A CHARACTERIZATION OF THE HARMONIC OPERATOR MEAN AS AN EXTENSION OF ANDO'S THEOREM

Jun Ichi Fujii* and Masahiro Nakamura**

Received December 12, 2005

Abstract. We show that the (weighted) harmonic operator mean is characterized as an operator mean m satisfying $F(AmB) \leq F(A)mF(B)$ for every operator monotone function F on $(0, \infty)$ based on the numerical means. We also show the non-affine representing function $f_m(x) = 1\,m\,x$ of an operator mean m is an extreme point of the set of representing functions F with $F \circ f_m \lneqq f_m \circ F$.

1 Introduction. Let us consider the arithmetic operator mean $A\nabla B = (A + B)/2$ for a pair of positive (invertible) operators A and B acting on a Hilbert space H. Then a real function F is *operator concave* if

$$F(A \nabla B) \geq F(A) \nabla F(B)$$

holds. It is known that every operator monotone function f on $(0, \infty)$ satisfying

$$f(A) \leq f(B) \qquad \text{whenever} \quad 0 \leq A \leq B$$

is operator concave. The harmonic operator mean ! is defined by

$$A\ !\ B = \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1}$$

and T.Ando [1, Theorem III.5] showed the contrastive result to the above:

Theorem (Ando)**1.** *If F is positive operator monotone, then*

$$F(A\ !\ B) \leq F(A)\ !\ F(B).$$

In this note, based on this inequality, we discuss when

$$F(A\ m\ B) \leq F(A)\ m\ F(B)$$

holds not only for numerical means but also operator means in the sense of Kubo and Ando [5] which can be constructed as

$$A\,m\,B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} \tag{1}$$

for a positive operator monotone function f on $(0, \infty)$ with $f(1) = 1$.

2000 *Mathematics Subject Classification.* 47A64, 47A63, 26A51, 26D07.
Key words and phrases. operator mean, operator monotone function, concave function.

2 Numerical mean. Let $M(a,b)$ be a positive homogeneous mean for positive numbers a and b. According to the Kubo-Ando theory [5], the operations $M^\circ(a,b) = M(b,a)$ and $M^*(a,b) = M(1/a, 1/b)^{-1}$ are called the *transpose* and *adjoint* for M respectively.

The symbols ∇_w, $\#_w$ and $!_w$ denote the arithmetic, geometric and harmonic means respectively for $0 < w < 1$:

$$\nabla_w(a,b) = (1-w)a + wb, \quad \#_w(a,b) = a^{1-w}b^w \quad \text{and} \quad !_w(a,b) = \frac{ab}{wa + (1-w)b}.$$

Then

$$\nabla_w^* = !_w, \quad !_w^* = \nabla_w \quad \text{and} \quad \#_w^* = \#_w$$

and these means are all symmetric for $w = 1/2$, i.e., $M^\circ = M$.

These operations * and $^\circ$ are also applied to the *representing function* $f_M(x) = M(1,x)$ for M:

$$f^*(x) = \frac{1}{f\left(\frac{1}{x}\right)} \quad \text{and} \quad f^\circ(x) = xf\left(\frac{1}{x}\right).$$

Note that the normalized condition $M(a,a) = a$ is equivalent to $f_M(1) = 1$. By homogeneity, such means are reconstructed by the representing functions:

$$M(a,b) = af_M(b/a) = bf_M^\circ(a/b).$$

Here we assume that f_M is positive, monotone-increasing and concave. Then so is f_M°. In fact, it is clear that f_M° is positive and monotone-increasing. The concavity follows from

$$\begin{aligned} f_M^\circ((1-w)x + wy) &= ((1-w)x + wy)f_M\left(\frac{1}{(1-w)x + wy}\right) \\ &= ((1-w)x + wy)f_M\left(\frac{(1-w)x\frac{1}{x} + wy\frac{1}{y}}{(1-w)x + wy}\right) \\ &\geqq (1-w)xf_M\left(\frac{1}{x}\right) + wyf_M\left(\frac{1}{y}\right) = (1-w)f_M^\circ(x) + wf_M^\circ(y). \end{aligned}$$

The adjoint f_M^* is also positive and monotone-increasing, but it is not always concave as in the following example:

Example 1. Put $F(x) = \sqrt{x} \wedge x$. Then $F(1/x) = (1/\sqrt{x}) \wedge (1/x)$ and hence

$$F^*(x) = \sqrt{x} \vee x,$$

which is not concave in a neighborhood of 1.

Moreover the concavity of F^* is equivalent to Ando's type theorem:

Lemma 2.1. *Let F be a positive monotone-increasing concave function on $(0,\infty)$. Then F^* is concave if and only if*

$$F(!_w(a,b)) \leqq !_w(F(a), F(b))$$

for all $a, b > 0$.

Proof. The concavity of F^* is written by

$$\frac{1}{F\left(\frac{1}{(1-w)x+wy}\right)} = F^*((1-w)x+wy) \geqq (1-w)F^*(x)+wF^*(y) = (1-w)\frac{1}{F\left(\frac{1}{x}\right)}+w\frac{1}{F\left(\frac{1}{y}\right)}.$$

By putting $a = 1/x$ and $b = 1/y$, it is equivalent to

$$F(!_w(a,b)) = F\left(\frac{1}{(1-w)x+wy}\right) \leqq \frac{F(a)F(b)}{(1-w)F(b)+wF(a)} = !_w(F(a),F(b)).$$

Thus the equivalence is shown. □

Here we restrict ourselves to the homogeneous numerical means M with the representing functions f_M satisfying

- **(i)** f_M, f_M^* and f_M° are positive monotone-increasing concave functions.
- **(ii)** f_M is normalized: $f_M(1) = 1$ (i.e., $M(a,a) = a$).

Note that (i) implies that the above means do not include trivial means: $M_\ell(a,b) = a$ and $M_r(a,b) = b$.

Next we consider when

$$F(M(a,b)) \leqq M(F(a),F(b)) \tag{2}$$

holds. Note that it is equivalent to

$$F^*(M^*(a,b)) \geqq M^*(F^*(a),F^*(b)). \tag{2'}$$

In spite of the above situation, it holds for a special pair of a mean $M \neq !_w$ and a function F. In fact, putting $F(x) = \sqrt{x}$ and $M(a,b) = \sqrt{ab}$, the geometric mean. Then

$$F(M(a,b)) = \sqrt[4]{ab} = M(F(a),F(b)).$$

But, considering the case that F^* is affine, we can characterize the harmonic mean in such means, which is an extension of Ando's theorem:

Theorem 2.2. *A homogeneous mean M in the above sense is the harmonic one if and only if*

$$F(M(a,b)) \leqq M(F(a),F(b))$$

for all positive monotone-increasing concave functions F on $(0,\infty)$ with the concave adjoint F^* and positive numbers a and b.

Proof. It follows from the above lemma that (2) holds for $M = !_w$. Suppose $M \neq !_w$ and (2) holds. Then $M^* \neq \nabla_w$, so that there exists x with

$$\frac{M^*(1,1)+M^*(1,x)}{2} = \frac{1+f_M^*(x)}{2} < f_M^*\left(\frac{1+x}{2}\right) = M^*\left(1,\frac{1+x}{2}\right).$$

Applying $F^*(x) = (1+x)/2$ to $(2')$, we have

$$\frac{1+f_M^*(x)}{2} = F\left(M^*(1,x)\right) \geqq M\left(1,\frac{1+x}{2}\right) = f_M\left(\frac{1+x}{2}\right).$$

This contradiction shows $M^* = \nabla_w$, that is, $M = !_w$. □

3 Operator mean. Next we discuss the case of operators. The harmonic operator mean with a weight w is defined by

$$A\ !_w B = ((1-w)A^{-1} + wB^{-1})^{-1}$$

and the arithmetic one with w is $A\nabla_w B = (1-w)A + wB$ for positive invertible operators A and B on a Hilbert space. In general, operator means here stand for the Kubo-Ando operator means defined by (1). Note that the representing function $f_m(x) = 1\, m\, x$ of a nontrivial operator mean m is a positive monotone-increasing concave function and so are f_m^* and f_m°. Now we have a characterization of the harmonic operator mean $!_w$:

Theorem 3.1. *A nontrivial operator mean m is the (weighted) harmonic (resp. arithmetic) one if and only if*

$$F(A\ m\ B) \le F(A)\ m\ F(B) \qquad \Big(\textit{resp. } F(A\ m\ B) \ge F(A)\ m\ F(B)\Big) \tag{3}$$

for all positive operator monotone functions F and positive operators A and B.

Finally we observe noncommutative examples. As we state above, for commuting operators A and B, we have

$$\sqrt{A \# B} = \sqrt[4]{AB} = \sqrt{A} \# \sqrt{B},$$

where # is the geometric operator mean

$$A \# B = A^{1/2}\sqrt{A^{-1/2}BA^{-1/2}}A^{1/2}.$$

But it does not hold in general and moreover we can give examples:

$$\sqrt{A \# B} \le \sqrt{A} \# \sqrt{B} \quad \text{and} \quad \sqrt{C \# B} \ge \sqrt{C} \# \sqrt{B}.$$

Recall the following formula in [2]:

$$S = \begin{pmatrix} x & \overline{y} \\ y & z \end{pmatrix},\ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{imply} \quad S \# P = \sqrt{\frac{xz - |y|^2}{z}}P.$$

Put

$$S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ A = S_1^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad B = P.$$

Then $S_1 \#\ B = P$ and $S_1^2 \#\ P = \frac{1}{\sqrt{2}}P$, and hence

$$\sqrt{A \# B} = \sqrt{S_1^2 \#\ B} = \frac{1}{\sqrt[4]{2}}P \le P = S_1 \#\ P = \sqrt{A} \# \sqrt{B}.$$

Next, put

$$S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ C = S_2^2 = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad \text{and} \quad B = P.$$

Then we have $S_2 \#\ P = \sqrt{\frac{3}{2}}P$ and $S_2^2 \#\ P = \sqrt{\frac{16}{5}}P$, so that

$$\sqrt{C \# B} = \sqrt{S_2^2 \#\ P} = \frac{2}{\sqrt[4]{5}}P \ge \sqrt{\frac{3}{2}}P = S_2 \#\ P = \sqrt{C} \# \sqrt{B}.$$

These examples show the difficulty to discuss the class of functions F satisfying (3). So we discuss another related class in the next section.

4 A class of functions. Finally we consider the set MF of all representing functions of operator means, that is, positive operator monotone functions f_m on $(0, \infty)$ with $f_m(1) = 1$. Note that $!_w$ and ∇_w belong to the boundary of MF, which corresponds to [3, Theorem 8] and [5, Theorem 4.5]:

Theorem 4.1. *The weighted arithmetic means* ∇_w (*resp. harmonic ones* $!_w$) *are the largest* (*resp. smallest*) *operator means whose representing functions satisfy* $f'_m(1) = w$.

Proof. Every representing function f_m is concave and differentiable, so we have

$$f_m(x) \leqq f'_m(1)(x-1) + f_m(1) = 1 - f'_m(1) + f'_m(1)x = f_{\nabla_w}(x)$$

for all $x > 0$, which shows $m \leq \nabla_w$. Therefore $!_w = \nabla_w^*$ are the smallest since $m \leq n$ implies $m^* \geq n^*$ for all operator means m and n. □

Let $S(m) = S(f_m)$ be the set of all $F \in MF$ satisfying

$$F \circ f_m \leq f_m \circ F, \tag{4}$$

which is derived from the case $A = 1$ in (3):

$$F(f_m(B)) = F(1\, m\, B) \leq F(1)\, m\, F(B) = 1\, m\, F(B) = f_m(F(B)).$$

Then $S(m)$ is a closed convex subset of MF with the maximal extreme points ∇_w by the above theorem. Since the equality in (4) holds, we have f_m itself belongs to $S(m)$. This suggests that m occupies an extremal position in $S(m)$. The above argument shows that $S(\,!_w)$ coincides with MF and $S(\nabla_w) = \{f_{\nabla_w} | 0 < w < 1\}$. In other words, by Theorem 4, the smallest class of $S(\,!_w)$ and $S(\nabla_w)$ is $\{f_{!_w}\}$ and $\{f_{\nabla_w}\}$ respectively. In particular, these means are extreme points of MF. Moreover it is valid in general, which is another variation of Ando's theorem:

Theorem 4.2. *If f_m be the non-affine representing function for an operator mean m, then it is an extreme point of $S(m)$:*

$$f_m \in \text{ext } S(m).$$

Proof. Let $(F_1 + F_2)/2 = f_m$ for $F_k \in S(m)$. Then, putting $y = f_m(x)$ for each $x > 0$, we have

$$\begin{aligned} f_m(y) &= \frac{F_1 + F_2}{2}(y) = \frac{F_1 + F_2}{2}(f_m(x)) = \frac{F_1(f_m(x)) + F_2(f_m(x))}{2} \\ &\leqq \frac{f_m(F_1(x)) + f_m(F_2(x))}{2} \leqq f_m\left(\frac{F_1(x) + F_2(x)}{2}\right) = f_m(f_m(x)) = f_m(y). \end{aligned}$$

Therefore, the equality holds and hence $F_1(x) = F_2(x)$ by the strict concavity of f_m. Consequently $F_1 = F_2 = f_m$, which implies $f_m \in \text{ext } S(m)$. □

Moreover we conjecture that f_m is a minimal function for $S(m)$, that is, for all totally ordered path of representing functions f_{m_r} passing through f_m, (see [4])

$$f_m = \min\{f_{m_r} \mid f_{m_r} \in S(m)\}.$$

Though it is valid for $m = !_w$ and ∇_w, it is an open problem in general.

Recall that for the power mean $m_{r,w}$ for $|r| \leqq 1$, the representing function

$$f_{m_{r,w}}(x) = (1 - w + wx^r)^{1/r},$$

is operator monotone and hence the representing one of an operator mean. For a fixed weight w, it is monotone increasing for r (while the power operator mean $Am_{r,w}B$ is not always monotone increasing in the usual order for operators). For $r \to 0$, we obtain the geometric operator mean $\#_w$ with a weight w:

$$A \#_w B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^w A^{1/2}.$$

Now we can verify that the representing function $f_{\#_w}(x) = x^w$ is the smallest one in the power ones in $S(\#_w)$. In fact, the monotonicity of power means shows

$$\left(1-w+wx^{-r}\right)^{-1/r} \leqq \left(1-w+wx^{-wr}\right)^{-1/(wr)} \leqq (1-w+wx^{wr})^{1/(wr)} \leqq (1-w+wx^r)^{1/r}$$

for all $0 < r \leqq 1$. This is equivalent to

$$\left(1-w+wx^{-r}\right)^{-w/r} \leqq \left(1-w+wx^{-wr}\right)^{-1/r} \leqq (1-w+wx^{wr})^{1/r} \leqq (1-w+wx^r)^{w/r},$$

which shows $f_{\#_w} = \min\{f_{m_{r,w}} \mid f_{m_{r,w}} \in S(\#_w)\}$.

References

[1] T.Ando: "Topics on operator inequalities", Hokkaido Univ. Lecture Note, 1978.

[2] J.I.Fujii: *Arithmetico-geometric mean of operators*, Math. Japon., **23**(1978), 667–669.

[3] J.I.Fujii and M.Fujii: *Some remarks on operator means*, Math. Japon., **24**(1979), 335-339.

[4] J.I.Fujii, M.Nakamura and S.-E.Takahasi: *Cooper's approach to chaotic operator means*, to appear in Sci. Math. Japon..

[5] F.Kubo and T.Ando: *Means of positive linear operators*, Math. Ann., **248** (1980) 205–224.

* Department of Arts and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address : fujii@cc.osaka-kyoiku.ac.jp

** Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.