

## FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN MODULAR FUNCTION SPACES\*

S. DHOMPONGSA, T. DOMÍNGUEZ BENAVIDES, A. KAEWCHAROEN, AND  
B. PANYANAK<sup>1</sup>

Received November 30, 2005

ABSTRACT. The purpose of this paper is to study the existence of fixed points for multivalued nonexpansive mappings in modular function spaces. We apply our main result to obtain fixed point theorems for multivalued mappings in the Banach spaces  $L_1$  and  $l_1$ .

### 1. INTRODUCTION

The theory of modular spaces was initiated by Nakano [15] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [2, 3, 7, 8]). In particular, some fixed point theorems for (singlevalued) nonexpansive mappings in modular function spaces are given in [8]. In 1969, Nadler [14] established the multivalued version of Banach's contraction principle in metric spaces. Since then the metric fixed point theory for multivalued mappings has been rapidly developed and many of papers have appeared proving the existence of fixed points for multivalued nonexpansive mappings in special classes of Banach spaces (see, for instance, [4, 5, 9, 11]). In this paper, we study similar problems in the setting of modular function spaces. Namely, we prove that every  $\rho$ -contraction  $T : C \rightarrow F_\rho(C)$  has a fixed point where  $\rho$  is a convex function modular satisfying the  $\Delta_2$ -type condition and  $C$  is a nonempty  $\rho$ -bounded  $\rho$ -closed subset of  $L_\rho$ . By using this result, we can assert the existence of fixed points for multivalued  $\rho$ -nonexpansive mappings. Finally, we apply our main result to obtain fixed point theorems in the Banach space  $L_1$  (resp.  $l_1$ ) for multivalued mappings whose domains are compact in the topology of the convergence locally in measure (resp.  $w^*$ -topology).

### 2. PRELIMINARIES

We start by recalling some basic concepts about modular function spaces. For more details the reader is referred to [10, 12].

Let  $\Omega$  be a nonempty set and  $\Sigma$  be a nontrivial  $\sigma$ -algebra of subsets of  $\Omega$ . Let  $\mathcal{P}$  be a  $\delta$ -ring of subsets of  $\Omega$ , such that  $E \cap A \in \mathcal{P}$  for any  $E \in \mathcal{P}$  and  $A \in \Sigma$ . Let us assume that there exists an increasing sequence of sets  $K_n \in \mathcal{P}$  such that  $\Omega = \bigcup K_n$  (for instance,  $\mathcal{P}$  can be the class of sets of finite measure in a  $\sigma$ -finite measure space). By  $\mathcal{E}$  we denote the

---

2000 Mathematics Subject Classification. Primary 46E30; Secondary 47H09, 47H10.

Key words and phrases : Multivalued mappings, fixed points, Modular function spaces.

<sup>1</sup>Corresponding author.

\* Supported by Thailand Research Fund under grant BRG4780013. The second author was partially supported by DGES, Grant D.G.E.S. REF. PBMF2003-03893-C02-C01 and Junta de Andalucia, Grant FQM-127. The third and fourth authors were supported by the Royal Golden Jubilee program under grant PHD/0250/2545 and PHD/0251/2545, respectively.

linear space of all simple functions with supports from  $\mathcal{P}$ . By  $\mathcal{M}$  we will denote the space of all measurable functions, i.e., all functions  $f : \Omega \rightarrow \mathbb{R}$  such that there exists a sequence  $\{g_n\} \in \mathcal{E}$ ,  $|g_n| \leq |f|$ , and  $g_n(\omega) \rightarrow f(\omega)$  for all  $\omega \in \Omega$ .

Let us recall that a set function  $\mu : \Sigma \rightarrow [0, \infty]$  is called a  $\sigma$ -subadditive measure if  $\mu(\emptyset) = 0$ ,  $\mu(A) \leq \mu(B)$  for any  $A \subset B$  and  $\mu(\bigcup A_n) \leq \sum \mu(A_n)$  for any sequence of sets  $\{A_n\} \subset \Sigma$ . By  $1_A$  we denote the characteristic function of the set  $A$ .

**Definition 2.1.** A functional  $\rho : \mathcal{E} \times \Sigma \rightarrow [0, \infty]$  is called a function modular if :

- (P<sub>1</sub>)  $\rho(0, E) = 0$  for any  $E \in \Sigma$ ,
- (P<sub>2</sub>)  $\rho(f, E) \leq \rho(g, E)$  whenever  $|f(\omega)| \leq |g(\omega)|$  for any  $\omega \in \Omega$ ,  $f, g \in \mathcal{E}$ , and  $E \in \Sigma$ ,
- (P<sub>3</sub>)  $\rho(f, \cdot) : \Sigma \rightarrow [0, \infty]$  is a  $\sigma$ -subadditive measure for every  $f \in \mathcal{E}$ ,
- (P<sub>4</sub>)  $\rho(\alpha, A) \rightarrow 0$  as  $\alpha$  decreases to 0 for every  $A \in \mathcal{P}$ , where  $\rho(\alpha, A) = \rho(\alpha 1_A, A)$ ,
- (P<sub>5</sub>) if there exists  $\alpha > 0$  such that  $\rho(\alpha, A) = 0$ , then  $\rho(\beta, A) = 0$  for every  $\beta > 0$ ,
- (P<sub>6</sub>) for any  $\alpha > 0$ ,  $\rho(\alpha, \cdot)$  is order continuous on  $\mathcal{P}$ , that is,  $\rho(\alpha, A_n) \rightarrow 0$  if  $\{A_n\} \subset \mathcal{P}$  and decreases to  $\emptyset$ .

A  $\sigma$ -subadditive measure  $\rho$  is said to be additive if  $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$  whenever  $A, B \in \Sigma$  such that  $A \cap B = \emptyset$  and  $f \in \mathcal{M}$ .

The definition of  $\rho$  is then extended to  $f \in \mathcal{M}$  by

$$\rho(f, E) = \sup \{ \rho(g, E) : g \in \mathcal{E}, |g(\omega)| \leq |f(\omega)| \text{ for every } \omega \in \Omega \}.$$

**Definition 2.2.** A set  $E$  is said to be  $\rho$ -null if  $\rho(\alpha, E) = 0$  for every  $\alpha > 0$ . A property  $p(\omega)$  is said to hold  $\rho$ -almost everywhere ( $\rho$ -a.e.) if the set  $\{\omega \in \Omega : p(\omega) \text{ does not hold}\}$  is  $\rho$ -null. For example, we will say frequently  $f_n \rightarrow f$   $\rho$ -a.e.

Note that a countable union of  $\rho$ -null sets is still  $\rho$ -null. In the sequel we will identify sets  $A$  and  $B$  whose symmetric difference  $A \Delta B$  is  $\rho$ -null, similarly we will identify measurable functions which differ only on a  $\rho$ -null set.

In the above condition, we define the function  $\rho : \mathcal{M} \rightarrow [0, \infty]$  by  $\rho(f) = \rho(f, \Omega)$ . We know from [10] that  $\rho$  satisfies the following properties :

- (i)  $\rho(f) = 0$  if and only if  $f = 0$   $\rho$ -a.e.
  - (ii)  $\rho(\alpha f) = \rho(f)$  for every scalar  $\alpha$  with  $|\alpha| = 1$  and  $f \in \mathcal{M}$ .
  - (iii)  $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$  if  $\alpha + \beta = 1$ ,  $\alpha, \beta \geq 0$  and  $f, g \in \mathcal{M}$ .
- In addition, if the following property is satisfied
- (iii)'  $\rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g)$  if  $\alpha + \beta = 1$ ,  $\alpha, \beta \geq 0$  and  $f, g \in \mathcal{M}$ ,
- we say that  $\rho$  is a convex modular.

A function modular  $\rho$  is called  $\sigma$ -finite if there exists an increasing sequence of sets  $K_n \in \mathcal{P}$  such that  $0 < \rho(1_{K_n}) < \infty$  and  $\Omega = \bigcup K_n$ .

The modular  $\rho$  defines a corresponding modular space  $L_\rho$ , which is given by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$

In general the modular  $\rho$  is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an  $F$ -norm. Recall that a functional  $\|\cdot\| : X \rightarrow [0, \infty]$  defines an  $F$ -norm on a linear space  $X$  if and only if

- (1)  $\|x\| = 0$  if and only if  $x = 0$ ,
- (2)  $\|\alpha x\| = \|x\|$  whenever  $|\alpha| = 1$ ,
- (3)  $\|x + y\| \leq \|x\| + \|y\|$ ,
- (4)  $\|\alpha_n x_n - \alpha x\| \rightarrow 0$  if  $\alpha_n \rightarrow \alpha$  and  $\|x_n - x\| \rightarrow 0$ .

The modular space  $L_\rho$  can be equipped with an  $F$ -norm defined by

$$\|f\|_\rho = \inf \left\{ \alpha > 0 : \rho \left( \frac{f}{\alpha} \right) \leq \alpha \right\}.$$

We know from [10] that the linear space  $(L_\rho, \|\cdot\|_\rho)$  is a complete metric space. If  $\rho$  is convex, the formula

$$\|f\|_l = \inf \left\{ \alpha > 0 : \rho \left( \frac{f}{\alpha} \right) \leq 1 \right\}$$

defines a norm which is frequently called the Luxemburg norm. The formula

$$\|f\|_a = \inf \left\{ \frac{1}{k} (1 + \rho(kf)) : k > 0 \right\}$$

defines a different norm which is called the Amemiya norm. Moreover,  $\|\cdot\|_l$  and  $\|\cdot\|_a$  are equivalent norms. We can also consider the space

$$E_\rho = \{f \in \mathcal{M} : \rho(\alpha f, \cdot) \text{ is order continuous for all } \alpha > 0\}.$$

**Definition 2.3.** A function modular  $\rho$  is said to satisfy the  $\Delta_2$ -condition if

$$\sup_{n \geq 1} \rho(2f_n, D_k) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ whenever } \{f_n\} \subset \mathcal{M}, D_k \in \Sigma$$

$$\text{decreases to } \emptyset \text{ and } \sup_{n \geq 1} \rho(f_n, D_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

It is known that the  $\Delta_2$ -condition is equivalent to  $E_\rho = L_\rho$ .

**Definition 2.4.** A function modular  $\rho$  is said to satisfy the  $\Delta_2$ -type condition if there exists  $K > 0$  such that for any  $f \in L_\rho$  we have  $\rho(2f) \leq K\rho(f)$ .

In general, the  $\Delta_2$ -type condition and  $\Delta_2$ -condition are not equivalent, even though it is obvious that the  $\Delta_2$ -type condition implies the  $\Delta_2$ -condition.

**Definition 2.5.** Let  $L_\rho$  be a modular space.

- (1) The sequence  $\{f_n\} \subset L_\rho$  is said to be  $\rho$ -convergent to  $f \in L_\rho$  if  $\rho(f_n - f) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (2) The sequence  $\{f_n\} \subset L_\rho$  is said to be  $\rho$ -a.e. convergent to  $f \in L_\rho$  if the set  $\{\omega \in \Omega : f_n(\omega) \not\rightarrow f(\omega)\}$  is  $\rho$ -null.
- (3) A subset  $C$  of  $L_\rho$  is called  $\rho$ -closed if the  $\rho$ -limit of a  $\rho$ -convergent sequence of  $C$  always belongs to  $C$ .
- (4) A subset  $C$  of  $L_\rho$  is called  $\rho$ -a.e. closed if the  $\rho$ -a.e. limit of a  $\rho$ -a.e. convergent sequence of  $C$  always belongs to  $C$ .
- (5) A subset  $C$  of  $L_\rho$  is called  $\rho$ -compact if every sequence in  $C$  has a  $\rho$ -convergent subsequence in  $C$ .
- (6) A subset  $C$  of  $L_\rho$  is called  $\rho$ -a.e. compact if every sequence in  $C$  has a  $\rho$ -a.e. convergent subsequence in  $C$ .
- (7) A subset  $C$  of  $L_\rho$  is called  $\rho$ -bounded if

$$\text{diam}_\rho(C) = \sup\{\rho(f - g) : f, g \in C\} < \infty.$$

We know by [10] that under the  $\Delta_2$ -condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the  $\Delta_2$ -type condition. In the sequel we will assume that the modular function  $\rho$  is convex and satisfies the  $\Delta_2$ -type condition.

**Definition 2.6.** Let  $\rho$  be as above. We define a growth function  $\omega$  by

$$\omega(t) = \sup \left\{ \frac{\rho(tf)}{\rho(f)} : f \in L_\rho, 0 < \rho(f) < \infty \right\} \text{ for all } 0 \leq t < \infty.$$

The following properties of the growth function can be found in [3].

**Lemma 2.7.** *Let  $\rho$  be as above. Then the growth function  $\omega$  has the following properties :*

- (1)  $\omega(t) < \infty, \forall t \in [0, \infty)$ .
- (2)  $\omega : [0, \infty) \rightarrow [0, \infty)$  is a convex, strictly increasing function. So, it is continuous.
- (3)  $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$ .
- (4)  $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$ , where  $\omega^{-1}$  is the function inverse of  $\omega$ .

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

**Lemma 2.8** (T. Domínguez Benavides et al. [3]). *Let  $\rho$  be as above. Then*

$$\|f\|_\rho \leq \frac{1}{\omega^{-1}(1/\rho(f))} \text{ whenever } f \in L_\rho \setminus \{0\}.$$

The following lemma is a technical lemma which will be need because of lack of the triangular inequality.

**Lemma 2.9** (T. Domínguez Benavides et al. [3]). *Let  $\rho$  be as above,  $\{f_n\}$  and  $\{g_n\}$  be two sequences in  $L_\rho$ . Then*

$$\lim_{n \rightarrow \infty} \rho(g_n) = 0 \implies \limsup_{n \rightarrow \infty} \rho(f_n + g_n) = \limsup_{n \rightarrow \infty} \rho(f_n)$$

and

$$\lim_{n \rightarrow \infty} \rho(g_n) = 0 \implies \liminf_{n \rightarrow \infty} \rho(f_n + g_n) = \liminf_{n \rightarrow \infty} \rho(f_n).$$

In the same way as the Hausdorff distance defined on the family of bounded closed subsets of a metric space, we can define the analogue to the Hausdorff distance for modular function spaces. We will call  $\rho$ -Hausdorff distance even though it is not a metric.

**Definition 2.10.** Let  $C$  be a nonempty subset of  $L_\rho$ . We shall denote by  $F_\rho(C)$  the family of nonempty  $\rho$ -closed subsets of  $C$  and by  $K_\rho(C)$  the family of nonempty  $\rho$ -compact subsets of  $C$ . Let  $H_\rho(\cdot, \cdot)$  be the  $\rho$ -Hausdorff distance on  $F_\rho(L_\rho)$ , i.e.,

$$H_\rho(A, B) = \max \left\{ \sup_{f \in A} \text{dist}_\rho(f, B), \sup_{g \in B} \text{dist}_\rho(g, A) \right\}, \quad A, B \in F_\rho(L_\rho),$$

where  $\text{dist}_\rho(f, B) = \inf \{ \rho(f - g) : g \in B \}$  is the  $\rho$ -distance between  $f$  and  $B$ . A multivalued mapping  $T : C \rightarrow F_\rho(L_\rho)$  is said to be a  $\rho$ -contraction if there exists a constant  $k \in [0, 1)$  such that

$$(2.1) \quad H_\rho(Tf, Tg) \leq k\rho(f - g), \quad f, g \in C.$$

If (2.1) is valid when  $k = 1$ , then  $T$  is called  $\rho$ -nonexpansive. A function  $f \in C$  is called a fixed point for a multivalued mapping  $T$  if  $f \in Tf$ .

### 3. MAIN RESULTS

We begin stating the Banach Contraction Principle for multivalued mappings in modular function spaces.

**Theorem 3.1.** *Let  $\rho$  be a convex function modular satisfying the  $\Delta_2$ -type condition,  $C$  a nonempty  $\rho$ -bounded  $\rho$ -closed subset of  $L_\rho$ , and  $T : C \rightarrow F_\rho(C)$  a  $\rho$ -contraction mapping, i.e., there exists a constant  $k \in [0, 1)$  such that*

$$(3.1) \quad H_\rho(Tf, Tg) \leq k\rho(f - g), \quad f, g \in C.$$

*Then  $T$  has a fixed point.*

**Proof.** Let  $f_0 \in C$  and  $\alpha \in (k, 1)$ . Since  $Tf_0$  is nonempty, there exists  $f_1 \in Tf_0$  such that  $\rho(f_0 - f_1) > 0$  (otherwise  $f_0$  is a fixed point of  $T$ ). In view of (3.1), we have

$$\text{dist}_\rho(f_1, Tf_1) \leq H_\rho(Tf_0, Tf_1) \leq k\rho(f_0 - f_1) < \alpha\rho(f_0 - f_1).$$

Since  $\text{dist}_\rho(f_1, Tf_1) = \inf\{\rho(f_1 - g) : g \in Tf_1\}$ , it follows that there exists  $f_2 \in Tf_1$  such that

$$\rho(f_1 - f_2) < \alpha\rho(f_0 - f_1).$$

Similarly, there exists  $f_3 \in Tf_2$  such that

$$\rho(f_2 - f_3) < \alpha\rho(f_1 - f_2).$$

Continuing in this way, there exists a sequence  $\{f_n\}$  in  $C$  satisfying  $f_{n+1} \in Tf_n$  and

$$\begin{aligned} \rho(f_n - f_{n+1}) &< \alpha\rho(f_{n-1} - f_n) \\ &< \alpha^2(\rho(f_{n-2} - f_{n-1})) \\ &< \dots \\ &< \alpha^{n-1}(\rho(f_1 - f_2)) \\ &< \alpha^n(\rho(f_0 - f_1)) \\ &\leq \alpha^n \text{diam}_\rho(C). \end{aligned}$$

Let  $M = \text{diam}_\rho(C)$ , then

$$\frac{1}{\alpha^n M} < \frac{1}{\rho(f_n - f_{n+1})}.$$

By Lemma 2.7, we have

$$\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right) < \omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right).$$

It follows that

$$\frac{1}{\omega^{-1}\left(\frac{1}{\rho(f_n - f_{n+1})}\right)} < \frac{1}{\left(\omega^{-1}\left(\frac{1}{\alpha}\right)\right)^n \omega^{-1}\left(\frac{1}{M}\right)}.$$

By Lemma 2.8, we obtain

$$\|f_n - f_{n+1}\|_\rho < \left(\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)}\right)^n \cdot \frac{1}{\omega^{-1}\left(\frac{1}{M}\right)}.$$

Since  $\omega^{-1}$  is strictly increasing, we have  $\frac{1}{\omega^{-1}\left(\frac{1}{\alpha}\right)} < 1$ . This implies that  $\{f_n\}$  is a Cauchy sequence in  $(L_\rho, \|\cdot\|_\rho)$ . Since  $(L_\rho, \|\cdot\|_\rho)$  is a complete metric space, there exists  $f \in L_\rho$  such that  $\{f_n\}$  is  $\|\cdot\|_\rho$ -convergent to  $f$ . Since under the  $\Delta_2$ -type condition, norm convergence and modular convergence are identical,  $\{f_n\}$  is  $\rho$ -convergent to  $f$  and  $f \in C$  because  $C$  is  $\rho$ -closed. Since  $f_n \in Tf_{n-1}$ , we have

$$(3.2) \quad \text{dist}_\rho(f_n, Tf) \leq H_\rho(Tf_{n-1}, Tf) \leq k\rho(f_{n-1} - f) \longrightarrow 0.$$

We observe that, for each  $n$ , there exists  $g_n \in Tf$  such that

$$(3.3) \quad \rho(f_n - g_n) \leq \text{dist}_\rho(f_n, Tf) + \frac{1}{n}.$$

Thus, (3.2) and (3.3) imply that  $\lim_{n \rightarrow \infty} \rho(f_n - g_n) = 0$ . By Lemma 2.9,

$$\limsup_{n \rightarrow \infty} \rho(g_n - f) = \limsup_{n \rightarrow \infty} \rho(g_n - f_n + f_n - f) = \limsup_{n \rightarrow \infty} \rho(f_n - f) = 0.$$

Since  $Tf$  is  $\rho$ -closed, we can conclude that  $f \in Tf$ . □

The following results will be very useful in the proof of our main theorem.

**Theorem 3.2** (M. A. Khamsi [7]). *Let  $\{f_n\} \subset L_\rho$  be  $\rho$ -a.e. convergent to 0. Assume there exists  $k > 1$  such that*

$$\sup_{n \geq 1} \rho(kf_n) = M < \infty.$$

Let  $g \in E_\rho$ , then we have

$$\liminf_{n \rightarrow \infty} \rho(f_n + g) = \liminf_{n \rightarrow \infty} \rho(f_n) + \rho(g).$$

The following lemma guarantees that every nonempty  $\rho$ -compact subset of  $L_\rho$  attains a nearest point.

**Lemma 3.3.** *Let  $\rho$  be a convex function modular satisfying the  $\Delta_2$ -type condition,  $f \in L_\rho$ , and  $K$  a nonempty  $\rho$ -compact subset of  $L_\rho$ . Then there exists  $g_0 \in K$  such that*

$$\rho(f - g_0) = \text{dist}_\rho(f, K).$$

**Proof.** Let  $m = \text{dist}_\rho(f, K)$ . For each  $n \in \mathbb{N}$ , there exists  $g_n \in K$  such that

$$m - \frac{1}{n} \leq \rho(f - g_n) \leq m + \frac{1}{n}.$$

By the  $\rho$ -compactness of  $K$ , we can assume, by passing through a subsequence, that  $g_n \xrightarrow{\rho} g_0 \in K$ . By Lemma 2.9, we obtain

$$\begin{aligned} m &= \limsup_{n \rightarrow \infty} \rho(g_n - f) = \limsup_{n \rightarrow \infty} \rho(g_n - g_0 + g_0 - f) \\ &= \limsup_{n \rightarrow \infty} \rho(g_0 - f) \\ &= \rho(g_0 - f). \end{aligned}$$

□

We can now state our main theorem.

**Theorem 3.4.** *Let  $\rho$  be a convex function modular satisfying the  $\Delta_2$ -type condition,  $C$  a nonempty  $\rho$ -a.e. compact  $\rho$ -bounded convex subset of  $L_\rho$ , and  $T : C \rightarrow K_\rho(C)$  a  $\rho$ -nonexpansive mapping. Then  $T$  has a fixed point.*

**Proof.** Fix  $f_0 \in C$ . For each  $n \in \mathbb{N}$ , the  $\rho$ -contraction  $T_n : C \rightarrow F_\rho(C)$  is defined by

$$T_n(f) = \frac{1}{n}f_0 + \left(1 - \frac{1}{n}\right)Tf, \quad f \in C.$$

By Theorem 3.1, we can conclude that  $T_n$  has a fixed point, say  $f_n$ . It is easy to see that

$$\text{dist}_\rho(f_n, Tf_n) \leq \frac{1}{n} \text{diam}_\rho(C) \rightarrow 0.$$

Because of  $\rho$ -a.e. compactness of  $C$ , we can assume, by passing through a subsequence, that  $f_n \xrightarrow{\rho\text{-a.e.}} f$  for some  $f \in C$ . By Lemma 3.3, for each  $n \in \mathbb{N}$ , there exists  $g_n \in Tf_n$  and  $h_n \in Tf$  such that

$$\rho(f_n - g_n) = \text{dist}_\rho(f_n, Tf_n)$$

and

$$\rho(g_n - h_n) = \text{dist}_\rho(g_n, Tf) \leq H_\rho(Tf_n, Tf) \leq \rho(f_n - f).$$

Because of  $\rho$ -compactness of  $Tf$ , we can assume, by passing through a subsequence, that  $h_n \xrightarrow{\rho} h \in Tf$ . Since  $\rho$  satisfies the  $\Delta_2$ -type condition, there exists  $K > 0$  such that

$\rho(2(f_n - f)) \leq K\rho(f_n - f)$  for all  $n \in \mathbb{N}$ .  
 This implies that

$$\sup_{n \geq 1} \rho(2(f_n - f)) \leq K \sup_{n \geq 1} \rho(f_n - f) < \infty.$$

By Theorem 3.2 and Lemma 2.9, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \rho(f_n - f) + \rho(f - h) &= \liminf_{n \rightarrow \infty} \rho(f_n - f + f - h) \\ &= \liminf_{n \rightarrow \infty} \rho(f_n - h) \\ &= \liminf_{n \rightarrow \infty} \rho(f_n - g_n + g_n - h_n + h_n - h) \\ &= \liminf_{n \rightarrow \infty} \rho(g_n - h_n) \\ &\leq \liminf_{n \rightarrow \infty} \rho(f_n - f). \end{aligned}$$

It follows that  $\rho(f - h) = 0$  and then we have  $f = h \in Tf$ . □

Consider the space  $L_p(\Omega, \mu)$  for a  $\sigma$ -finite measure  $\mu$  with the usual norm. Let  $C$  be a bounded closed convex subset of  $L_p$  for  $1 < p < \infty$  and  $T : C \rightarrow K(C)$  a multivalued nonexpansive mapping. Because of uniform convexity of  $L_p$ , it is known that  $T$  has a fixed point. For  $p = 1$ ,  $T$  can fail to have a fixed point even in the singlevalued case for a weakly compact convex set  $C$  (see [1]). However, since  $L_1$  is a modular space where  $\rho(f) = \int_{\Omega} |f| d\mu = \|f\|$  for all  $f \in L_1$ , Theorem 3.4 implies the existence of a fixed point when we define mappings on a  $\rho$ -a.e. compact  $\rho$ -bounded convex subset of  $L_1$ . Thus we can state :

**Corollary 3.5.** *Let  $(\Omega, \mu)$  be as above,  $C \subset L_1(\Omega, \mu)$  a nonempty bounded convex set which is compact for the topology of the convergence locally in measure, and  $T : C \rightarrow K(C)$  a nonexpansive mapping. Then  $T$  has a fixed point.*

**Proof.** Under the above hypothesis,  $\rho$ -a.e. compact sets and compact sets in the topology of the convergence locally in measure are identical (see [2]). Consequently, Theorem 3.4 can be applied to obtain a fixed point for  $T$ . □

In the case of the space  $l_1$  we also can obtain a bounded closed convex set  $C$  and a nonexpansive mapping  $T : C \rightarrow C$  which is fixed point free. Indeed, consider the following easy and well known example :  
 Let

$$C = \left\{ \{x_n\} \in l_1 : 0 \leq x_n \leq 1 \text{ and } \sum_{n=1}^{\infty} x_n = 1 \right\}.$$

Define a nonexpansive mapping  $T : C \rightarrow C$  by

$$T(x) = (0, x_1, x_2, x_3, \dots) \text{ where } x = \{x_n\}.$$

Then  $T$  is a fixed point free. However, if we consider  $L_{\rho} = l_1$  where  $\rho(x) = \|x\|$ ,  $\forall x \in l_1$ . Then  $\rho$ -a.e. convergence and  $w^*$ -convergence are identical on bounded subsets of  $l_1$  (see [3]). This fact leads us to obtain the following corollary :

**Corollary 3.6.** *Let  $C$  be a nonempty  $w^*$ -compact convex subset of  $l_1$  and  $T : C \rightarrow K(C)$  a nonexpansive mapping. Then  $T$  has a fixed point.*

**Proof.** By the above argument, we know that  $\rho$ -a.e. compact bounded sets and  $w^*$ -compact sets are identical. Then we can apply Theorem 3.4 to assert the existence of a fixed point of  $T$ . □

In fact Corollary 3.5 and 3.6 are consequences of a general result: Assume that  $X$  is a linear normed space and  $\tau$  is a Hausdorff topology on  $X$ . We say that  $X$  satisfies the strict  $\tau$ -Opial property if

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for each sequence  $\{x_n\}$  in  $X$  which converges to  $x$  for the topology  $\tau$  and each  $y \neq x$ . Following the same argument as in [11] it is easy to prove the following theorem:

**Theorem 3.7.** *Let  $X$  be a Banach space,  $C$  a convex bounded sequentially  $\tau$ -compact subset of  $X$ , and  $T : C \rightarrow K(C)$  a nonexpansive mapping. If  $X$  satisfies the strict  $\tau$ -Opial property, then  $T$  has a fixed point.*

When  $X$  is a modular function space equipped with either Luxemburg or Amemiya norm, we can consider the topology  $\tau$  of convergence  $\rho$ -a.e. In this case, Theorem 3.7 yields to the following:

**Theorem 3.8.** *Let  $\rho$  be a convex additive  $\sigma$ -finite function modular satisfying the  $\Delta_2$ -type condition. Assume that  $L_\rho$  is equipped either with Luxemburg or Amemiya norm. Let  $C$  be a nonempty  $\rho$ -a.e. compact  $\rho$ -bounded convex subset of  $L_\rho$ , and  $T : C \rightarrow K(C)$  a nonexpansive mapping. Then  $T$  has a fixed point.*

**Proof.** From [6] (Theorem 4.1 and 4.3),  $X$  satisfies the uniform Opial property with respect to the topology of  $\rho$ -a.e. convergence. Since  $\rho$ -a.e. compact sets and  $\rho$ -a.e. sequentially compact sets are identical for this topology (see [2]), we can deduce the result from Theorem 3.7  $\square$

**Remark.** In the case of the space  $L^1(\Omega)$  we have

$$\rho(f) = \int_{\Omega} |f| d\mu = \|f\|_1 = \|f\|_a$$

and we can deduce Corollary 3.5 and 3.6 from Theorem 3.8.

#### ACKNOWLEDGEMENT

This work was conducted while the third and fourth authors were visiting Universidad de Sevilla. We are very grateful to the Department of Mathematical Analysis and Professor T. Domínguez Benavides for the hospitality. We also would like to thank A. Kaewkhao for his helpful conversation during the preparation of the manuscript.

#### REFERENCES

1. D. E. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* **82** (1981), 423-424.
2. T. Domínguez Benavides, M. A. Khamsi and S. Samadi, Asymptotically nonexpansive mappings in modular function spaces, *J. Math. Anal. Appl.* **265** (2002), 249-263.
3. T. Domínguez Benavides, M. A. Khamsi and S. Samadi, Asymptotically regular mappings in modular function spaces, *Sci. Math. Jpn.* **53**, No. 2 (2001), 295-304, :e4, 239-248.
4. T. Domínguez Benavides and P. Lorenzo Ramírez, Fixed point theorems for multivalued nonexpansive mappings without uniform convexity, *Abs. Appl. Anal.* 2003:**6** (2003), 375-386.
5. T. Domínguez Benavides and P. Lorenzo Ramírez, Asymptotic centers and fixed points for multivalued nonexpansive mappings, *Annal. Univ. Marie Curie-Sklodowska.* **LVIII** (2004), 37-45.
6. M. A. Japón, Some geometrical properties in modular spaces and applications to fixed point theory, *J. Math. Anal. Appl.* **295** (2004), 576-594.
7. M. A. Khamsi, *Fixed point theory in modular function spaces*, Recent Advance on Metric Fixed Point Theory, pp. 31-58 (Universidad de Sevilla, Sevilla, 1996).
8. M. A. Khamsi, W. M. Kozłowski and S. Reich, Fixed point theory in modular function spaces, *Nonlinear Anal.* **14** (1990), 935-953.
9. W. A. Kirk and S. Massa, Remarks on asymptotic and Chebyshev centers, *Houston J. Math.* **16** (3) (1990), 357-364.



10. W. M. Kozłowski, *Modular function spaces* (Dekker, New York/Basel, 1988).
11. T. C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, *Bull. Amer. Math. Soc.* **80** (1974), 1123-1126.
12. J. Musielak, *Orlicz and Modular spaces*, Lecture Notes in Mathematics, Vol. **1034** (Springer, Berlin, 1983).
13. J. Musielak and W. Orlicz, On modular spaces, *Studia Math.* **18** (1959), 591-597.
14. S. B. Nadler, Jr., Multivalued contraction mappings, *Pacific J. Math.* **30** (1969), 475-488.
15. H. Nakano, *Modular semi-ordered spaces* (Tokyo, 1950).

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200,  
THAILAND

*E-mail address:* sompong@chiangmai.ac.th (S. Dhompongsa)

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, P.O. BOX 1160, 41080 SEVILLA,  
SPAIN

*E-mail address:* tomasd@us.es (T. Domínguez Benavides)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200,  
THAILAND

*E-mail address:* akaewcharoen@yahoo.com (A. Kaewcharoen)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200,  
THAILAND

*E-mail address:* g4565152@cm.edu (B. Panyanak)