_\mu x$. Therefore, we also have $QT(t)x = Qx$ for all $x \in C$ and $t \in S$. By the definition of Q, we obtain that $Qx \in F(\mathcal{S})$ for all $x \in C$. We also obtain that $Qz = z$ for all $z \in F(\mathcal{S})$ (see [36]). Hence, $Q^2x = Qx = T(t)Qx$ for all $x \in C$ and $t \in S$. This completes the proof. ☐

5. Strong convergence of implicit iterations

In this section, we assume that S is a commutative semigroup. Let C be a nonempty weakly compact convex subset of a Banach space E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup of C. We consider the following iteration procedure (see [37]):

$$x_1 = x \in C, \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{\mu_n} x_n \tag{12}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Lemma 5.1 ([7])**.** *Let C be a nonempty weakly compact convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let $\{\mu_n\}$ be a sequence of means on S and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_1 = x \in C, \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{\mu_n} x_n$$

for every $n \in \mathbb{N}$. Then, $\|x_{n+1} - w\| \le \|x_n - w\|$ and $\lim_{n\to\infty} \|x_n - w\|$ exists for each $w \in F(\mathcal{S})$.

Using Lemma 5.1, we prove the following strong convergence theorem.

Theorem 5.2. *Let C be a nonempty compact convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ which is $\mathcal{S}$-admissible. Let $\{\mu_n\}$ be a sequence of means on S which is asymptotically invariant and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and* $\lim_{n\to\infty} \alpha_n = 0$. *Let $x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_1 = x \in C, \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{\mu_n} x_n$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$.

Proof. By the definition of $\{x_n\}$, we have

$$(x_n - T_{\mu_n} x_n) = \alpha_n (x_{n-1} - T_{\mu_n} x_n).$$

So, it follows that

$$\begin{aligned}\|x_n - T_{\mu_n} x_n\| &\le \alpha_n \|x_{n-1} - T_{\mu_n} x_n\| \\ &\le 2\alpha_n M,\end{aligned}$$

where $M = \sup_{z\in C}\|z\|$. So, by $\lim_{n\to\infty}\alpha_n = 0$, we have $\lim_{n\to\infty}\|T_{\mu_n}x_n - x_n\| = 0$. Since C is compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $z \in C$. Since

$$\begin{aligned}\overline{\lim_{k\to\infty}}\|T_{\mu_{n_k}}z - z\| &\le \overline{\lim_{k\to\infty}}\{\|T_{\mu_{n_k}}z - T_{\mu_{n_k}}x_{n_k}\| + \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|\}\\ &\le \overline{\lim_{k\to\infty}}\{2\|z - x_{n_k}\| + \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\|\} = 0,\end{aligned}$$

we obtain $\underline{\lim}_{n\to\infty}\|T_{\mu_n}z - z\| \le \lim_{k\to\infty}\|T_{\mu_{n_k}}z - z\| = 0$. So, by Theorem 4.4, we have $z \in F(\mathcal{S})$. By Lemma 5.1, there exists $\lim_{n\to\infty}\|x_n - z\|$. Then, we obtain

$$\lim_{n\to\infty}\|x_n - z\| = \lim_{k\to\infty}\|x_{n_k} - z\| = 0.$$

This completes the proof. □

6. Applications

Throughout this section, we assume that C is a nonempty compact convex subset of a Banach space E and $\{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty}\alpha_n = 0$. Using Theorem 5.2, we can prove some strong convergence theorems as in [36].

Theorem 6.1. *Let T be a nonexpansive mapping from C into itself and let $x \in C$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 6.2. *Let T be as in Theorem 6.1 and let $x \in C$. Let $\{q_{n,m} : n, m \in \mathbb{N}\}$ be a sequence of real numbers such that $q_{n,m} \ge 0$, $\sum_{m=0}^{\infty} q_{n,m} = 1$ for each $n \in \mathbb{N}$ and $\lim_n \sum_{m=0}^{\infty}|q_{n,m+1} - q_{n,m}| = 0$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)\sum_{m=0}^{\infty} q_{n,m}T^m x_n$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 6.3. *Let T and U be commutative, nonexpansive mappings from C into itself and let $x \in C$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{i,j=0}^{n} T^i U^j x_n$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of T and U.

Theorem 6.4. *Let $\mathcal{S} = \{T(t) : t \in [0,\infty)\}$ be a nonexpansive semigroup on C such that the functions $t \mapsto \langle T(t)x, x^*\rangle$ and $t \mapsto \|T(t)x - y\|$ are measurable for each $x, y \in C$ and $x^* \in E^*$. Let $x \in C$ and let $\{s_n\}$ be a sequence of positive real numbers with $s_n \to \infty$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(t)x_n\,dt$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Theorem 6.5. *Let $\mathcal{S}$ be as in Theorem 6.4 and let $x \in C$. Let $\{r_n\}$ be a sequence of positive real numbers with $r_n \to 0$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)\, r_n \int_0^\infty e^{-r_n t} T(t) x_n \, dt$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Theorem 6.6. *Let $\mathcal{S}$ be as in Theorem 6.4 and let $x \in C$. Let $\{q_n\}$ be a sequence of measurable functions from $[0,\infty)$ into itself such that $\int_0^\infty q_n(t)\,dt = 1$ for each $n \in \mathbb{N}$, $\lim_n q_n(t) = 0$ for almost every $t \geq 0$, $\lim_n \int_0^\infty |q_n(t+s) - q_n(t)|\,dt = 0$ for all $s \geq 0$ and there exists $r \in L^1_{\mathrm{loc}}[0,\infty)$ such that $\sup_n q_n(t) \leq r(t)$ for almost every $t \geq 0$, where $r \in L^1_{\mathrm{loc}}[0,\infty)$ means a restriction of r on $[0,s]$ belongs to $L^1[0,s]$ for each $s > 0$. Let $\{x_n\}$ be the sequence defined by*

$$x_1 = x, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n) \int_0^\infty q_n(t) T(t) x_n \, dt$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

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(S. Atsushiba) Department of Mathematics, Shibaura Institute of Technology, Fukasaku, Minuma-ku, Saitama-City, Saitama 337–8570, Japan
E-mail address: atusiba@sic.shibaura-it.ac.jp

(W. Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, O-okayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp