NOTE ON THE BRONSHTEIN THEOREM CONCERNING HYPERBOLIC POLYNOMIALS

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Abstract. Using the argument of Bronshtein, we give proofs of some inequalities related to hyperbolic polynomials whose coefficients are not sufficiently smooth. We give also the correction of previously announced estimates.

1 Introduction We call a polynomial which has only real roots a hyperbolic polynomial. Consider a hyperbolic polynomial with parameter t

$$P(t,\tau) = \tau^M + \sum_{h=1}^{M} A_h(t)\tau^{M-h}.$$

We assume the multiplicity of roots is at most m. Bronshtein [2] showed Lipschitz continuity of roots with respect to t under the assumption that all the coefficients are C^m. Using this, Bronshtein [1] drew the estimate

$$\left|\frac{\partial_t P(t,\tau)}{P(t,\tau)}\right| \le C|\Im\tau|^{-1}, \quad 1 \ge |\Im\tau| > 0$$

and showed the Gevrey wellposedness of Cauchy problem. Using the idea of [1], Ohya-Tarama [3] considered Cauchy problem for a hyperbolic operator with coefficients that are κ-Hölder continuous with respect to the time variable. In the case where $2 \ge \kappa > 1$, a modified version of the estimate above under the assumption $m \ge 2$

$$\left|\frac{\partial_t P(t,\tau)}{P(t,\tau)}\right| \le C|\Im\tau|^{-\frac{m}{\kappa}}, \quad 1 \ge |\Im\tau| > 0,$$

(for the precise statement see Theorem 1.1 and the remark after Corollary 1.3), is used. Concerning the regularity of roots, we see that the roots are α-Hölder continuous with $\alpha = \min\{1, \kappa/m\}$ if the coefficients belong to C^κ. Although the proofs of these results have already been given by Wakabayashi [5] with some extension, we give here our proofs directly based on the idea of Bronshtein [2]. In the course of proof, we show also that the estimate above and Hölder continuity of roots are equivalent.

Theorem 1.1. *Let $T > 0$, m and M positive integers with $2 \le m \le M$, r_0 a positive integer and $\gamma \in (0,1]$. Let*

$$P(t,\tau) = \tau^M + \sum_{h=1}^{M} A_h(t)\tau^{M-h}$$

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be a hyperbolic polynomial with coefficients $A_h(t)$ $(h = 1, \ldots, M)$ in $C^{r_0,\gamma}([-T, T])$. We assume that the multiplicity of roots of $P(t, \tau)$ is at most m. Then for any $S \in (0, T)$ there exists a positive constant C such that for $j = 1, \ldots, \min\{m-1, r_0\}$

$$\left| \frac{\partial_t^j P(t,\tau)}{P(t,\tau)} \right| \le C|\Im\tau|^{-j\max\{1,\frac{m}{r_0+\gamma}\}} \tag{1.1}$$

when $\tau \in \mathbb{C}$ with $\Im\tau \in (0, 1]$ and $t \in [-S, S]$.

Here the constant C may depend on S but it can be chosen uniformly when coefficients $A_j(t)$ belong to a bounded set in $C^{r_0,\gamma}([-T, T])$ as long as $P(t, \tau)$ is hyperbolic and the multiplicity of roots is at most m uniformly. That is to say, in the case of $m < M$, let $P_\theta(t, \tau) = \tau^M + \sum_{h=1}^M A_{h,\theta}(t)\tau^{M-h}$ be a hyperbolic polynomial with a parameter $\theta \in \Theta$ where $\{A_{h,\theta}(t) \mid h = 1, \ldots, M, \ \ \theta \in \Theta\}$ is a bounded set in $C^{r_0,\gamma}([-T, T])$. Assume that, using the factorization $P_\theta(t, \tau) = \prod_{l=1}^M (\tau - \lambda_{l,\theta}(t))$ with $\lambda_{1,\theta}(t) \le \lambda_{2,\theta}(t) \le \cdots \le \lambda_{M,\theta}(t)$, we have for $l = 1, \ldots, M-m$

$$\lambda_{l+m,\theta}(t) - \lambda_{l,\theta}(t) \ge D, \quad |t| \le T, \ \theta \in \Theta. \tag{1.2}$$

with some positive constant D. Then we have the uniform estimate (1.1) for $P_\theta(t, \tau)$ with $\theta \in \Theta$.

Here we denote by $C^{k,\alpha}([-T, T])$ with a non-negative integer k and $\alpha \in (0, 1]$ the space of function $f(t)$ on $[-T, T]$ that has continuous derivatives up to order k and whose k-th derivative $f^{(k)}(t)$ is α-Hölder continuous, that is to say,

$$|f^{(k)}(s) - f^{(k)}(t)| \le C|s-t|^\alpha, \quad s, t \in [-T, T]$$

with some constant C.

Remark 1.1. When a hyperbolic polynomial $P(t) = \tau^M + \sum_{h=1}^M A_h(t)\tau^{M-h}$ with coefficients $A_h(t)$ in $C^{r_0,\gamma}([-T, T])$ have only roots with the multiplicity at most m with $m < M$, we see, using the factorization $P(t, \tau) = \prod_{l=1}^M (\tau - \lambda_l(t))$ with $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_M(t)$, that

$$\min_{|t|\le T, \ l=1,\ldots,M-m} (\lambda_{l+m}(t) - \lambda_l(t)) > 0,$$

which follows from the continuity of roots $\lambda_l(t)$ with respect to the variable t.

For the case where coefficients depend on two variables t and x, we have the following.

Corollary 1.2. *Let*

$$P(t, x, \tau) = \tau^M + \sum_{h=1}^M A_h(t, x)\tau^{M-h}$$

be a hyperbolic polynomial with coefficients $A_h(t, x)$ $(h = 1, \ldots, M)$ in $C^{(r_0,\gamma_0),(r_1,\gamma_1)}([-T, T] \times [-L, L])$. We assume that the multiplicity of roots of $P(t, x, \tau)$ is at most m. Then for any $S \in (0, T)$ and $L_0 \in (0, L)$ there exists a positive constant C such that for $j = 0, \ldots, \min\{m-1, r_0\}$ and $k = 0, \ldots, r_1$ satisfying $1 \le j + k \le m$

$$\left| \frac{\partial_t^j \partial_x^k P(t,x,\tau)}{P(t,x,\tau)} \right| \le C|\Im\tau|^{-j\max\{1,\frac{m}{r_0+\gamma}\}-k\max\{1,\frac{m}{r_1+\gamma_1}\}} \tag{1.3}$$

when $\tau \in \mathbb{C}$ with $\Im\tau \in (0, 1]$, $t \in [-S, S]$ and $x \in [-L_0, L_0]$.

If we consider the case $\tau^2 - (t - T)$, we cannot expect the estimate (1.1) near the extremity of interval, $t = T$ or $t = -T$. In this case we have the following simple corollary.

Corollary 1.3. *Under the assumption of Theorem 1.1 with $r_0 = 1$, we have*

$$\left|\frac{\partial_t P(t,\tau)}{P(t,\tau)}\right| \le C(|t+T||t-T|)^{-\frac{\gamma}{1+\gamma}} |\Im\tau|^{-\max\{1,\frac{m}{1+\gamma}\}} \tag{1.4}$$

when $t \in (-T, T)$ and $\tau \in \mathbb{C}$ with $0 < |\Im\tau| \le 1$.

Here we remark, taking into account the singularity appearing in (1.4) at $t = \pm T$, that the statement of Lemme 14.1 of [3] is not always valid unless the operator has an extension to an open set keeping regularity of coefficients, hyperbolicity and the maximal multiplicity of roots. Then we have to modify some arguments in [3] unless we consider Cauchy problem in an open interval of the variable t. The necessary modification is given in the forth coming paper [4].

The Hölder continuity of roots are given in the following form.

Theorem 1.4. *Under the assumption of Theorem 1.1, we see that the roots $\lambda_l(t)$ $l = 1, \dots, M$ are locally Hölder continuous with the index $\min\{1, \frac{r_0+\gamma}{m}\}$ on the open interval $(-T, T)$ that is to say, $\lambda_l(t) \in \cap_{S\in(0,T)} C^{\min\{1,\frac{r_0+\gamma}{m}\}}([-S, S])$.*

Furthermore as Theorem 1.1, for a bounded family of coefficients $A_j(t)$ in $C^{r_0,\gamma}([-T, T])$, the roots $\lambda_j(t)$ $j = 1, \dots, M$ of $P(t,\tau)$ form a bounded set in $C^{\min\{1,\frac{r_0+\gamma}{m}\}}([-S, S])$ for any $S \in (0, T)$, as long as $P(t,\tau)$ is hyperbolic and the multiplicity of roots is uniformly at most m.

As mentioned above, the two theorems above have already been proven by Wakabayashi [5] with some extension for Theorem 1.1.

Our interest is, in one hand, to show that the idea of Bronshtein [2] is applicable for their proofs and in other hand, to show that Theorem 1.1 and Theorem 1.4 is equivalent.

In the next section we recall some properties of a hyperbolic polynomial. In the section 3, we prove the key proposition on the behavior of coefficients with a parameter. Using the results in the section 3, we give the proof of Theorem 1.1 in the section 4. Two Corollaries are proven in the section 5. Theorem 1.4 is proven in the section 6 where we give two proofs of the key lemma, Lemma 6.2. One proof is based on the results in the section 3, while another proof is based only Theorem 1.1. Finally in the section 7 we prove Theorem 1.1 only assuming that Theorem 1.4 is valid.

For $a(t) \in C^{k,\alpha}([-t_0, t_0])$ we denote by $\|a(\cdot)\|_{\dot{C}^{k,\alpha}}$

$$\|a(\cdot)\|_{\dot{C}^{k,\alpha}} = \sup_{-t_0\le s<t\le t_0} \frac{|a^{(k)}(t) - a^{(k)}(s)|}{|t-s|^\alpha}$$

Then the norm of $a(t)$ in $C^{k,\alpha}$ is given by $\max_{t\in[-T,T],\ j=0,\dots,k} |a^{(j)}(t)| + \|a(\cdot)\|_{\dot{C}^{k,\alpha}}$. In the following, we use C, D suffixed or not in order to denote constants that may be different line by line. As mentioned in the section 3, there is a distinction of usage between C and D in the section 3.

2 Properties of hyperbolic polynomials In this section we collect some known properties of hyperbolic polynomials (see Bronshtein [2]). In this section we call a polynomial in τ with real coefficients

$$p(\tau) = \sum_{j=0}^{m} a_j \tau^{m-j},$$

that is not identically zero, hyperbolic when $p(\tau) \neq 0$ for any $\tau \in \mathbb{C}\backslash\mathbb{R}$, that is to say, all the zeros of $p(\tau)$ are real. We remark that nonzero real constant is also hyperbolic by the definition.

Lemma 2.1. *If $p(\tau) = \sum_{j=0}^{m} a_j \tau^{m-j}$ is hyperbolic, then $q(\tau) = \sum_{j=0}^{m} a_j \tau^j$ is also hyperbolic.*

Proof. When $p(\tau)$ is monomial, say, $p(\tau) = a_j \tau^{m-j}$ with $a_j \neq 0$, then $q(\tau) = a_j \tau^j$ that is clearly hyperbolic. In the case where $p(\tau)$ is not monomial, we may assume that $p(\tau)$ can be written in the following way

$$p(\tau) = B\tau^{k_1}\Pi_{j=1}^{k_2}(\tau - \mu_j)$$

where k_1 and k_2 are non negative integer satisfying $k_1 + k_2 \leq m$ and B and μ_j $(1 \leq j \leq k_2)$ are non zero real numbers. Since $q(\tau) = \tau^m p(\tau^{-1})$, we have

$$q(\tau) = B\tau^{m-k_1-k_2}\Pi_{j=1}^{k_2}(1 - \mu_j\tau),$$

which implies that $q(\tau)$ is hyperbolic. □

Lemma 2.2. *Let $p(\tau) = \sum_{j=0}^{m} a_j \tau^{m-j}$ be a hyperbolic polynomial with $m \geq 1$ and $a_0 \neq 0$. Then we have the followings.*

1). *For $0 \leq k \leq m$, $\frac{d^k}{d\tau^k}p(\tau)$ is also hyperbolic.*

2).

$$\left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0} \geq 0 \tag{2.1}$$

and

$$\frac{|a_j|}{|a_0|} \leq (3m)^{\frac{j}{2}} \max\left\{\left|\frac{a_1}{a_0}\right|, \left|\frac{a_2}{a_0}\right|^{\frac{1}{2}}\right\}^j \tag{2.2}$$

for $j \geq 1$.

3). *When $a_k \neq 0$ with $k \in \{1, \ldots, m-2\}$, we have*

$$\left(\frac{a_{k+1}}{a_k}\right)^2 - \frac{a_{k+2}}{a_k} \geq 0 \tag{2.3}$$

and

$$\frac{|a_{k+j}|}{|a_k|} \leq (3m^3)^{\frac{j}{2}} \max\left\{\left|\frac{a_{k+1}}{a_k}\right|, \left|\frac{a_{k+2}}{a_k}\right|^{\frac{1}{2}}\right\}^j \tag{2.4}$$

for $m - k \geq j \geq 1$.

4). *When $a_k \neq 0$ with $k \in \{2, \ldots, m\}$, we have*

$$\frac{|a_{k-j}|}{|a_k|} \leq (3m^3)^{\frac{j}{2}} \max\left\{\left|\frac{a_{k-1}}{a_k}\right|, \left|\frac{a_{k-2}}{a_k}\right|^{\frac{1}{2}}\right\}^j \tag{2.5}$$

for $k \geq j \geq 1$.

Proof. The assertion 1). may be evident. But for completeness, we give a proof. Let $p(\tau) = a_0 \prod_{q=1}^{Q} (\tau - \epsilon_q)^{m_q}$ where $m_q \geq 1$ and $\epsilon_q < \epsilon_{q+1}$ for $1 \leq q \leq Q-1$. Then we have $p'(\tau) = p_1(\tau) p_2(\tau)$ where $p_1(\tau) = a_0 \prod_{q=1}^{Q} (\tau - \epsilon_q)^{m_q - 1}$ and $p_2(\tau) = \sum_{q=1}^{Q} m_q \prod_{r \neq q} (\tau - \epsilon_r)$. Since $p_2(\epsilon_q) p_2(\epsilon_{q+1}) < 0$ for $1 \leq q \leq Q-1$, we see that $p_2(\tau)$ is hyperbolic, while $p_1(\tau)$ is evidently hyperbolic. Then $p'(\tau)$ is hyperbolic.

Let

$$p(\tau) = a_0 \Pi_{j=1}^{m} (\tau - \mu_j)$$

with $\mu_j \in \mathbb{R}$. Since

$$\frac{a_j}{a_0} = (-1)^j \sum_{0 \leq l_1 < l_2 < \ldots < l_j \leq m} \mu_{l_1} \mu_{l_2} \cdots \mu_{l_j}, \tag{2.6}$$

we have

$$\sum_{i=1}^{m} \mu_i^2 = \left(\frac{a_1}{a_0} \right)^2 - \frac{2a_2}{a_0} \tag{2.7}$$

and

$$\left| \frac{a_j}{a_0} \right| \leq \left(\sum_{i=1}^{m} |\mu_i| \right)^j. \tag{2.8}$$

We see that (2.7) and $\mu_i \in \mathbb{R}$ imply (2.1). Noting that

$$\left(\frac{a_1}{a_0} \right)^2 - 2\frac{a_2}{a_0} \leq 3 \max\{ \left| \frac{a_1}{a_0} \right|^2, \left| \frac{a_2}{a_0} \right| \}$$

and that $\sum_{i=1}^{m} |\mu_i| \leq m^{1/2} (\sum_{j=1}^{m} \mu_i^2)^{1/2}$ we obtain (2.2) from (2.7) and (2.8).

Let

$$q(\tau) = \sum_{j=0}^{m} a_j \tau^j,$$

that is hyperbolic thanks to Lemma 2.1. When $a_k \neq 0$ with $k \geq 1$, the assertion 1). implies that $q_k(\tau) = \frac{d^k}{d\tau^k} q(\tau)$ is hyperbolic. Hence Lemma 2.1 implies that $r(\tau) = \tau^{m-k} q_k(\tau^{-1})$ is also hyperbolic. Since

$$r(\tau) = \sum_{j=k}^{m} \frac{j!}{(j-k)!} a_j \tau^{m-j},$$

we see from (2.1), if $a_k \neq 0$, that

$$\left(\frac{(k+1)a_{k+1}}{a_k} \right)^2 - 2\frac{(k+2)(k+1)a_{k+2}}{2a_k} \geq 0,$$

from which we obtain (2.3). While (2.2) implies that

$$\left| \frac{(k+j)! a_{k+j}}{k! j! a_k} \right| \leq (3m)^{\frac{j}{2}} \left(max\{ \left| \frac{(k+1)a_{k+1}}{a_k} \right|, \left| \frac{(k+2)(k+1)a_{k+2}}{2a_k} \right|^{\frac{1}{2}} \} \right)^j.$$

Then noting that

$$\max\{\left|\frac{(k+1)a_{k+1}}{a_k}\right|, \left|\frac{(k+2)(k+1)a_{k+2}}{2a_k}\right|^{\frac{1}{2}}\} \leq m \max\{\left|\frac{a_{k+1}}{a_k}\right|, \left|\frac{a_{k+2}}{a_k}\right|^{\frac{1}{2}}\},$$

we obtain (2.4).

When $a_k \neq 0$ with $k \in \{2, \dots, m\}$, the degree of $q(\tau)$ is at least k. Then we see that the estimate (2.4) applied to $q(\tau)$ implies that (2.5) is valid. □

We see from the assertion 3). of Lemma 2.2 that the following lemma is valid.

Lemma 2.3. *Let $p(\tau) = \sum_{j=0}^{m} a_j \tau^{m-j}$ be hyperbolic with $a_0 \neq 0$. If $a_{j_0} = 0,\ a_{j_0+1} = 0$ for some j_0, then $a_j = 0$ for any j satisfying $j_0 \leq j \leq m$.*

Proof. Indeed, let k_0 be the largest integer j satisfying $a_j \neq 0$ and $j < j_0$. The assumption $a_{j_0} = 0,\ a_{j_0+1} = 0$ implies $a_{k_0+1} = 0$ and $a_{k_0+2} = 0$. Then the estimate (2.4) shows that $a_j = 0$ for any j satisfying $k_0 < j \leq m$. □

The following lemma is due to Bronshtein [2].

Lemma 2.4. *Let $p(\tau) = \sum_{j=0}^{m} a_j \tau^{m-j}$ be hyperbolic with $a_0 \neq 0$. Let k_0 be the largest integer j satisfying $a_j \neq 0$. Then there exists a subset M of $\{0, 1, \dots, k_0\}$ satisfying the followings.*

1). $0, k_0 \in \mathsf{M}$. *If $j \in \mathsf{M}$ and $0 \leq j \leq k_0 - 1$, then $j+1 \in \mathsf{M}$ or $j+2 \in \mathsf{M}$.*

2). *$a_j \neq 0$ for any $j \in \mathsf{M}$*

3). *If $j, j+1 \in \mathsf{M}$, then $(\frac{a_{j+1}}{a_j})^2 > |\frac{a_{j+2}}{a_j}|$. If $j, j+2 \in \mathsf{M}$ and $j+1 \notin \mathsf{M}$, then $(\frac{a_{j+1}}{a_j})^2 \leq |\frac{a_{j+2}}{a_j}|$.*

Here we consider $a_j = 0$ for $j > m$ if necessary.

Proof. We assume that $k_0 > 0$. We construct the sequence $j_0,\ j_1,\ \dots$ in $\{0, 1, \dots, k_0\}$ inductively. Let $j_0 = 0$. If $a_{j_0+1}^2 > |a_{j_0} a_{j_0+2}|$, we put $j_1 = j_0 + 1$. Otherwise we put $j_1 = j_0 + 2$. We note that, if $j_1 \leq k_0$, $a_{j_1} \neq 0$. Indeed when $j_1 = j_0 + 1$, we see $a_{j_1} \neq 0$ from the definition of j_1. Consider the case where $j_1 = j_0 + 2$. Then we have $a_{j_0+1}^2 \leq |a_{j_0} a_{j_0+2}|$. Thus if $a_{j_1} = 0$, we have $a_{j_0+1} = 0$. Hence $a_{j_0+1} = a_{j_0+2} = 0$ from which and Lemma 2.3 follows that $a_k = 0$ for $k \geq j_0 + 1$. That is not consistent with the assumption $j_1 \leq k_0$ and $a_{k_0} \neq 0$. Hence in the case where $j_1 = j_0 + 2$ also, we have $a_{j_1} \neq 0$. If $j_1 < k_0$, we put

$$j_2 = \begin{cases} j_1 + 1 & \text{if } a_{j_1+1}^2 > |a_{j_1} a_{j_1+2}| \\ j_1 + 2 & \text{otherwise.} \end{cases}$$

Noting $a_{k_0} \neq 0$ and $a_{k_0+1} = 0$, we see that $a_{k_0}^2 > |a_{k_0-1} a_{k_0+1}|$, where we put $a_j = 0$ for $j > m$ if necessary. Then we see that $j_2 \leq k_0$. Hence we see that $a_{j_2} \neq 0$ by the same way as for $a_{j_1} \neq 0$. We repeat this procedure of the construction of a sequence $j_0, j_1, \dots$ until we have $j_s = k_0$ with some s. Now we define the subset M by $\mathsf{M} = \{j_0, j_1, \dots, j_s\}$. From the construction of a sequence $j_0, j_1, \dots, j_s$ we see that the subset M satisfies the desired properties. □

For an element $l < k_0$ in M, we denote by $\bar{l}$ the smallest element in $\mathsf{M} \cap \{l+1, \dots, k_0\}$. We see from the property 1) of M in Lemma 2.4 that $\bar{l} - l$ is 1 or 2. Similarly for $l \in \mathsf{M}$ with $l > 0$, we denote by $\underline{l}$ the largest element in $\mathsf{M} \cap \{0, \dots, l-1\}$. Then we have $\underline{(\bar{l})} = l$. Using this notation we have the following lemma.

Lemma 2.5. *Under the assumption of Lemma 2.4, there exist a constant K depending only on the degree of the polynomial $p(\tau)$ such that we have the following estimates, where M is a set given in Lemma 2.4;*

$$\max\{\left|\frac{a_{l+1}}{a_l}\right|, \left|\frac{a_{l+2}}{a_l}\right|^{\frac{1}{2}}\} = \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}} \quad \textit{for any } l \in \mathsf{M} \textit{ with } k_0 > l \ , \tag{2.9}$$

$$\max\{\left|\frac{a_{l-1}}{a_l}\right|, \left|\frac{a_{l-2}}{a_l}\right|^{\frac{1}{2}}\} \le K\left|\frac{a_{\underline{l}}}{a_l}\right|^{\frac{1}{l-\underline{l}}} \quad \textit{for any } l \in \mathsf{M} \textit{ with } 2 \le l \ , \tag{2.10}$$

$$|a_{\bar{l}}|^l \le K|a_l|^{\bar{l}}|a_0|^{l-\bar{l}} \quad \textit{for any } l \in \mathsf{M} \textit{ with } 0 < l < k_0 \ , \tag{2.11}$$

and

$$\Big(\frac{|a_{\bar{h}}|}{|a_h|}\Big)^{\frac{1}{\bar{h}-h}} \le K\Big(\frac{|a_{\bar{l}}|}{|a_l|}\Big)^{\frac{1}{\bar{l}-l}} \quad \textit{for any } l, h \in \mathsf{M} \textit{ with } l < h < k_0 \ . \tag{2.12}$$

Proof. In this proof , we denote by K an arbitrary constant depending only on the degree of $p(\tau)$.

Noting the property 3) of M in Lemma 2.4 and the definition of $\bar{l}$, we see that (2.9) is valid.

Let $l \ge 2$ be in M. If $\underline{l} = l-2$, then we see from the property 3) of M in Lemma 2.4 that $(|a_{\underline{l}}||a_l|)^{1/2} \ge |a_{l-1}|$ which implies that $\max\{(\frac{|a_{\underline{l}}|}{|a_l|})^{1/2}, \frac{|a_{l-1}|}{|a_l|}\}$ is equal to $(\frac{|a_{\underline{l}}|}{|a_|})^{1/2} = (\frac{|a_{\underline{l}}|}{|a_l|})^{\frac{1}{l-\underline{l}}}$. Then if $\underline{l} = l-2$, we have (2.10). Assume now $\underline{l} = l-1$. Then $(|a_{\underline{l}}|/|a_l|)^{l-\underline{l}} = |a_{l-1}|/|a_l|$. When $l-2 \in \mathsf{M}$, then we have $l-2, l-1 \in \mathsf{M}$ which and the property 3) of M in Lemma 2.4 imply $(|a_l||a_{l-2}|)^{1/2} < |a_{l-1}|$. Hence $\max\{(\frac{|a_{l-2}|}{|a_l|})^{1/2}, \frac{|a_{l-1}|}{|a_l|}\}$ is equal to $\frac{|a_{l-1}|}{|a_l|}$. We have (2.10). On the other hand, when $l-2 \notin \mathsf{M}$, it follows from $l \ge 2$ and the property 1) of M in Lemma 2.4 that $l-3 \in \mathsf{M}$. Noting that $l-3, l-1 \in \mathsf{M}$ and $l-2 \notin \mathsf{M}$, that is $\overline{l-3} = l-1$, we see from (2.9) with $l = \overline{l-3}$ that $\max\{(\frac{|a_{l-1}|}{|a_{l-3}|})^{1/2}, \frac{|a_{l-2}|}{|a_{l-3}|}\}$ is equal to $(\frac{|a_{l-1}|}{|a_{l-3}|})^{1/2}$. Then the estimate (2.4) with $k = l-3$ and $j = 3$ implies

$$\frac{|a_l|}{|a_{l-3}|} \le K\Big(\frac{|a_{l-1}|}{|a_{l-3}|}\Big)^{\frac{3}{2}},$$

from which and

$$\frac{|a_{l-2}|}{|a_{l-3}|} \le \Big(\frac{|a_{l-1}|}{|a_{l-3}|}\Big)^{\frac{1}{2}}$$

we obtain

$$|a_{l-2}| \le K\frac{|a_{l-1}|^2}{|a_l|}.$$

Hence

$$\max\{\Big(\frac{|a_{l-2}|}{|a_l|}\Big)^{\frac{1}{2}}, \frac{|a_{l-1}|}{|a_l|}\} \le K(\frac{|a_l|}{|a_{l-1}|}).$$

Therefore if $\underline{l} = l-1$, we have also (2.10).

For $l \in \mathsf{M}$ with $0 < l < k_0$, we have $\bar{l} \geq 2$. Then from (2.10) we obtain

$$\max\{(\frac{|a_{\bar{l}-2}|}{|a_{\bar{l}}|})^{\frac{1}{2}}, \frac{|a_{\bar{l}-1}|}{|a_{\bar{l}}|}\} \leq K(\frac{|a_l|}{|a_{\bar{l}}|})^{\frac{1}{\bar{l}-l}}.$$

Then we see from (2.5) with $k = j = \bar{l}$,

$$\frac{|a_0|}{|a_{\bar{l}}|} \leq K(\frac{|a_l|}{|a_{\bar{l}}|})^{\frac{\bar{l}}{\bar{l}-l}}.$$

Then we obtain (2.11).

In order to show (2.12) we have only to consider the case $h = \bar{l}$. We note that

$$\max\{\frac{|a_{l+1}|}{|a_l|}, (\frac{|a_{l+2}|}{|a_l|})^{\frac{1}{2}}\} = (\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{1}{\bar{l}-l}}.$$

Then we see from (2.4) with $k = l$ that

$$\frac{|a_{l+j}|}{|a_l|} \leq K(\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{j}{\bar{l}-l}}.$$

Hence

$$\frac{|a_{l+j}|}{|a_{\bar{l}}|} \leq K(\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{j}{\bar{l}-l}-1}.$$

Let $h = \bar{l}$. If $\bar{h} = h + i$, then $\bar{h} = l + \bar{l} - l + i$. Therefore we obtain

$$\frac{|a_{\bar{h}}|}{|a_h|} \leq K(\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{i}{\bar{l}-l}}.$$

Then we get

$$(\frac{|a_{\bar{h}}|}{|a_h|})^{\frac{1}{\bar{h}-h}} \leq K(\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{1}{\bar{l}-l}}.$$

□

3 Properties of coefficients of hyperbolic polynomials In this section, we show the properties of coefficients of a hyperbolic polynomial $p(t,\tau)$ with a parameter t.

Let

$$p(t,\tau) = \tau^m + \sum_{j=1}^{m} A_j(t)\tau^{m-j},$$

where $m \geq 2$ and $A_j(t) \in C^{r_0,\gamma}([-t_0, t_0])$ $(1 \leq j \leq m)$ with a positive integer r_0 and $0 < \gamma \leq 1$.

In the following we put $a_j = A_j(0)$, $a_j^{(l)} = \frac{d^l A_j}{dt^l}(0)$ with $a_0 = A_0(t) = 1$. Let k_0 be the largest integer satisfying $a_j \neq 0$. Let the subset M of $\{0, 1, \ldots, k_0\}$ be the subset given in Lemma 2.2 applied to the hyperbolic polynomial $p(0,\tau) = \sum_{j=0}^{m} a_j \tau^{m-j}$. As in the section 2, for $l \in \mathsf{M}$ with $l < k_0$ we denote by $\bar{l}$ the smallest integer in $\mathsf{M} \cap \{l+1, \ldots, k_0\}$.

Then we have the following proposition which is due to Bronshtein [2] in the case $m \leq r_0 + \gamma$.

Proposition 3.1. *Under the setting above, assuming that $k_0 \geq 1$, we see that for any $l \in \mathsf{M}$ with $l < k_0$ there exists a positive constant $C_l \in (0,1]$ satisfying*

$$C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{(\frac{1}{\bar{l}-l})\max\{1,\frac{h}{r_0+\gamma}\}} \leq t_0 \text{ for } \; h = l+1,\dots,m$$

such that we have, when $|t| \leq C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{(\frac{1}{\bar{l}-l})\max\{1,\frac{\bar{l}}{r_0+\gamma}\}}$,

$$\left|\frac{A_l(t)}{a_l} - 1\right| \leq \frac{1}{2} \tag{3.1}$$

$$\left|\frac{A_{\bar{l}}(t)}{a_{\bar{l}}} - 1\right| \leq \frac{1}{2} \tag{3.2}$$

and when $|t| \leq C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{(\frac{1}{\bar{l}-l})\max\{1,\frac{l+2}{r_0+\gamma}\}}$

$$\max\{\frac{|A_{l+1}(t)|}{|A_l(t)|}, (\frac{|A_{l+2}(t)|}{|A_l(t)|})^{\frac{1}{2}}\} \leq 4 \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}} \tag{3.3}$$

where we put $A_{m+1}(t) = a_{m+1} = 0$ if necessary. Furthermore we have

$$\left|\frac{A_h(t)}{a_l}\right| \leq K \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}}, \quad |t| \leq C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{(\frac{1}{\bar{l}-l})\max\{1,\frac{h}{r_0+\gamma}\}} \tag{3.4}$$

when $m \geq h \geq l+1$ with a constant K depending only on the degree of polynomial m.

Here the positive constants above C_l can be chosen uniformly for the coefficients $A_j(t)$ belonging to a bounded set in $C^{r_0,\gamma}([-t_0,t_0])$ as long as $P(t,\tau)$ is hyperbolic. That is to say for any $M > 0$ there exist the positive constants δ_1 and δ_2, that are independent of k_0 and M, such that we have $\delta_1 \leq C_l \leq \delta_2$ when the $C^{r_0,\gamma}$-norm of all the coefficients of a hyperbolic polynomial $P(t,\tau)$ are equal or inferior to M.

For the proof of Proposition 3.1 we prepare two lemmas.

Lemma 3.2. *Let $a(t)$ be a real valued function belonging to $C^{k,\alpha}([-t_0,t_0])$ with a positive integer k, $0 < \alpha \leq 1$ and $t_0 > 0$.*

If we have with some $\delta_0 \in (0,t_0]$ and some constant C_0

$$|a(t)| \leq C_0 \qquad (t \in [-\delta_0,\delta_0]),$$

then we have the estimate of the derivative $a^{(j)}(0)$ $(1 \leq j \leq k)$; for $0 < \delta \leq \delta_0$

$$|a^{(j)}(0)| \leq \Gamma(C_0 + \|a(\cdot)\|_{\dot{C}^{k,\alpha}}\delta^{k+\alpha})\delta^{-j}, \tag{3.5}$$

where the constant Γ depends only on k and α.

If we have with some $\delta_0 \in (0,t_0]$

$$a(t) \geq 0 \qquad (t \in [-\delta_0,\delta_0]),$$

then we have the following estimate ; for $0 < \delta \leq \delta_0$

$$|a^{(1)}(0)| \leq a(0)\delta^{-1} + \sum_{j=2}^{k} |a^{(j)}(0)|\frac{\delta^{j-1}}{j!} + \|a(\cdot)\|_{\dot{C}^{k,\alpha}}\delta^{k+\alpha-1}. \tag{3.6}$$

Proof. From Taylor's formula

$$a(t) = \sum_{j=0}^{k} \frac{a^{(j)}(0)}{j!} t^j + \frac{t^k}{(k-1)!} \int_0^1 (1-\tau)^{k-1} (a^{(k)}(t\tau) - a^{(k)}(0))\, d\tau, \tag{3.7}$$

we see that for $t \in [-\delta, \delta]$

$$|\sum_{j=0}^{k} \frac{a^{(j)}(0)}{j!} t^j - a(t)| \le \|a(\cdot)\|_{\dot{C}^{k,\alpha}} \delta^{k+\alpha}.$$

When $|a(t)| \le C_0$ $(|t| \le \delta_0)$, for $\delta \in (0, \delta]$ we get

$$|\sum_{j=1}^{k} \frac{t^j}{j!} a^{(j)}(0)| \le 2C_0 + \|a(\cdot)\|_{\dot{C}^{k,\alpha}} \delta^{k+\alpha} \qquad (t \in [-\delta, \delta]).$$

Hence by picking $t = l\delta/k$ $(l = 1, \ldots, k)$, we see $\frac{a^{(j)}(0)\delta^j}{j!k^j}$ satisfy the following linear equations; for $l = 1, \ldots, k$

$$\sum_{j=1}^{k} l^j \frac{a^{(j)}(0)\delta^j}{j!k^j} = D_l,$$

where we have $|D_l| \le 2C_0 + \|a(\cdot)\|_{\dot{C}^{k,\alpha}} \delta^{k+\alpha}$ Since $k \times k$ matrix whose (l, j) element is l^j has its inverse whose (i, j) element we denote by $d_{i,j}$, we have

$$\frac{a^{(j)}(0)\delta^j}{j!k^j} = \sum_{l=1}^{k} d_{j,l} D_l$$

from which we obtain the estimates (3.5).

If $a(t) \ge 0$, we have from (3.7)

$$-a^{(1)}(0)\frac{t}{|t|} \le a(0)|t|^{-1} + \sum_{j=2}^{k} |a^{(j)}(0)| \frac{|t|^{j-1}}{j!} + \|a(\cdot)\|_{\dot{C}^{k,\alpha}} |t|^{k+\alpha-1}$$

from which we obtain the estimate (3.6). □

Next we show the following lemma.

Lemma 3.3. *Under the setting of Proposition 3.1, let* $l \in \mathsf{M}$ *with* $0 \le l < k_0$ *and* $h \in \{l+1, \ldots, m\}$. *Assume that there exists a positive constant* $C_l \in (0, 1]$ *satisfying* $C_l(|a_{\bar{l}}|/|a_l|)|^{(\frac{1}{\bar{l}-l})\max\{1, \frac{h}{r_0+\gamma}\}} \le t_0$ *such that we have*

$$\left|\frac{A(t)}{a_l}\right| \le K \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}}, \quad |t| \le C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{(\frac{1}{\bar{l}-l})\max\{1, \frac{h}{r_0+\gamma}\}} \tag{3.8}$$

where $A(t) \in C^{r_0,\gamma}([-t_0, t_0]))$ *with a positive integer* r_0 *and* γ.

Then we have

$$|A^{(j)}(0)| \le D|a_l| \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}(h-l-j\max\{1, \frac{h}{r_0+\gamma}\})} \tag{3.9}$$

for $j = 1, \ldots, \min\{r_0, h-1\}$. *Here the constant* D *depends only on* C_l, K, $C^{r_0,\gamma}$*-norm of* $A(t)$ *and the degree* m *of hyperbolic polynomial* $\tau^m + \sum_{h=1}^{m} a_h \tau^{m-h}$.

Proof. First we remark that we have

$$|a_{\bar{l}}|^{l} \leq D_{00}|a_l|^{\bar{l}}. \tag{3.10}$$

Indeed when $0 < l < k_0$, (2.11) and $a_0 = 1$ imply (3.10) with $D_{00} = K$, while in the case where $l = 0$, we obtain (3.10) with $D_{00} = 1$ from $a_0 = 1$. From (3.8) and (3.5) applied to $A(t)$ with $k = \min\{h-1, r_0\}$ and $\alpha = \min\{h, r_0+\gamma\} - k$ we obtain, noting $k + \alpha = \min\{h, r_0+\gamma\}$, for $j = 1, \cdots, \min\{h-1, r_0\}$

$$\left|\frac{A^{(j)}(0)}{a_l}\right| \leq \Gamma(K\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}} \delta^{-j} + \frac{\|A\|_{\dot{C}^{k,\alpha}}}{|a_l|}\delta^{\min\{h,r_0+\gamma\}-j}) \tag{3.11}$$

with $\delta = C_l \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}\max\{1,\frac{h}{r_0+\gamma}\}}$. Since $\max\{1, \frac{h}{r_0+\gamma}\}\min\{h, r_0+\gamma\} = h$, (3.10) implies

$$\delta^{\min\{h,r_0+\gamma\}} \leq C_l^{\min\{h,r_0+\gamma\}} D_{00}^{\frac{1}{\bar{l}-l}} |a_l| \left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}}.$$

Then we obtain (3.9) with D given by

$$D = \Gamma(KC_l^{-j} + C_l^{\min\{h,r_0+\gamma\}-j} D_{00}^{\frac{1}{\bar{l}-l}}\|A\|_{\dot{C}^{k,\alpha}}).$$

□

Proof of Proposition 3.1. In this proof we denote positive constants, that are not less than 1, depending only on the degree of $p(t,\tau)$ by K and we use Γ and Γ with some suffix in order to denote positives constants that depend only on $r_0+\gamma$: the index of the regularity of the coefficients $A_j(t)$ $(1 \leq j \leq m)$. Furthermore C, C with some suffix, D and D with some suffix are used in order to denote positives constants that may depend also on the $C^{r_0,\gamma}$-norm of the coefficients $A_j(t)$ $(1 \leq j \leq m)$. Here D or D with some suffix are used to denote positive constants which are bounded by the $C^{r_0,\gamma}$ norms of coefficients $A_j(t)$, while C or C with some suffix are used for positive constants that are inferior or equal to 1 and whose inverses are bounded by the $C^{r_0,\gamma}$ norms of coefficients $A_j(t)$.

Considering, if necessary, the linear change of the variable t, we assume that

$$t_0 = 1.$$

Then $A_j(t) \in C^{r_0,\gamma}([-1,1])$. Recall

$$a_j = A_j(0) \text{ and } a_j^{(l)} = \frac{d^l A_j}{dt^l}(0).$$

Note $a_0 = 1$ and that we have $|\frac{a_{\bar{0}}}{a_0}|^{1/\bar{0}} = \max\{|\frac{a_1}{a_0}|, |\frac{a_2}{a_0}|^{1/2}\}$ from (2.9). Then from (2.12), we have for any $l \in \mathsf{M}$ with $l < k_0$

$$(\frac{|a_{\bar{l}}|}{|a_l|})^{\frac{1}{\bar{l}-l}} \leq D_0 \tag{3.12}$$

where $D_0 = K\max\{|a_1|, |a_2|^{1/2}\}$.

First we consider the case where $l = 0$.

When $0, 1 \in \mathsf{M}$, that is to say $\bar{0} = 1$, we have $|a_2| < |a_1|^2$ and $a_1 \neq 0$. Since $A_1(t) \in C^{0,1}([-1,1])$, we see that, with $C_{1,1} = \min\{\frac{1}{1+2\|A_1(\cdot)\|_{\dot{C}^{0,1}}}, \frac{1}{1+|a_1|}\}$, $|A_1(t) - a_1| \leq \frac{|a_1|}{2}$ when $|t| \leq C_{1,1}|a_1|$. Here we remark that $C_{1,1}|a_1| \leq 1$.

Hence

$$\Big|\frac{A_1(t)}{a_1} - 1\Big| \le \frac{1}{2}, \qquad |t| \le C_{1,1}|a_1|. \tag{3.13}$$

Then $A_1(t)^2 \le 4a_1^2$ when $|t| \le C_{1,1}|a_1|$. Since (2.1) implies $2A_2(t) \le A_1(t)^2$, we see that

$$4a_1^2 - 2A_2(t) \ge 0, \qquad |t| \le C_{1,1}|a_1|. \tag{3.14}$$

Note $A_2(t) \in C^{r_0,\gamma}([-1,1])$ and

$$|4a_1^2 - 2A_2(0)| = |4a_1^2 - 2a_2| \le 6{a_1}^2. \tag{3.15}$$

In the case where $2 \ge r_0 + \gamma$, since $r_0 \ge 1$ and $1 \ge \gamma > 0$, we have $r_0 = 1$. Taylor's formula (3.7) with $k = 1$ and $\alpha = \gamma$, implies

$$|A_2(t) - a_2| \le |a_2^{(1)} t| + \|A_2(\cdot)\|_{\dot{C}^{r_0,\gamma}} |t|^{r_0+\gamma}, \qquad |t| \le 1,. \tag{3.16}$$

From (3.15) we see that the estimate (3.6) applied to $4a_1^2 - 2A_2(t)$ with $k = 1$ and $\alpha = \gamma$ and $\delta = (C_{1,1}|a_1|)^{\frac{2}{r_0+\gamma}}$ implies, with $D_{1,1} = (3 + C_{1,1}^2 \|A_2(\cdot)\|_{\dot{C}^{r_0,\gamma}}) C_{1,1}^{-\frac{2}{r_0+\gamma}}$,

$$|a_2^{(1)}| \le D_{1,1} |a_1|^{2-\frac{2}{r_0+\gamma}}.$$

Then we see that with $C_{1,2} = \min\{\frac{1}{1+2D_{1,1}+2\|A_2(\cdot)\|_{\dot{C}^{r_0,\gamma}}}, \frac{1}{(1+|a_1|)^2}\}$,

$$|A_2(t) - a_2| \le a_1^2 \qquad \text{when } |t| \le C_{1,2}|a_1|^{\frac{2}{r_0+\gamma}}$$

where we remark that $(1+|a_1|)^2 C_{1,2} \le 1$ and $\frac{2}{r_0+\gamma} \le 2$ show that $C_{1,2}|a_1|^{\frac{2}{r_0+\gamma}} \le 1$. Hence we have

$$|A_2(t)| \le 2a_1^2, \qquad |t| \le C_{1,2}|a_1|^{\frac{2}{r_0+\gamma}}.$$

If $2 \le r_0 + \gamma$, we see from (3.6) applied to $4a_1^2 - 2A_2(t)$ with $k = \alpha = 1$ and $\delta = C_{1,1}|a_1|$, (3.14) and (3.15) that, with $D_{1,2} = (3C_{1,1}^{-1} + C_{1,1}\|A_2(\cdot)\|_{\dot{C}^{1,1}})$

$$|a_2^{(1)}| \le D_{1,2}|a_1|$$

which and the following estimate drawn from Taylor's formula (3.7) with $k = \alpha = 1$

$$|A_2(t) - a_2| \le |a_2^{(1)} t| + \|A_2(\cdot)\|_{\dot{C}^{1,1}} |t|^2, \qquad |t| \le 1 \tag{3.17}$$

imply with $C_{1,3} = \min\{\frac{1}{1+2D_{1,2}+2\|A_2(\cdot)\|_{\dot{C}^{1,1}}}, \frac{1}{1+|a_1|}\}$ that satisfies $C_{1,3}|a_1| \le 1$,

$$|A_2(t) - a_2| \le a_1^2, \qquad |t| \le C_{1,3}|a_1|.$$

Then

$$|A_2(t)| \le 2a_1^2, \qquad |t| \le C_{1,3}|a_1|.$$

Therefore with

$$C_0 = \begin{cases} \min\{C_{1,1}(1+|a_1|)^{-1}, C_{1,2}\} & (2 \ge r_0 + \gamma) \\ \min\{C_{1,1}, C_{1,3}\} & (2 \le r_0 + \gamma), \end{cases}$$

we see that

$$\left|\frac{A_1(t)}{a_1}-1\right|\le\frac{1}{2},\quad |t|\le C_0|a_1|^{\max\{1,\frac{1}{r_0+\gamma}\}}, \tag{3.18}$$

and

$$\max\{|A_1(t)|,|A_2(t)|^{\frac{1}{2}}\}\le 2|a_1|,\quad |t|\le C_0|a_1|^{\max\{1,\frac{2}{r_0+\gamma}\}}.$$

The estimate above and (2.2) implies that for $j=2,\dots,m$

$$|A_j(t)|\le K|a_1|^j \qquad \text{when } |t|\le C_0|a_1|^{\max\{1,\frac{2}{r_0+\gamma}\}}.$$

Putting

$$C_{0,0}=\frac{C_0}{(1+|a_1|)^m}$$

we see that $C_{0,0}\le C_0$ and $C_{0,0}|a_1|^{\max\{1,\frac{j}{r_0+\gamma}\}}\le C_0|a_1|^{\max\{1,\frac{2}{r_0+\gamma}\}}$ when $2\le j\le m$. Hence we have $C_{0,0}|a_1|^{\max\{1,\frac{j}{r_0+\gamma}\}}\le 1$ $(1\le j\le m)$ and following estimates;

$$\left|\frac{A_1(t)}{a_1}-1\right|\le\frac{1}{2},\quad |t|\le C_{0,0}|a_1|^{\max\{1,\frac{1}{r_0+\gamma}\}},$$

$$\max\{|A_1(t)|,|A_2(t)|^{\frac{1}{2}}\}\le 2|a_1|,\quad |t|\le C_{0,0}|a_1|^{\max\{1,\frac{2}{r_0+\gamma}\}}$$

and

$$|A_j(t)|\le K|a_1|^j \qquad |t|\le C_{0,0}|a_1|^{\max\{1,\frac{j}{r_0+\gamma}\}}.$$

when $1\le j\le m$.

On the other hand, when $0,2\in \mathsf{M}$ and $1\notin \mathsf{M}$, that is to say, $\overline{0}=2$, we have $|a_1^2|\le|a_2|$ and $a_2\neq 0$. Since $A_1(t)\in C^{0,1}([-1,1])$, we have with $C_{2,1}=\min\{\frac{1}{1+2\|A_1(\cdot)\|_{\dot{C}^{0,1}}},\frac{1}{1+|a_2|}\}$, which implies $C_{2,1}|a_2|^{1/2}\le 1$,

$$|A_1(t)-a_1|\le|a_2|^{\frac{1}{2}},\qquad |t|\le C_{2,1}|a_2|^{\frac{1}{2}}.$$

Then

$$A_1(t)^2\le 4|a_2|,\qquad |t|\le C_{2,1}|a_2|^{\frac{1}{2}}. \tag{3.19}$$

Since (2.1) implies $2A_2(t)\le A_1(t)^2$, we see that

$$4|a_2|-2A_2(t)\ge 0,\qquad |t|\le C_{2,1}|a_2|^{\frac{1}{2}}. \tag{3.20}$$

Note $A_2(t)\in C^{r_0,\gamma}([-1,1])$ and

$$|4|a_2|-2A_2(0)|=|4|a_2|-2a_2|\le 6|a_2|. \tag{3.21}$$

In the case where $2\ge r_0+\gamma$, the estimate (3.6) applied to $4|a_2|-2A_2(t)$ with $k=r_0=1$, $\alpha=\gamma$ and $\delta=(C_{2,1}|a_2|^{1/2})^{\frac{2}{r_0+\gamma}}$ implies that with $D_{2,1}=(3+C_{2,1}^2\|A_2(\cdot)\|_{\dot{C}^{r_0,\gamma}})C_{2,1}^{-\frac{2}{r_0+\gamma}}$

$$|a_2^{(1)}|\le D_{2,1}|a_2|^{1-\frac{1}{r_0+\gamma}}$$

from which and (3.16) we see that with $C_{2,2} = \min\{\frac{1}{1+4D_{2,1}+4\|A_2(\cdot)\|_{\dot{C}^{r_0,\gamma}}}, \frac{1}{1+|a_2|}\}$, that satisfies $C_{2,2}(1+|a_2|) \leq 1$,

$$|A_2(t) - a_2| \leq \frac{1}{2}|a_2|, \qquad |t| \leq C_{2,2}|a_2|^{\frac{1}{r_0+\gamma}}. \tag{3.22}$$

If $2 \leq r_0 + \gamma$, the estimate (3.6) applied to $4|a_2| - 2A_2(t)$ with $k = \alpha = 1$ and $\delta = C_{2,1}|a_2|^{\frac{1}{2}}$, (3.20) and (3.21) imply that with $D_{2,2} = (3C_{2,1}^{-1} + C_{2,1}\|A_2(\cdot)\|_{\dot{C}^{1,1}})$

$$|a_2^{(1)}| \leq D_{2,2}|a_2|^{\frac{1}{2}}$$

from which and (3.17) we obtain with $C_{2,3} = \min\{\frac{1}{1+4D_{2,2}+4\|A_2(\cdot)\|_{\dot{C}^{1,1}}}, \frac{1}{1+|a_2|}\}$, that satisfies $C_{2,3}|a_2| \leq 1$,

$$|A_2(t) - a_2| \leq \frac{1}{2}|a_2|, \qquad |t| \leq C_{2,3}|a_2|^{\frac{1}{2}}. \tag{3.23}$$

Therefore with

$$C_0 = \begin{cases} \min\{C_{2,1}(1+|a_2|)^{-1}, C_{2,2}\} & (2 \geq r_0+\gamma) \\ \min\{C_{2,1}, C_{2,3}\} & (2 \leq r_0+\gamma), \end{cases}$$

we see that, when $|t| \leq C_0|a_2|^{\max\{\frac{1}{2}, \frac{1}{r_0+\gamma}\}}$

$$\max\{|A_1(t)|, |A_2(t)|^{\frac{1}{2}}\} \leq 2|a_2|^{\frac{1}{2}}.$$

The estimates above and (2.2) show that for $j = 2, \dots, m$

$$|A_j(t)| \leq K|a_2|^{\frac{j}{2}}, \qquad |t| \leq C_0|a_2|^{\max\{\frac{1}{2}, \frac{1}{r_0+\gamma}\}}.$$

Putting

$$C_{0,0} = \frac{C_0}{(1+|a_2|)^m}$$

we see that $C_{0,0} \leq C_0$ and $C_{0,0}|a_2|^{\frac{1}{2}\max\{1, \frac{j}{r_0+\gamma}\}} \leq C_0|a_2|^{\frac{1}{2}\max\{1, \frac{2}{r_0+\gamma}\}}$ when $2 \leq j \leq m$. Hence $C_{0,0}|a_2|^{\frac{1}{2}\max\{1, \frac{j}{r_0+\gamma}\}} \leq 1$ when $1 \leq j \leq m$. Then it follows from (3.19), (3.22), (3.23) and the estimates above that we have following estimates;

$$\left|\frac{A_2(t)}{a_2} - 1\right| \leq \frac{1}{2}, \quad |t| \leq C_{0,0}|a_2|^{\frac{1}{2}\max\{1, \frac{2}{r_0+\gamma}\}},$$

$$\max\{|A_1(t)|, |A_2(t)|^{\frac{1}{2}}\} \leq 2|a_2|^{\frac{1}{2}}, \quad |t| \leq C_{0,0}|a_2|^{\frac{1}{2}\max\{1, \frac{2}{r_0+\gamma}\}}$$

and

$$|A_j(t)| \leq K|a_2|^{\frac{j}{2}} \qquad |t| \leq C_{0,0}|a_2|^{\frac{1}{2}\max\{1, \frac{j}{r_0+\gamma}\}}.$$

when $1 \leq j \leq m$.

Then we see that the assertion of Proposition 3.1 is valid when $l = 0$.

In the following , by the induction we show that the assertion for l of Proposition 3.1 is valid.

Let $l_0 \in \mathsf{M}$ with $0 < l_0 < k_0$. Assume that the assertion of Proposition 3.1 for $l \in \mathsf{M}$ with $0 \le l < l_0$ is valid. Then the assumption of Lemma 3.3 with $t_0 = 1$ is satisfied for $l \in \mathsf{M}$ with $0 < l < l_0$ and $A(t) = A_h(t)$ with $l + 1 \le h \le m$.

We see from (2.9), (2.11) and (2.12)

$$\max\{\left|\frac{a_{l_0+1}}{a_{l_0}}\right|, \left|\frac{a_{l_0+2}}{a_{l_0}}\right|^{\frac{1}{2}}\} = \left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}}, \tag{3.24}$$

$$|a_{\overline{l_0}}|^{l_0} \le K|a_{l_0}|^{\overline{l_0}} \tag{3.25}$$

and

$$\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}} \le K\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}}. \tag{3.26}$$

Since $\overline{(\underline{l_0})} = l_0$, we see from (3.9) with $l = \underline{l_0}$, $h = l_0 + 1$ and $A(t) = A_{l_0+1}(t)$ that

$$|a_{l_0+1}^{(j)}| \le D|a_{\underline{l_0}}|\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}(l_0+1-\underline{l_0}-j\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$$

from which and (3.26) we see

$$|a_{l_0+1}^{(j)}| \le DK|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}(1-j\max\{1,\frac{l_0+1}{r_0+\gamma}\})} \tag{3.27}$$

for $j = 1, \ldots, \min\{l_0, r_0\}$.

We see from Taylor's formula (3.7) that, with $k = \min\{l_0, r_0\}$ and $\alpha = \min\{l_0 + 1, r_0 + \gamma\} - k$,

$$|A_{l_0+1}(t) - a_{l_0+1}| \le \Big(\sum_{j=1}^{k} |a_{l_0+1}^{(j)}||t|^j + \|A_{l_0+1}(\cdot)\|_{\dot{C}^{k+\alpha}}|t|^{\min\{l_0+1,r_0+\gamma\}}\Big) \tag{3.28}$$

when $|t| \le 1$. Noting that (3.25) implies $(|a_{\overline{l_0}}|/|a_{l_0}|)^{(l_0)/(\overline{l_0}-l_0)} \le K|a_{l_0}|$, we see from (3.27) that, when $|t| \le \delta\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$ with $\delta > 0$, the right hand side of (3.28) is not larger than

$$\Big(\sum_{j=1}^{k} DK\delta^j + \|A_{l_0+1}(\cdot)\|_{\dot{C}^{k+\alpha}}K\delta^{\min\{l_0+1,r_0+\gamma\}}\Big)|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}}$$

which is not larger than $\frac{1}{2}|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}}$ if $\delta \le 1$ and

$$2(kDK + \|A_{l_0+1}(\cdot)\|_{\dot{C}^{k+\alpha}}K)\delta \le 1.$$

Noting (3.12) with $l = l_0$, we see that

$$|A_{l_0+1}(t) - a_{l_0+1}| \le \frac{1}{2}|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}} \tag{3.29}$$

when $|t| \le C_1\big(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\big)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$ with

$$C_1 = \min\{\frac{1}{(1+D_0)^{(l_0+1)}}, \frac{1}{2(kDK + \|A_{l_0+1}(\cdot)\|_{\dot{C}^{k+\alpha}}K)}\}.$$

Here we note

$$C_1\big(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\big)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}} \le 1.$$

On the other hand, from (3.9) with $h = l_0+2$, $l = \underline{l_0}$ and $A(t) = A_{l_0+2}(t)$, we have

$$|a^{(j)}_{l_0+2}| \le D|a_{\underline{l_0}}|\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}(l_0+2-\underline{l_0}-j\max\{1,\frac{l_0+2}{r_0+\gamma}\})}$$

from which and (3.26) we see

$$|a^{(j)}_{l_0+2}| \le DK|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}(2-j\max\{1,\frac{l_0+2}{r_0+\gamma}\})} \tag{3.30}$$

for $j = 2, \dots, \min\{l_0+1, r_0\}$.

Since we assume that the assertion of Proposition 3.1 for $l = \underline{l_0}$ is valid, we have

$$\left|\frac{A_{l_0}(t)}{a_{l_0}} - 1\right| \le \frac{1}{2} \tag{3.31}$$

when $|t| \le C_{\underline{l_0}}\big(|a_{l_0}|/|a_{\underline{l_0}}|\big)^{\max\{1,l_0/(r_0+\gamma)\}/(l_0-\underline{l_0})}$. Note that from (3.26) and (3.12) with $l = l_0$ we see that, when $|t| \le \frac{C_{\underline{l_0}}}{(1+K)^{l_0}(1+D_0)}\big(|a_{\overline{l_0}}|/|a_{l_0}|\big)^{\max\{1,(l_0+1)/(r_0+\gamma)\}/(\overline{l_0}-l_0)}$, the estimate (3.31) holds. Then by setting

$$C_2 = \min\{C_1, \frac{C_{\underline{l_0}}}{(1+K)^{l_0}(1+D_0)}\},$$

we see that (3.29) and (3.31) hold when $|t| \le C_2\big(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\big)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$. We see from (3.31) that $A_{l_0}(t) \ne 0$ when $|t| \le C_2\big(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\big)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$, which and (2.3) implies

$$\left(\frac{A_{l_0+1}(t)}{A_{l_0}(t)}\right)^2 - \frac{A_{l_0+2}(t)}{A_{l_0}(t)} \ge 0. \tag{3.32}$$

We see from (3.24) that $|a_{l_0+1}| \le |a_{l_0}|(|a_{\overline{l_0}}|/|a_{l_0}|)^{1/(\overline{l_0}-l_0)}$. Then we see from (3.29), (3.31) and (3.32) that we have

$$8\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{2}{\overline{l_0}-l_0}} - \frac{A_{l_0+2}(t)}{a_{l_0}} \ge 0 \tag{3.33}$$

when $|t| \le C_2\big(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\big)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$. Then, noting (3.24), it follows from (3.6) applied to (3.33) with $k = \min\{l_0+1, r_0\}$, $\alpha = \min\{l_0+2, r_0+\gamma\} - k$ and

$$\delta = \delta_1(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,\frac{l_0+2}{r_0+\gamma}\}/(\overline{l_0}-l_0)}$$

that

$$\left|\frac{a_{l_0+2}^{(1)}}{a_{l_0}}\right| \le \big(9\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{2}{\overline{l_0}-l_0}}\delta_1^{-1} + \sum_{j=2}^{k}\frac{|a_{l_0+2}^{(j)}|}{|a_{l_0}|}\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{j}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}}\delta_1^{j-1} + \frac{1}{|a_{l_0}|}\|A_{l_0+2}(\cdot)\|_{\dot{C}^{k+\alpha}}\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{l_0+2}{\overline{l_0}-l_0}}\delta_1^{\min\{l_0+2,r_0+\gamma\}-1}\big)\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{-\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}} \tag{3.34}$$

where $\delta_1 = C_2/(1+D_0)$. Here we note that the estimate (3.12) with $l = l_0$ implies $\delta_1(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,\frac{l_0+2}{r_0+\gamma}\}/(\overline{l_0}-l_0)}$ is less than $C_2(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,\frac{l_0+1}{r_0+\gamma}\}/(\overline{l_0}-l_0)}$. Noting that we have $(|a_{\overline{l_0}}|/|a_{l_0}|)^{l_0/(\overline{l_0}-l_0)} \le (K+1)|a_{l_0}|$ from (3.25), we see from (3.30) that the right hand side of (3.34) is not larger than

$$D_2\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}(2-\max\{1,\frac{l_0+2}{r_0+\gamma}\})} \tag{3.35}$$

with

$$D_2 = 9\delta_1^{-1} + \sum_{j=2}^{k}\delta_1^{j-1}DK + \|A_{l_0+2}(\cdot)\|_{\dot{C}^{k+\alpha}}\delta_1^{\min\{l_0+2,r_0+\gamma\}-1}(K+1).$$

Then we have $|a_{l_0+2}^{(1)}| \le D_2|a_{l_0}|(|a_{\overline{l_0}}|/|a_{l_0}|)^{(2-\max\{1,\frac{l_0+2}{r_0+\gamma}\})/(\overline{l_0}-l_0)}$, ¿from which and Taylor's formula (3.7), taking account of (3.30), we see that, with $k = \min\{l_0+1, r_0\}$ and $\alpha = \min\{l_0+2, r_0+\gamma\} - k$,

$$|A_{l_0+2}(t) - a_{l_0+2}| \le D_3|a_{l_0}|\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{2}{\overline{l_0}-l_0}}\sum_{j=1}^{k}|t|^j\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}(-j\max\{1,\frac{l_0+2}{r_0+\gamma}\})} + \|A_{l_0+2}(\cdot)\|_{\dot{C}^{k+\alpha}}|t|^{\min\{l_0+2,r_0+\gamma\}}, \quad |t| \le 1 \tag{3.36}$$

where

$$D_3 = D_2 + DK\max\{0, k-1\}.$$

Again we remark that $\min\{l_0+2, r_0+\gamma\}\max\{1,(l_0+2)/(r_0+\gamma)\}$ is equal to l_0+2 and $(|a_{\overline{l_0}}|/|a_{l_0}|)^{l_0/(\overline{l_0}-l_0)} \le (K+1)|a_{l_0}|$. Then when $|t| \le \delta_2(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,(l_0+2)/(r_0+\gamma)\}/(\overline{l_0}-l_0)}$, the right hand side of (3.36) is not larger than

$$\big(D_3\sum_{j=1}^{k}\delta_2^j + \|A_{l_0+2}(\cdot)\|_{\dot{C}^{k+\alpha}}(K+1)\delta_2^{\min\{l_0+2,r_0+\gamma\}}\big)|a_{l_0}|\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{2}{\overline{l_0}-l_0}}.$$

Therefore we see that

$$|A_{l_0+2}(t) - a_{l_0+2}| \le \frac{1}{2}|a_{l_0}|\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{2}{\overline{l_0}-l_0}} \tag{3.37}$$

when $|t| \le C_3(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,(l_0+2)/(r_0+\gamma)\}}$ where

$$C_3 = \min\{\frac{1}{(1+D_0)^{l_0+2}}, \frac{1}{2(D_3k + \|A_{l_0+2}(\cdot)\|_{\dot{C}^{k+\alpha}}(K+1))}\}.$$

Here we note that $C_3 \le 1$ and that the estimate (3.12) with $l = l_0$ implies that $C_3(|a_{\overline{l_0}}|/|a_{l_0}|)^{\max\{1,(l_0+2)/(r_0+\gamma)\}} \le 1$. Furthermore we obtain for $j \ge 1$

$$\frac{1}{(1+D_0)}\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{j+1}{r_0+\gamma}\}} \le \left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{j}{r_0+\gamma}\}}.$$

Thus by setting

$$C_{l_0} = \frac{1}{(1+D_0)^m}\min\{C_2, C_3\}$$

we see that

$$C_{l_0}\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{j}{r_0+\gamma}\}} \le C_2\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}} \le 1 \tag{3.38}$$

for $j = l_0+1, \dots, m$ and

$$C_{l_0}\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{j}{r_0+\gamma}\}} \le C_3\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}} \le 1 \tag{3.39}$$

for $j = l_0+2, \dots, m$. Then when $|t| \le C_{l_0}\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}}$, we have (3.29),(3.31) and (3.37).

Taking account of (2.9), we see that $\max\{|a_{l_0+1}|/|a_{l_0}|, (|a_{l_0+2}|/|a_{l_0}|)^{\frac{1}{2}}\}$ is equal to $(|a_{\overline{l_0}}|/|a_{l_0}|)^{1/(\overline{l_0}-l_0)}$. Thus we obtain from (3.29) and (3.37)

$$\left|\frac{A_{l_0+1}(t)}{a_{l_0}}\right| \le 2\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}} \tag{3.40}$$

and

$$\left|\frac{A_{l_0+2}(t)}{a_{l_0}}\right| \le 2\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{2}{\overline{l_0}-l_0}} \tag{3.41}$$

which and (3.31) imply that

$$\max\{\left|\frac{A_{l_0+1}(t)}{A_{l_0}(t)}\right|, \left|\frac{A_{l_0+2}(t)}{A_{l_0}(t)}\right|^{\frac{1}{2}}\} \le 4\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}}$$

when $|t| \le C_{l_0}\left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}}$. Therefore, taking account of (3.31), (3.38), (3.39), (3.40) and (3.41), we obtain from (2.3)

$$\left|\frac{A_h(t)}{a_{l_0}}\right| \le K\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{h-l_0}{\overline{l_0}-l_0}}, \quad h = l_0+1, \dots, m$$

when $|t| \le C_{l_0} \left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{h}{r_0+\gamma}\}}$. Finally we see from (3.29) and (3.37) that if $\overline{l_0} = l_0+1$, we have

$$\left|\frac{A_{l_0+1}(t)}{a_{l_0+1}} - 1\right| \le \frac{1}{2}$$

when $|t| \le C_{l_0} \left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+1}{r_0+\gamma}\}}$ otherwise we have

$$\left|\frac{A_{l_0+2}(t)}{a_{l_0+2}} - 1\right| \le \frac{1}{2}$$

when $|t| \le C_{l_0} \left(\frac{|a_{\overline{l_0}}|}{|a_{l_0}|}\right)^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}}$. Then we see that the assertion of Proposition 3.1 is valid for $l = l_0$. Then the proof of Proposition 3.1 is completed. □

Next we show a lemma on the behavior of coefficients $A_j(t)$ for $j > k_0$.

Lemma 3.4. *Under the same setting as in Proposition 3.1, without the restriction $k_0 > 0$, we assume $k_0 \le m-2$. Then we have*

$$a_h^{(j)} = 0$$

when $j \le r_0$ and $h - j\max\{1, \frac{2}{r_0+\gamma}\} > k_0$.

Proof. We use the same notations as in the proof of Proposition 3.1. By the definition of k_0, we have $a_{k_0} \ne 0$ and $a_h = 0$ for $h \ge k_0+1$. Then, from the continuity of $A_{k_0}(t)$ we see that there exists $C > 0$ such that

$$\left|\frac{A_{k_0}(t)}{a_{k_0}} - 1\right| \le \frac{1}{2}, \quad |t| \le C.$$

Hence from (2.3), we get

$$\left(\frac{A_{k_0+1}(t)}{A_{k_0}(t)}\right)^2 - \frac{A_{k_0+2}(t)}{A_{k_0}(t)} \ge 0 \ \ |t| \le C.$$

Since $A_{k_0+1}(0) = a_{k_0+1} = 0$, we have $|A_{k_0+1}(t)| \le D|t|$. Then we have

$$\frac{4D^2t^2}{a_{k_0}^2} - \frac{A_{k_0+2}(t)}{a_{k_0}} \ge 0 \ \ |t| \le C,$$

which and $A_{k_0+2}(0) = a_{k_0+2} = 0$ imply that $a_{k_0+2}^{(1)} = 0$. Then, since $A_{k_0+2}(t)$ belongs to $C^{r_0+\gamma}([-t_0, t_0])$, we have $|A_{k_0+2}(t)| \le D|t|^{\min\{2, r_0+\gamma\}}$. Hence we have

$$\max\{\left|\frac{A_{k_0+1}(t)}{A_{k_0}(t)}\right|, \left|\frac{A_{k_0+2}(t)}{A_{k_0}(t)}\right|^{\frac{1}{2}}\} \le D\frac{|a_{k_0}|+1}{|a_{k_0}|}|t|^{\min\{1,\frac{r_0+\gamma}{2}\}}, \quad |t| \le C.$$

Then from (2.4) we obtain for $h = k_0+1, \dots, m$

$$\left|\frac{A_h(t)}{a_{k_0}}\right| \le D\left(\frac{1+|a_{k_0}|}{|a_{k_0}|}\right)^{h-k_0}|t|^{(h-k_0)\min\{1,\frac{r_0+\gamma}{2}\}}, \quad |t| \le C.$$

Hence we have $A_h^{(j)}(0) = 0$ when $j \le r_0$ and $j < (h-k_0)\min\{1, \frac{r_0+\gamma}{2}\}$. Then

$$a_h^{(j)} = 0 \ \text{ when } j \le r_0 \text{ and } h - j\max\{1, \frac{2}{r_0+\gamma}\} > k_0.$$

□

4 Proof of Theorem 1.1 First we show the following lemma.

Lemma 4.1. *Let m be a positive integer and $T > 0$. Let*

$$p(t,\tau) = \tau^m + \sum_{h=1}^{m} A_h(t)\tau^{m-h}$$

be a hyperbolic polynomial in τ whose coefficients $A_h(t)$ belong to $C^{r_0,\gamma}([-T,T])$ with a positive integer r_0 and $< \gamma \leq 1$. For any S satisfying $0 < S < T$, there exists a positive constant C such that we have for $j = 1, \ldots, \min\{m-1, r_0\}$

$$\left| \frac{\partial_t^j p(t,\tau)}{p(t,\tau)} \right| \leq C|\Im\tau|^{-j\max\{1, \frac{m}{r_0+\gamma}\}}, \tag{4.1}$$

when $|t| \leq S$ and $\tau \in \mathbb{C}\backslash\mathbb{R}$ satisfies $|\Im\tau| \leq 1$.

Here the constant C, that may depend on S, can be chosen uniformly for a bounded family of coefficients $A_h(t)$ in $C^{r_0,\gamma}([-T,T])$ as long as $p(t,\tau)$ is hyperbolic.

Remark 4.1. Since $p(t,\tau)$ is hyperbolic in τ, we have the factorization:

$$p(t,\tau) = \prod_{l=1}^{m}(\tau - \lambda_l(t)) \tag{4.2}$$

with $\lambda_l(t) \in \mathbb{R}$. Hence $|p(t,\tau)| \geq |\Im\tau|^m$. For $j = 1, \ldots, r_0$ we have

$$|\partial_t^j p(t,\tau)| \leq D(|\tau|+1)^{m-1} \quad |t| \leq T.$$

Set $D_0 = \max_{|t|\leq T}(A_1(t)^2 - 2A_2(t))$. Then we see from (2.7) that for $j = 1, \ldots, m$

$$|\lambda_l(t)| \leq D_0^{\frac{1}{2}} \quad |t| \leq T.$$

Then it follows from (4.2) that $|p(t,\tau)| \geq 2^{-m}|\tau|^m$ when $|\tau| \geq 2D_0^{\frac{1}{2}}$ and $|t| \leq T$. Therefore we have for $j = 1, \ldots, r_0$

$$\left| \frac{\partial_t^j p(t,\tau)}{p(t,\tau)} \right| \leq \begin{cases} D|\Im\tau|^{-m} & \text{when } \Im\tau \neq 0,\ |\tau| \leq 2D_0^{\frac{1}{2}} \text{ and } |t| \leq T \\ D|\tau|^{-1} & \text{when } |\tau| \geq 2D_0^{\frac{1}{2}} \text{ and } |t| \leq T. \end{cases} \tag{4.3}$$

Proof of Lemma 4.1. When $m = 1$, (4.3) implies (4.1). In the following we assume $m \geq 2$. Taking account of the remark above, we have only consider the case $|\Re\tau| \leq 2D_0^{\frac{1}{2}}$. For $s \in [-S,S]$ and $\tau_0 \in [-2D_0^{\frac{1}{2}}, 2D_0^{\frac{1}{2}}]$ we set

$$p_{s,\tau_0}(t,\tau) = p(s+t, \tau_0+\tau).$$

Then $p_{s,\tau_0}(t,\tau)$ is hyperbolic and we have

$$p_{s,\tau_0}(t,\tau) = \tau^m + \sum_{h=1}^{m} A_{h,s,\tau_0}(t)\tau^{m-h}$$

where

$$\{A_{h,s,\tau_0}(t) \mid h = 1, \ldots, m \quad s \in [-S,S] \text{ and } \tau_0 \in [-2D_0^{\frac{1}{2}}, 2D_0^{\frac{1}{2}}]\}$$

is a bounded set in $C^{r_0,\gamma}([-t_0,t_0])$ with $t_0=T-S$.

Set

$$a_h=A_{h,s,\tau_0}(0) \ \text{ and } \ a_h^{(j)}=A_{h,s,\tau_0}^{(j)}(0).$$

Since $p_{s,\tau_0}(0,\tau)=\tau^m+\sum_{h=1}^m a_h\tau^{m-h}$ is hyperbolic, we have $p_{s,\tau_0}(0,\tau)=\prod_{l=1}^m(\tau-\lambda_l)$ with $\lambda_l\in\mathbb{R}$. Then we see from (2.6) that for real μ

$$\begin{aligned}|p_{s,\tau_0}(0,i\mu)| &\geq 2^{-m}\prod_{l=1}^m(|\mu|+|\lambda_l|)\\ &\geq 2^{-m}(|\mu|^m+\sum_{h=1}^m|a_h||\mu|^{m-h}).\end{aligned} \tag{4.4}$$

Now we show for $j=1,\dots,\min\{m-1,r_0\}$,

$$\left|\frac{\partial_t^j p_{s,\tau_0}(0,i\mu)}{p_{s,\tau_0}(0,i\mu)}\right|\leq D|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma}\}},\qquad 0<|\mu|\leq 1. \tag{4.5}$$

In order to draw the estimate above, we consider the estimate of $|a_h^{(j)}\mu^{m-h}/p_{s,\tau_0}(0,i\mu)|$.

Let k_0 be the largest h satisfying $a_h\neq 0$. Let M be a subset of $\{0,1,\dots,k_0\}$ defined in Lemma 2.4 applied to our $p_{s,\tau_0}(0,\tau)=\tau^m+\sum_{h=1}^m a_h\tau^{m-h}$. First we remark that it follows from Lemma 3.4 that

$$a_h^{(j)}=0$$

when $j\geq 1$ and $h-k_0>j\max\{1,\frac{h}{r_0+\gamma}\}$. For we see that $j\geq 1$ and $h-k_0>j\max\{1,\frac{h}{r_0+\gamma}\}$ imply $h\geq 2$ and $j\leq r_0$. On the other hand, when $\min\{m-1,r_0\}\geq j\geq 1$ and $h-j\max\{1,\frac{h}{r_0+\gamma}\}\leq 0$, noting $|\mu|^{-h}\leq|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma}\}}$ for $|\mu|\leq 1$, and (4.4), we obtain

$$\left|\frac{a_h^{(j)}\mu^{m-h}}{p_{s,\tau_0}(0,i\mu)}\right|\leq 2^m|a_h^{(j)}||\mu|^{-j\max\{1,\frac{m}{r_0+\gamma}\}},\quad |\mu|\leq 1.$$

Finally in the case where $\min\{m-1,r_0\}\geq j\geq 1$ and $k_0\geq h-j\max\{1,\frac{h}{r_0+\gamma}\}>0$, there exists $l\in\mathsf{M}$ such that we have

$$l<h-j\max\{1,\frac{h}{r_0+\gamma}\}\leq\bar{l}. \tag{4.6}$$

Thanks to (3.4) of Proposition 3.1, we have

$$\left|\frac{A_{h,s,\tau_0}(t)}{a_l}\right|\leq K\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}}$$

when $|t|\leq C\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}\max\{1,\frac{h}{r_0+\gamma}\}}$. From (3.5) applied to $A_{h,s,\tau_0}(t)$ with $k=\min\{h-1,r_0\}$ and $\alpha=\min\{h,r_0+\gamma\}-k$ with

$$\delta=C\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}\max\{1,\frac{h}{r_0+\gamma}\}}$$

we obtain

$$|a_h^{(j)}|\leq\Gamma\big(K|a_l|\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{h-l}{\bar{l}-l}}+\|A_{h,s,\tau_0}(\cdot)\|_{\dot{C}^{k,\alpha}}\delta^{\min\{h,r_0+\gamma\}}\big)\delta^{-j}.$$

Just as the proof of Proposition 3.1, we obtain from (2.11)

$$\left|\frac{a_{\bar{l}}}{a_l}\right|^h \leq K|a_l|^{\bar{l}-l}\left|\frac{a_{\bar{l}}}{a_l}\right|^{h-l}.$$

Then

$$|a_h^{(j)}| \leq D|a_l|\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}(h-l-j\max\{1,\frac{h}{r_0+\gamma}\})}$$

with

$$D = \Gamma(K + K^{\frac{1}{\bar{l}-l}}\|A_{h,s,\tau_0}(\cdot)\|_{\dot{C}^{k,\alpha}}C^{\min\{h,r_0+\gamma\}})C^{-j}.$$

Since (4.6) implies

$$1 \geq \frac{1}{\bar{l}-l}(h-l-j\max\{1,\frac{h}{r_0+\gamma}\}) > 0,$$

we have

$$|a_h^{(j)}| \leq D|a_l|^{1-\sigma}|a_{\bar{l}}|^{\sigma}$$

with

$$\sigma = \frac{1}{\bar{l}-l}(h-l-j\max\{1,\frac{h}{r_0+\gamma}\}).$$

Hence by Young's inequality we have

$$\frac{|a_h^{(j)}||\mu|^{m-h}}{|a_l||\mu|^{m-l}+|a_{\bar{l}}||\mu|^{m-\bar{l}}} \leq D|\mu|^{-h+(1-\sigma)l+\sigma\bar{l}}.$$

Since $-h+(1-\sigma)l+\sigma\bar{l} = -j\max\{1,\frac{h}{r_0+\gamma}\}$, ¿from the estimate above and (4.4) we obtain

$$\frac{|a_h^{(j)}||\mu|^{m-h}}{|p_{s,\tau_0}(0,i\mu)|} \leq D|\mu|^{-j\max\{1,\frac{h}{r_0+\gamma}\}},$$

since $h \leq m$

$$\leq D|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma}\}}, \qquad |\mu| \leq 1.$$

Then we have the desired estimate (4.5).

Since, according to Proposition 3.1, the constant above D can be chosen uniformly when $|s| \leq S$ and $|\tau_0| \leq 2D_0^{\frac{1}{2}}$, we obtain from (4.5), for $j = 1,\ldots,\min\{m-1,r_0\}$,

$$\left|\frac{\partial_t^j p(t,\tau)}{p(t,\tau)}\right| \leq D|\Im\tau|^{-j\max\{1,\frac{m}{r_0+\gamma}\}}$$

when $|s| \leq S$, $|\Re\tau| \leq 2D_0^{\frac{1}{2}}$ and $0 < |\Im\tau| \leq 1$. We remark that the constant D can be chosen uniformly for a bounded family of coefficients $A_j(t)$ in $C^{r_0,\gamma}([-T,T])$, for Proposition 3.1 claims that the constant C appearing in the argument above has an estimate $\delta_1 \leq C \leq \delta_2$ by using two positive constants δ_1 and δ_2 that can be chosen uniformly for a bounded family of coefficients $A_h(t)$ in $C^{r_0,\gamma}([-T,T])$ although they may depend on S. □

The following lemma is obvious.

Lemma 4.2. *Let*

$$P(t,\tau)=\tau^M+\sum_{h=1}^{M}A_h(t)\tau^{M-h}$$

be a hyperbolic polynomial in τ whose coefficients $A_h(t)\in C^{r_0,\gamma}([-T,T])$ with a positive integer r_0 and $\gamma\in(0,1]$. We assume that the multiplicity of its roots is at most m with $m<M$. That is to say, assuming that

$$P(t,\tau)=\prod_{l=1}^{M}(\tau-\lambda_l(t))$$

where $\lambda_1(t)\le\lambda_2(t)\le\cdots\le\lambda_M(t)$, we suppose that there exists a positive constant Δ such that we have

$$\lambda_{m+l}(t)-\lambda_l(t)\ge\Delta \tag{4.7}$$

for $l=1,\dots,M-m$ and $t\in[-T,T]$. Then there exists a positive constant δ such that for any $s\in[-T,T]$, we have a decomposition

$$P(t,\tau)=\prod_{k=1}^{M_s}p_{k,s}(t,\tau)\quad t\in[-T,T]\cap[s-\delta,s+\delta]$$

where each polynomial $p_{k,s}(t,\tau)$ has coefficients in $C^{r_0,\gamma}([-T,T]\cap[s-\delta,s+\delta])$ and the degree at most m. Furthermore for a bounded family of $A_h(t)$ in $C^{r_0,\gamma}([-T,T])$, we have uniform $C^{r_0,\gamma}$ estimates of coefficients of $p_{k,s}(t,\tau)$ if $P(t,\tau)$ is hyperbolic and we have uniform estimates (4.7).

Proof. For the completeness we give a proof. Since in the case where $M=1$ the assertion of Lemma 4.2 is evident, we assume $M\ge 2$,. Let $D_0=\max_{|t|\le T}(|A_1(t)^2-2A_2(t)|^{\frac{1}{2}})$. Then we have $|\lambda_l(t)|\le D_0$. It follows we from (4.7) that for $s\in[-T,T]$, there exist $l_1,l_2,\dots,l_{s_M}=M$ such that $l_1<l_2<\cdots<l_{s_M}=M$, $l_1\le m$, $l_{k+1}-l_k\le m$ $(k=1,\cdots,s_M-1)$ and $\lambda_{l_k+1}(s)-\lambda_{l_k}(s)\ge\frac{\Delta}{m+1}$ $(k=1,\dots,s_M-1)$. Let $\nu_k=\frac{\lambda_{l_k+1}(s)+\lambda_{l_k}(s)}{2}$ $(k=1,\dots,s_M-1)$. Then we have $|P(s,\nu_k)|\ge\big(\frac{\Delta}{2(m+1)}\big)^M$. Since $|P(t,\tau)-P(s,\tau)|\le D|t-s|(|\tau|+1)^{M-1}$, we see that

$$|P(t,\nu_k)|\ge\frac{1}{2}\Big(\frac{\Delta}{2(m+1)}\Big)^M,\quad t\in[-T,T]\cap[s-\delta,s+\delta]$$

if $\delta>0$ satisfies

$$D(D_0+1)^{M-1}\delta\le\frac{1}{2}\Big(\frac{\Delta}{2(m+1)}\Big)^M.$$

Hence

$$p_{k,s}(t,\tau)=\prod_{l=l_{k-1}+1}^{l_k}(\tau-\lambda_l(t))$$

with $l_0 = 0$, satisfies desired properties. For $\sum_{l=l_{k-1}+1}^{l_k} \lambda_l(t)^r$ $(r = 1, \dots, l_k - l_{k-1} - 1)$ can be given by the contour integral of $\frac{\tau^r \partial_\tau P(t,\tau)}{P(t,\tau)}$ along a closed curve, enclosing only $\lambda_l(t)$ $(l = l_{k-1}, \dots, l_k)$, on which we have

$$|P(t,\tau)| \geq \frac{1}{2}\Big(\frac{\Delta}{2(m+1)}\Big)^M.$$

Then we see that the coefficients of $p_{k,s}(t,\tau)$ belong to $C^{r_0,\gamma}([-T,T] \cap [s-\delta, s+\delta])$. □

We see that Theorem 1.1 follows immediately from Lemma 4.1 and Lemma 4.2. Indeed, according to Lemma 4.2, we see that $\partial_t^j P(t,\tau)/P(t,\tau))$ with $j \leq \min\{m-1, r_0\}$ is given by a sum of products $\partial_t^{j_k} p_{k,s}(t,\tau)/p_{k,s}(t,\tau)$ with $\sum_{1 \leq k \leq M_s} j_k = j$. Since the degree of $p_{k,s}(t,\tau)$ is not larger than m, we see from Lemma 4.1 and (4.3) that $|\partial_t^{j_k} p_{k,s}(t,\tau)/p_{k,s}(t,\tau)|$ is not larger than $D|\Im\tau|^{-j_k \max\{1, m/(r_0+\gamma)\}}$ when $0 < |\Im\tau| \leq 1$. Then we have the estimate (1.1).

5 Proof of Corollaries 1.2 and 1.3 We prove Corollary 1.2 using Theorem 1.1. As the proof of Theorem 1.1, we have only consider the case of $|\Re\tau| \leq 2D_0$ where

$$D_0 = \max_{(t,x)\in[-T,T]\times[-L,L]} \sqrt{M(A_1^2(t,x) - 2A_2(t,x))}.$$

Let

$$\begin{aligned} P(t,x,\tau) =& \tau^M + \sum_{h=1}^{M} A_h(t,x)\tau^{M-h} \\ =& \prod_{l=1}^{M} (\tau - \lambda_l(t,x)) \end{aligned}$$

where $\lambda_1(t,x) \leq \dots \leq \lambda_M(t,x)$. Since $A_h(t,x)$ is continuous in $[-T,T] \times [-L,L]$ and the multiplicity of roots is at most m, we see that, if $m < M$,

$$C_1 = \min_{l=1,\dots,M-m \quad (t,x)\in[-T,T]\times[-L,L]} (\lambda_{l+m}(t,x) - \lambda_l(t,x)) \tag{5.1}$$

is positive. Then Theorem 1.1 shows that for any $S \in (0,T)$ and $L_0 \in (0,L)$ there exists a positive constant C such that for $j = 1, \dots, \min\{m-1, r_0\}$

$$\left|\frac{\partial_t^j P(t,x,\tau)}{P(t,x,\tau)}\right| \leq C|\Im\tau|^{-j\max\{1, \frac{m}{r_0+\gamma_0}\}} \quad (t,x) \in [-S,S] \times [-L,L] \tag{5.2}$$

and for $\kappa = 1, \dots, \min\{m-1, r_1\}$

$$\left|\frac{\partial_x^\kappa P(t,x,\tau)}{P(t,x,\tau)}\right| \leq C|\Im\tau|^{-\kappa\max\{1, \frac{m}{r_1+\gamma_1}\}} \quad (t,x) \in [-T,T] \times [-L_0,L_0] \tag{5.3}$$

when $\tau \in \mathbb{C}$ with $\Im\tau \in [-1,1]$. We note that the hyperbolicity of $P(t,x,\tau)$ and the assumption on the multiplicity of roots (5.1) imply

$$|P(t,x,\tau+i\mu)| \geq \Big(\frac{C_1}{2}\Big)^{M-m} |\mu|^m \tag{5.4}$$

when $\tau \in \mathbb{R}$ and $\mu \in [-1,1]$. For $\mu \in [-(L-L_0), L-L_0] \cap [-1,1]$, $\tau \in [-2D_0, 2D_0]$ and $(t,x) \in [-T,T] \times [-L_0, L_0]$ we define a function $f(y)$ on $[-1,1]$ by

$$f(y) = P(t, x + y|\mu|^{\max\{1, \frac{m}{r_1+\gamma_1}\}}, \tau + i\mu).$$

By Taylor 's formula (3.7) with $k = \min\{m-1, r_1\}$ and $\alpha = \min\{m, r_1+\gamma_1\} - k$, we have for $y \in [-1,1]$

$$|f(y)| \le |f(0)| + \sum_{j=1}^{k} |f^{(j)}(0)| + \|f(\cdot)\|_{\dot{C}^{k,\alpha}}.$$

Since (5.3) implies, for $j = 0, \dots, m-1$,

$$|f^{(j)}(0)| \le C|P(t,x,\tau+i\mu)|,$$

noting $\|f(\cdot)\|_{\dot{C}^{k,\alpha}([-1,1])} \le D|\mu|^{\max\{1,\frac{m}{r_1+\gamma_1}\}\min\{m, r_1+\gamma_1\}} = D|\mu|^m$, we obtain from (5.4)

$$|P(t, x + y|\mu|^{\max\{1, \frac{m}{r_1+\gamma_1}\}}, \tau + i\mu)| \le D|P(t,x,\tau+i\mu)|. \tag{5.5}$$

For $(t,x) \in [-S,S] \times [-L_0, L_0]$, $\mu \in [-(L-L_0), L-L_0] \cap [-1,1]$, $\tau \in [-2D_0, 2D_0]$ and $j = 1, \dots, \min\{m-1, r_0\}$ we define a function $g_j(y)$ on $[-1,1]$ by

$$g_j(y) = \partial_t^j P(t, x + y|\mu|^{\max\{1, \frac{m}{r_1+\gamma_1}\}}, \tau + i\mu).$$

Then from (5.2) and (5.5) we obtain

$$|g_j(y)| \le D|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma_0}\}}|P(t,x,\tau+i\mu)| \quad |y| \le 1.$$

Hence from (3.5) applied to $g_j(y)$ with $k = \min\{m-1, r_1\}$, $\alpha = \min\{m, r_1+\gamma_1\}$ and $\delta = 1$ we see that for $\kappa = 1, \dots, \min\{m-1, r_1\}$

$$|g_j^{(\kappa)}(0)| \le \Gamma(D|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma_0}\}}|P(t,x,\tau+i\mu)| + \|g_j(\cdot)\|_{\dot{C}^{k,\alpha}}).$$

Since $k + \alpha = \min\{m, r_1+\gamma_1\}$, we have

$$\begin{aligned} \|g_j(\cdot)\|_{\dot{C}^{k,\alpha}} &\le D|\mu|^{\max\{1,\frac{m}{r_1+\gamma_1}\}\min\{m,r_1+\gamma_1\}} \\ &= D|\mu|^m \\ &\le D|P(t,x,\tau+i\mu)| \end{aligned}$$

Then we have

$$|\partial_x^\kappa \partial_t^j P(t,x,\tau+i\mu)| \le D|\mu|^{-\kappa\max\{1,\frac{m}{r_1+\gamma_1}\}}\big(|\mu|^{-j\max\{1,\frac{m}{r_0+\gamma_0}\}} + 1\big)|P(t,x,\tau+i\mu)|.$$

Noting $|\mu| \le 1$, we obtain the desired estimate when $|\mu| \le \min\{L-L_0, 1\}$. In the cse of $L - L_0 < 1$, the estimate for $L - L_0 < |\mu| \le 1$ follows from (5.4). The proof of Corollary 1.2 is completed.

Proof of Corollary 1.3.

First note that for $a(t) \in C^{1,\gamma}([0,2T])$ with $T > 0$, the function $\tilde{a}(t)$ defined by

$$\tilde{a}(t) = a(|t|^{1+\gamma}), \qquad |t| \le (2T)^{\frac{1}{1+\gamma}}$$

belongs to $C^{1,\gamma}([-(2T)^{\frac{1}{1+\gamma}}, (2T)^{\frac{1}{1+\gamma}}])$. For we have $||t|^\gamma - |s|^\gamma| \le |t-s|^\gamma$ if $0 < \gamma \le 1$.

Near $t = T$, we consider the hyperbolic polynomial $\tilde{P}(s,\tau)$ defined by

$$\tilde{P}(s,\tau) = P(T - |s|^{1+\gamma}, \tau) \quad |s| \le (2T)^{\frac{1}{1+\gamma}}.$$

Theorem 1.1 shows that

$$\left| \frac{\partial_s \tilde{P}(s,\tau)}{\tilde{P}(s,\tau)} \right| \le C|\Im\tau|^{-\max\{1, \frac{m}{1+\gamma}\}}$$

when $|s| \le T^{\frac{1}{1+\gamma}}$ and $\tau \in \mathbb{C}$ with $|\Im\tau| \le 1$. Since we have

$$\partial_s \tilde{P}(s,\tau) = -(1+\gamma)s^\gamma \partial_t P(T - s^{1+\gamma}, \tau)$$

for $s > 0$, we see that

$$\partial_t P(t,\tau) = \frac{-1}{1+\gamma}(T-t)^{-\frac{\gamma}{1+\gamma}} \partial_s \tilde{P}(s,\tau)$$

with $s = (T-t)^{\frac{1}{1+\gamma}}$ if $T > t \ge 0$. Then we get the estimate (1.4) when $T > t \ge 0$. Similarly we obtain the estimate (1.4) when $0 \ge t > -T$. The proof of Corollary 1.3 is completed. ☐

We remark that the proof of Corollary 1.3 can be applied to a hyperbolic polynomial with several parameters as that considered in Corollary 1.2.

6 Proof of Theorem 1.4 According to Lemma 4.2, it is sufficient to prove the following Proposition.

Proposition 6.1. *Let m be a positive integer and $T > 0$. Let*

$$p(t,\tau) = \tau^m + \sum_{h=1}^{m} A_h(t)\tau^{m-j}$$

be a hyperbolic polynomial with coefficients $A_h(t) \in C^{r_0,\gamma}([-T,T])$ with a positive integer r_0 and $\gamma \in (0,1]$.

Let $\lambda_l(t)$ $(l = 1,\dots,m)$ be roots of $p(t,\tau)$ numbered in increasing order $\lambda_l(t) \ge \lambda_{l-1}(t)$ $(l = 1,\dots,m-1)$.

Then for any $S \in (0,T)$, we see that $\lambda_j(t) \in C^{0,\min\{1,\frac{r_0+\gamma}{m}\}}([-S,S])$ for $j = 1,\dots,m$.

Here $C^{0,\min\{1,\frac{r_0+\gamma}{m}\}}$-norm of the roots $\lambda_l(t)$ in $[-S,S]$ is uniformly bounded for a bounded family of $A_h(t)$ in $C^{r_0,\gamma}([-T,T])$ as long as $p(t,\tau)$ is hyperbolic.

Proof. Since it is evident in the case of $m = 1$, in the following we assume $m \ge 2$. Set $t_0 = T - S$. For any $s \in [-S,S]$ and any root $\lambda_{l_0}(s)$ $(l_0 = 1,\dots,m)$ of $p(s,\tau)$, set

$$p_{s,l_0}(t,\tau) = p(t+s, \tau + \lambda_{l_0}(s)) = \tau^m + \sum_{h=1}^{m} A_{s,l_0,h}(t)\tau^{m-j}.$$

Then we see that $p_{s,l_0}(0,0) = 0$. We remark that $\{A_{s,l_0,h}(t) \mid s \in [-S,S],\ l_0, h = 1,\dots,m\}$ is a bounded set in $C^{r_0,\gamma}([-t_0,t_0])$. Proposition 6.1 follows from the following lemma.

Lemma 6.2. *Let m be a positive integer greater than 1 and $t_0 > 0$. Let*

$$p(t,\tau) = \tau^m + \sum_{h=1}^{m} A_h(t)\tau^{m-h}$$

be a hyperbolic polynomial with coefficients $A_h(t) \in C^{r_0,\gamma}([-t_0,t_0])$. *with a positive integer* r_0 *and* $\gamma \in (0,1]$

Assume $p(0,0)=0$. *Then there exists a constant* $C \in (0,t_0]$ *such that for any* $\sigma \in (0,1]$ *we have some constant* $\nu \in (0,1]$ *so that*

$$p(t,\nu\sigma) \neq 0 \quad \textit{and} \quad p(t,-\nu\sigma) \neq 0 \quad \textit{when } |t| \le C\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}.$$

Here for a bounded family of $A_j(t)$ *in* $C^{r_0,\gamma}([-t_0,t_0])$, *the constant above* C *has the estimate* $\sigma_1 \le C \le \sigma_2$ *with some positive* σ_1 *and* σ_2, *as long as* $p(t,\tau)$ *is hyperbolic.*

Now assuming that Lemma 6.2 is valid, we continue the proof of Proposition 6.1. Indeed applying Lemma 6.2 to $p_{s,j_0}(t,\tau)$ and noting $p_{s,j_0}(t,\lambda_{j_0}(t+s)-\lambda_{j_0(s)})=0$, we see from the continuity of $\lambda_{j_0}(t)$ that there exists a positive constant $C\in(0,t_0]$ such that for any $\sigma\in(0,1]$, we have

$$|\lambda_{j_0}(t+s)-\lambda_{j_0}(s)| \le \sigma, \quad |t| \le C\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}.$$

Then we have

$$|\lambda_{j_0}(t)-\lambda_{j_0}(s)| \le D|t-s|^{\min\{1,\frac{r_0+\gamma}{m}\}} \text{ when } t,s\in[-S,S] \text{ satisfy } |t-s|\le C$$

with $D = C^{-\min\{1,\frac{r_0+\gamma}{m}\}}$. Hence we see that $\lambda_{j_0}(t) \in C^{0,\min\{1,\frac{r_0+\gamma}{m}\}}([-S,S])$. According to Lemma 6.2, the constant above D can be chosen uniformly for a bounded family of $A_j(t)$ in $C^{r_0,\gamma}([-T,T])$. Then we are done. □

In the following we give two proofs of Lemma 6.2. First one uses Proposition 3.1. Second one depends only on Theorem 1.1.

First proof of Lemma 6.2. Set $a_h = A_h(0)$ and $a_h^{(j)} = A_h^{(j)}(0)$ with $a_0 = A_0(t) = 1$. Let k_0 be the largest h satisfying $a_h \neq 0$. Since $p(0,0)=0$, we see that $k_0 < m$. Let M be the subset of $\{0,\dots,k_0\}$ satisfying the properties mentioned in Lemma 2.4 applied to our polynomial $\tau^m + \sum_{h=1}^m a_h\tau^{m-l}$. Recall that for the element l in M satisfying $l<k_0$, $\overline{l}$ is the smallest element of $\mathsf{M}\cap\{l+1,\dots,k_0\}$, while for the element l in M satisfying $0<l$, $\underline{l}$ is the largest element of $\mathsf{M}\cap\{0,\dots,l-1\}$. We note that $\overline{(\underline{l})}=l$. Using the constant K appearing in Lemma 2.5 and Proposition 3.1 applied to $p(0,\tau)$ and $p(t,\tau)$ respectively, where we assume $K \ge (3m^3)^m$ taking (2.5) of Lemma 2.2 into account, we set

$$K_l = (8K+2)(4^5(K+1)^3)^l \tag{6.1}$$

for $l\in\mathsf{M}$. Then $K_0 = 8K+2$.

First we consider the case where $k_0>0$. For $\sigma\in(0,1]$, let l_0 be the smallest element $l\in\mathsf{M}$ satisfying

$$\left|\frac{a_{\overline{l}}}{a_l}\right|^{\frac{1}{\overline{l}-l}} \le \frac{\sigma}{K_l}.$$

If $l_0=0$, noting $a_0=1$ we have $|a_{\overline{0}}|^{1/\overline{0}} \le \sigma/K_0$. The estimate (3.4) of Proposition 3.1 implies, for $j=1,\dots,m$,

$$|A_h(t)| \le K|a_{\overline{0}}|^{\frac{h}{\overline{0}}}, \quad |t| \le C|a_{\overline{0}}|^{\frac{1}{\overline{0}}\max\{1,\frac{h}{r_0+\gamma}\}}. \tag{6.2}$$

Lemma 3.3 with $l=0$ and $A(t)=A_h(t)$ implies

$$|a_h^{(j)}| \le D|a_{\overline{0}}|^{\frac{1}{\overline{0}}(h-j\max\{1,\frac{h}{r_0+\gamma}\})} \quad \text{for } j=1,\dots,\min\{h-1,r_0\}.$$

Noting $h-j\max\{1,\frac{h}{r_0+\gamma}\}\geq 0$ for $j=1,\dots,\min\{h-1,r_0\}$ and $|a_{\overline{0}}|^{1/\overline{0}}\leq\sigma/K_0$, we have

$$|a_h^{(j)}|\leq D\sigma^{(h-j\max\{1,\frac{h}{r_0+\gamma}\})}.$$

Then by Taylor's formula (3.7) with $k=\min\{h-1,r_0\}$ and $\alpha=\min\{h,r_0+\gamma\}-k$, noting $\min\{h,r_0+\gamma\}\max\{1,\frac{h}{r_0+\gamma}\}=h$, we get

$$|A_h(t)-a_h|\leq D\sigma^h\mu$$

when $|t|\leq\mu\sigma^{\max\{1,\frac{h}{r_0+\gamma}\}}$ where $\mu\leq\min\{1,t_0\}$. Note that (6.2) and $|a_{\overline{0}}|^{1/\overline{0}}\leq\sigma/K_0$ imply $|a_h|\leq K\frac{\sigma^h}{K_0^h}$. We see that with $C_0=\min\{\frac{1}{(D+1)K_0^m},1,t_0\}$

$$|A_h(t)|\leq 2\frac{K\sigma^h}{K_0^h}\ ,\quad |t|\leq C_0\sigma^{\max\{1,\frac{h}{r_0+\gamma}\}}.$$

Hence it follows from $K_0=8K+2$ and $\sigma\in(0,1]$, we see that

$$\sum_{h=1}^m|A_h(t)||\sigma|^{m-h}\leq\frac{\sigma^m}{2}$$

when $|t|\leq C_0\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Therefore we have

$$|p(t,\tau)-\tau^m|\leq\frac{|\tau|^m}{2}$$

when $\tau=\pm\sigma$ and $|t|\leq C_0\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$, from which we see

$$p(t,\sigma)\neq 0\quad\text{and}\quad p(t,-\sigma)\neq 0 \tag{6.3}$$

when $|t|\leq C_0\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$.

If $0<l_0<\overline{l_0}\leq k_0$, then

$$\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}}\leq\frac{\sigma}{K_{l_0}} \tag{6.4}$$

$$\left|\frac{a_{\overline{l}}}{a_l}\right|^{\frac{1}{\overline{l}-l}}>\frac{\sigma}{K_l} \tag{6.5}$$

for $l<l_0$ in M. Then, since $K_l<K_{l+1}$ and $a_0=1$, we have from (6.5)

$$|a_{l_0}|\geq\left(\frac{\sigma}{K_{\underline{l_0}}}\right)^{l_0}. \tag{6.6}$$

We remark that from (3.1), (3.2), (3.3) and (3.4) with $l=l_0$ and $l=\underline{l_0}$ we obtain

$$\max\{\left|\frac{A_{l_0}(t)}{a_{l_0}}-1\right|,\left|\frac{A_{\overline{l_0}}(t)}{a_{\overline{l_0}}}-1\right|\}\leq\frac{1}{2},\ \ |t|\leq C\Big|\frac{a_{\overline{l_0}}}{a_{l_0}}\Big|^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{\overline{l_0}}{r_0+\gamma}\}}, \tag{6.7}$$

$$\max\{\left|\frac{A_{l_0+1}(t)}{A_{l_0}(t)}\right|,\left|\frac{A_{l_0+2}(t)}{A_{l_0}(t)}\right|^{\frac{1}{2}}\}\leq 4\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}},\ \ |t|\leq C\Big|\frac{a_{\overline{l_0}}}{a_{l_0}}\Big|^{\frac{1}{\overline{l_0}-l_0}\max\{1,\frac{l_0+2}{r_0+\gamma}\}} \tag{6.8}$$

and

$$\left|\frac{A_h(t)}{a_{l_0}}\right| \le K \left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{h-l_0}{\overline{l_0}-l_0}} \qquad h = l_0+1,\dots,m \tag{6.9}$$

when $|t| \le C\left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\max\{1,\frac{h}{r_0+\gamma}\}/(\overline{l_0}-l_0)}$, and

$$\max\left\{\left|\frac{A_{\underline{l_0}}(t)}{a_{\underline{l_0}}}-1\right|,\left|\frac{A_{l_0}(t)}{a_{l_0}}-1\right|\right\} \le \frac{1}{2},\ \ |t| \le C\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}\max\{1,\frac{l_0}{r_0+\gamma}\}}, \tag{6.10}$$

$$\max\left\{\left|\frac{A_{\underline{l_0}+1}(t)}{A_{\underline{l_0}}(t)}\right|,\left|\frac{A_{\underline{l_0}+2}(t)}{A_{\underline{l_0}}(t)}\right|^{\frac{1}{2}}\right\} \le 4\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}},\ \ |t| \le C\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}\max\{1,\frac{\underline{l_0}+2}{r_0+\gamma}\}} \tag{6.11}$$

and

$$\left|\frac{A_h(t)}{a_{\underline{l_0}}}\right| \le K \left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{h-\underline{l_0}}{l_0-\underline{l_0}}} \qquad h = \underline{l_0}+1,\dots,m \tag{6.12}$$

when $|t| \le C\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\max\{1,\frac{h}{r_0+\gamma}\}/(l_0-\underline{l_0})}$.

Now we show

$$\max\left\{\left|\frac{A_{l_0-1}(t)}{A_{l_0}(t)}\right|,\left|\frac{A_{l_0-2}(t)}{A_{l_0}(t)}\right|^{\frac{1}{2}}\right\} \le 16(K+1)\left|\frac{a_{\underline{l_0}}}{a_{l_0}}\right|^{\frac{1}{l_0-\underline{l_0}}} \tag{6.13}$$

when $|t| \le C\left(\frac{\sigma}{K_{l_0}}\right)^{\max\{1,\frac{m}{r_0+\gamma}\}}$where $A_{-1}(t) = 0$. Indeed if $l_0 = 1$, we see that (6.13) follows from (6.6) and (6.10). Next consider the case where $l_0 > 1$. If $\underline{l_0} = l_0 - 2$, we see from (6.10) and (6.11) that

$$\max\left\{\left|\frac{A_{l_0-1}(t)}{A_{l_0}(t)}\right|,\left|\frac{A_{l_0-2}(t)}{A_{l_0}(t)}\right|^{\frac{1}{2}}\right\} \le 16\left|\frac{a_{\underline{l_0}}}{a_{l_0}}\right|^{\frac{1}{l_0-\underline{l_0}}} \tag{6.14}$$

when $|t| \le C\left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}\max\{1,\frac{\underline{l_0}+2}{r_0+\gamma}\}}$. Hence from $\frac{\sigma}{K_l} \le 1$ and (6.5) we obtain (6.13). Similarly, in the case where $\underline{l_0} = l_0 - 1$ and $\underline{l_0 - 1} = l_0 - 2$, note that (3.1) with $l = l_0 - 2$ and (6.10) imply

$$|A_{l_0-2}(t)| \le 2|a_{l_0-2}|,\quad |A_{l_0-1}(t)| \le 2|a_{l_0-1}|,\quad \frac{1}{2}|a_{l_0}| \le |A_{l_0}(t)| \tag{6.15}$$

when $|t| \le C\min\{\left|\frac{a_{l_0-1}}{a_{l_0-2}}\right|^{\max\{1,\frac{l_0-1}{r_0+\gamma}\}},\left|\frac{a_{l_0}}{a_{l_0-1}}\right|^{\max\{1,\frac{l_0}{r_0+\gamma}\}}\}$. Since $l_0-2, l_0-1 \in \mathsf{M}$, the property 3) of M in Lemma 2.4 implies $|a_{l_0-2}a_{l_0}| < a_{l_0-1}^2$. Then we obtain (6.13) from (6.5) and (6.15).

In the case where $\underline{l_0} = l_0 - 1$ and $\underline{l_0 - 1} = l_0 - 3$, (3.4) with $l = l_0 - 3$ and $h = l_0 - 2$ imply

$$\left|\frac{A_{l_0-2}(t)}{a_{l_0-3}}\right| \le K \left|\frac{a_{l_0-1}}{a_{l_0-3}}\right|^{\frac{1}{2}}, \ |t| \le C \left|\frac{a_{l_0-1}}{a_{l_0-3}}\right|^{\frac{1}{2}\max\{1, \frac{l_0-2}{r_0+\gamma}\}},$$

while we obtain $\left|\frac{a_{l_0}}{a_{l_0-1}}\right| \le K \left|\frac{a_{l_0-1}}{a_{l_0-3}}\right|^{\frac{1}{2}}$ from (2.12). Then we see from (6.5) and (6.10) that (6.13) is valid. For we have

$$\left|\frac{A_{l_0-2}(t)}{A_{l_0}(t)}\right| \le 2K \left|\frac{a_{l_0-1}a_{l_0-3}}{a_{l_0}^2}\right|^{\frac{1}{2}} \le 2K^2 \left|\frac{a_{l_0-1}}{a_{l_0}}\right|^2,$$

from which we obtain

$$\max\{\left|\frac{A_{l_0-1}(t)}{A_{l_0}(t)}\right|, \left|\frac{A_{l_0-2}(t)}{A_{l_0}(t)}\right|^{\frac{1}{2}}\} \le 4(K+1)\left|\frac{a_{l_0-1}}{a_{l_0}}\right|$$

when $|t| \le C \min\{\left|\frac{a_{l_0-1}}{a_{l_0-3}}\right|^{\frac{1}{2}\max\{1, \frac{l_0-2}{r_0+\gamma}\}}, \left|\frac{a_{l_0}}{a_{l_0-1}}\right|^{\max\{1, \frac{l_0}{r_0+\gamma}\}}\}$. Thus we obtain (6.13).

We note that it follows from (6.13) and (2.5) that for $h = 0, 1, \dots, l_0 - 1$ we have

$$\left|\frac{A_h(t)}{A_{l_0}(t)}\right| \le K\big(16(K+1)\left|\frac{a_{\underline{l_0}}}{a_{l_0}}\right|^{\frac{1}{l_0-\underline{l_0}}}\big)^{l_0-h}$$

from (6.5) with $l = \underline{l_0}$

$$\le K\big(16(K+1)\frac{K_{\underline{l_0}}}{\sigma}\big)^{l_0-h}$$

when $|t| \le C\big(\frac{\sigma}{K_{l_0}}\big)^{\max\{1, \frac{m}{r_0+\gamma}\}}$. Then with some $\nu_{l_0} \in (0, 1]$ that will be determined later,

$$\sum_{h=0}^{l_0-1} |A_h(t)||\nu_{l_0}\sigma|^{m-h} \le \big(\sum_{h=0}^{l_0-1} K\big(16(K+1)K_{\underline{l_0}}\nu_{l_0}\big)^{l_0-h}\big)|A_{l_0}(t)||\nu_{l_0}\sigma|^{m-l_0}, \tag{6.16}$$

if $16(K+1)^2 K_{\underline{l_0}}\nu_{l_0} \le \frac{1}{4}$

$$\le \frac{1}{3}|A_{l_0}(t)||\nu_{l_0}\sigma|^{m-l_0} \tag{6.17}$$

when $|t| \le C\big(\frac{\sigma}{K_{l_0}}\big)^{\max\{1, \frac{m}{r_0+\gamma}\}}$.

On the other hand, applying Lemma 3.3 with $l = l_0$ to (6.9)
we obtain, for $h = l_0 + 1, \dots, m$ and $j = 1, \dots, \min\{h - 1, r_0\}$,

$$|a_h^{(j)}| \le D|a_{l_0}| \left|\frac{a_{\overline{l_0}}}{a_{l_0}}\right|^{\frac{1}{\overline{l_0}-l_0}(h-l_0-j\max\{1, \frac{h}{r_0+\gamma}\})}, \tag{6.18}$$

while, applying Lemma 3.3 with $l = \underline{l_0}$ to (6.12) we obtain, for $h = l_0 + 1, \dots, m$ and $j = 1, \dots, \min\{h - 1, r_0\}$,

$$|a_h^{(j)}| \le D|a_{l_0}| \left|\frac{a_{l_0}}{a_{\underline{l_0}}}\right|^{\frac{1}{l_0-\underline{l_0}}(h-l_0-j\max\{1, \frac{h}{r_0+\gamma}\})}, \tag{6.19}$$

where we used $|a_{\underline{l_0}}| = |a_{l_0}|(|a_{l_0}|/|a_{\underline{l_0}}|)^{(\underline{l_0}-l_0)/(l_0-\underline{l_0})}$. We remark that from (6.4) and (6.5) with $l = \underline{l_0}$ we see $(|a_{\overline{l_0}}|/|a_{l_0}|)^{\frac{1}{\overline{l_0}-l_0}} \leq \delta/K_{l_0}$ and $(|a_{l_0}|/|a_{\underline{l_0}}|)^{-\frac{1}{l_0-\underline{l_0}}} \leq (\delta/K_{l_0})^{-1}$. The estimates above , (6.18) and (6.19) imply that, if $h = l_0+1, \dots, m$ and $j = 1, \dots, \min\{h-1, r_0\}$ we have

$$|a_h^{(j)}| \leq D|a_{l_0}|\sigma^{h-l_0-j\max\{1,\frac{h}{r_0+\gamma}\}}. \tag{6.20}$$

Indeed if $h - l_0 - j\max\{1, h/(r_0+\gamma)\} \geq 0$, we obtain (6.20) from (6.4) and (6.18). If $h - l_0 - j\max\{1, h/(r_0+\gamma)\} < 0$, we obtain (6.20) from (6.19) and (6.5) with $l = \underline{l_0}$.

Therefore Taylor's formula (3.7) with $k = \min\{h-1, r_0\}$ and $\alpha = \min\{h, r_0+\gamma\} - k$, (6.20) and (6.6) show

$$|A_h(t) - a_h| \leq D|a_{l_0}| \left(\sum_{j=1}^{k} \sigma^{h-l_0}(|t|\sigma^{-\max\{1,\frac{h}{r_0+\gamma}\}})^j + \sigma^{-l_0}|t|^{\min\{h,r_0+\gamma\}} \right)$$

when $|t| \leq t_0$. Since $\min\{h, r_0+\gamma\}\max\{1, \frac{h}{r_0+\gamma}\} = h$, we see

$$\sigma^{-l_0}|t|^{\min\{h,r_0+\gamma\}} = \sigma^{h-l_0}\left(|t|\sigma^{-\max\{1,\frac{h}{r_0+\gamma}\}}\right)^{\min\{h,r_0+\gamma\}}.$$

Hence if $|t| \leq \min\{\frac{1}{2}, \frac{K}{8DK_{l_0}^m}, t_0\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$, noting $|t|^{1+\kappa}\sigma^{(-1-\kappa)\max\{1,\frac{m}{r_0+\gamma}\}}$ is less than or equal to $\frac{K}{8DK_{l_0}^m}2^{-\kappa}$ for any $\kappa > 0$, we have

$$|A_h(t) - a_h| \leq \frac{K}{K_{l_0}^m}|a_{l_0}|\sigma^{h-l_0}.$$

From (6.4) and (6.9) we obtain

$$|a_h| \leq K|a_{l_0}|\big(\frac{\sigma}{K_{l_0}}\big)^{h-l_0}.$$

Then we have for $h = l_0+1, \dots, m$

$$|A_h(t)| \leq 2K|a_{l_0}|\big(\frac{\sigma}{K_{l_0}}\big)^{h-l_0}$$

when $|t| \leq \min\{\frac{1}{2}, \frac{K}{8DK_{l_0}^m}, t_0\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Since we see from (6.10), (6.5) with $l = \underline{l_0}$ and $K_{l_0} \geq K_{\underline{l_0}} \geq 1$ that

$$|a_{l_0}| \leq 2|A_{l_0}(t)|, \quad |t| \leq C\big(\frac{\sigma}{K_{l_0}}\big)^{\max\{1,\frac{m}{r_0+\gamma}\}},$$

we have for $h = l_0+1, \dots, m$

$$|A_h(t)| \leq 4K|A_{l_0}(t)|\big(\frac{\sigma}{K_{l_0}}\big)^{h-l_0}$$

when $|t| \leq \min\{\frac{1}{2}, \frac{K}{8DK_{l_0}^m}, t_0, C{K_{l_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Then we obtain

$$\sum_{h=l_0+1}^{m} |A_h(t)||\nu_{l_0}\sigma|^{m-h} \leq 4K\Big(\sum_{h=l_0+1}^{m} \big(\frac{1}{K_{l_0}\nu_{l_0}}\big)^{h-l_0}\Big)|A_{l_0}(t)||\nu_{l_0}\sigma|^{m-l_0},$$

if $\quad \frac{4K+1}{\nu_{l_0}K_{l_0}} \leq \frac{1}{4}$

$$\leq \frac{1}{3}|A_{l_0}(t)||\nu_{l_0}\sigma|^{m-l_0} \tag{6.21}$$

when $|t| \leq \min\{\frac{1}{2}, \frac{K}{8DK_{l_0}^m}, t_0, CK_{l_0}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Noting that $K_l = (8K+2)(4^5(K+1)^3)^l$, we see that by picking

$$\nu_{l_0} = \frac{1}{4^3(K+1)^2K_{\underline{l_0}}},$$

we have

$$\nu_{l_0}K_{l_0} = 4^2(K+1),$$

which implies

$$4^2(K+1)^2K_{\underline{l_0}}\nu_{l_0} = \frac{1}{4} \quad \text{and} \quad \frac{4K+1}{\nu_{l_0}K_{l_0}} \leq \frac{1}{4}.$$

Therefore, since (6.10) implies $A_{l_0}(t) \neq 0$ for $|t| \leq CK_{l_0}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$, from (6.17) and (6.21) we obtain

$$|p(t, \pm\nu_{l_0}\sigma)| \geq \frac{1}{3}|A_{l_0}(t)||\nu_{l_0}\sigma|^{m-l_0} > 0 \tag{6.22}$$

when $|t| \leq \min\{\frac{1}{2}, \frac{K}{8DK_{l_0}^m}, t_0, CK_{l_0}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$.

Next consider the case where $l_0 = k_0$. In this case we have for $l < k_0$ in M

$$\left|\frac{a_{\bar{l}}}{a_l}\right|^{\frac{1}{\bar{l}-l}} > \frac{\sigma}{K_l}. \tag{6.23}$$

Then $|a_{k_0}| \geq (\frac{\sigma}{K_{\underline{k_0}}})^{k_0}$ from (6.6). From (3.1), (3.2), (3.3) and (3.4) we obtain

$$\max\{\left|\frac{A_{\underline{k_0}}(t)}{a_{\underline{k_0}}} - 1\right|, \left|\frac{A_{k_0}(t)}{a_{k_0}} - 1\right|\} \leq \frac{1}{2}, \;\; |t| \leq C\left|\frac{a_{k_0}}{a_{\underline{k_0}}}\right|^{\frac{1}{k_0-\underline{k_0}}\max\{1,\frac{k_0}{r_0+\gamma}\}}, \tag{6.24}$$

$$\max\{\left|\frac{A_{\underline{k_0}+1}(t)}{A_{\underline{k_0}}(t)}\right|, \left|\frac{A_{\underline{k_0}+2}(t)}{A_{\underline{k_0}}(t)}\right|^{\frac{1}{2}}\} \leq 4\left|\frac{a_{k_0}}{a_{\underline{k_0}}}\right|^{\frac{1}{k_0-\underline{k_0}}}, \;\; |t| \leq C\left|\frac{a_{k_0}}{a_{\underline{k_0}}}\right|^{\frac{1}{k_0-\underline{k_0}}\max\{1,\frac{\underline{k_0}+2}{r_0+\gamma}\}} \tag{6.25}$$

and

$$\left|\frac{A_h(t)}{a_{\underline{k_0}}}\right| \leq K\left|\frac{a_{k_0}}{a_{\underline{k_0}}}\right|^{\frac{h-\underline{k_0}}{k_0-\underline{k_0}}} \qquad h = \underline{k_0}+1, \ldots, m \tag{6.26}$$

when $|t| \leq C\left|\frac{a_{k_0}}{a_{\underline{k_0}}}\right|^{\frac{1}{k_0-\underline{k_0}}\max\{1,\frac{h}{r_0+\gamma}\}}$.

Since the estimate (6.16) with $l_0 = k_0$ is derived from (6.24) and (6.25) as the case where $0 < l_0 < k_0$, we have for $h = 0, 1, \dots, k_0 - 1$

$$\left|\frac{A_h(t)}{A_{k_0}(t)}\right| \le K\big(16(K+1)\left|\frac{a_{\underline{k_0}}}{a_{k_0}}\right|^{\frac{1}{k_0-\underline{k_0}}}\big)^{k_0-h},$$

(6.23) with $l = \underline{k_0}$ implies

$$\le K\big(16(K+1)\frac{K_{\underline{k_0}}}{\sigma}\big)^{k_0-h}$$

when $|t| \le C\big(\frac{\sigma}{K_{k_0}}\big)^{\max\{1, \frac{m}{r_0+\gamma}\}}$. Then with some $\nu_{k_0} \in (0, 1]$ that will be determined later,

$$\sum_{h=0}^{k_0-1} |A_h(t)||\nu_{k_0}\sigma|^{m-h} \le (\sum_{h=0}^{k_0-1} K\big(16(K+1)K_{\underline{k_0}}\nu_{k_0}\big)^{k_0-h})|A_{k_0}(t)||\nu_{k_0}\sigma|^{m-k_0},$$

if $16(K+1)^2 K_{\underline{k_0}}\nu_{k_0} \le \frac{1}{4}$

$$\le \frac{1}{3}|A_{k_0}(t)||\nu_{k_0}\sigma|^{m-k_0} \tag{6.27}$$

when $|t| \le C\big(\frac{\sigma}{K_{k_0}}\big)^{\max\{1, \frac{m}{r_0+\gamma}\}}$.

Furthermore, as the derivation of the estimate (6.20), we obtain (6.19) with $l_0 = k_0$ for $h = k_0 + 1, \dots, m$ and $j = 1, \dots, \min\{h-1, r_0\}$ from Lemma 3.3 applied to (6.26) with $l = \underline{k_0}$. Then if $h = k_0+1, \dots, m$, $j = 1, \dots, \min\{h-1, r_0\}$ and $h - k_0 - j\max\{1, \frac{h}{r_0+\gamma}\} \le 0$, we obtain from (6.23) with $l = \underline{k_0}$

$$|a_h^{(j)}| \le D|a_{k_0}|\sigma^{h-k_0-j\max\{1, \frac{h}{r_0+\gamma}\}}. \tag{6.28}$$

While Lemma 3.4 shows that $a_h^{(j)} = 0$ when $h > k_0$ and $h - k_0 > j\max\{1, \frac{h}{r_0+\gamma}\}$. Hence (6.28) holds for any $j = 1, \dots, \min\{h-1, r_0\}$. Then from Taylor's formula (3.7) with $k = \min\{h-1, r_0\}$ and $\alpha = \min\{h, r_0+\gamma\}$ and the estimate $|a_{k_0}| \ge (\frac{\sigma}{K_{\underline{k_0}}})^{k_0}$, we get for $h > k_0$

$$|A_h(t)| \le D|a_{k_0}|\sigma^{h-k_0}(\sum_{j=1}^{k}(|t|\sigma^{-\max\{1, \frac{h}{r_0+\gamma}\}})^j$$
$$+ (|t|\sigma^{-\max\{1, \frac{h}{r_0+\gamma}\}})^{\min\{h, r_0+\gamma\}}), \quad |t| \le t_0.$$

Here we note $a_k = 0$ for $h > k_0$. When $|t| \le \min\{\frac{1}{2}, \frac{1}{8DK_{k_0}^m}, t_0\}\sigma^{\max\{1, \frac{m}{r_0+\gamma}\}}$, we have

$$|A_h(t)| \le \frac{1}{K_{k_0}^m}|a_{k_0}|\sigma^{h-k_0}.$$

Since we see from (6.24), (6.5) with $l = \underline{k_0}$ and $K_{k_0} \ge K_{\underline{k_0}} \ge 1$ that

$$|a_{k_0}| \le 2|A_{k_0}(t)|, \quad |t| \le C\big(\frac{\sigma}{K_{k_0}}\big)^{\max\{1, \frac{m}{r_0+\gamma}\}}.$$

Hence we have for $h = k_0 + 1, \dots, m$

$$|A_h(t)| \le 2|A_{k_0}(t)|\big(\frac{\sigma}{K_{k_0}}\big)^{h-k_0}$$

when $|t| \le \min\{\frac{1}{2}, \frac{K}{8DK_{k_0}^m}, C{K_{k_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Then we see that

$$\sum_{h=k_0+1}^{m} |A_h(t)|\nu_{k_0}\sigma|^{m-h} \le 2|A_{k_0}(t)|(\nu_{k_0}\sigma)^{m-k_0} \sum_{h=k_0+1}^{m} \big(\frac{1}{\nu_{k_0}K_{k_0}}\big)^{h-k_0},$$

if $\frac{1}{\nu_{k_0}K_{k_0}} \le \frac{1}{8}$

$$\le \frac{1}{3}|A_{k_0}(t)|(\nu_{k_0}\sigma)^{m-k_0} \tag{6.29}$$

when $|t| \le \min\{\frac{1}{2}, \frac{K}{8DK_{k_0}^m}, t_0, C{K_{k_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Since, by picking

$$\nu_{k_0} = \frac{1}{4^3(K+1)^2 K_{\underline{k_0}}},$$

we have from $K_{k_0} = 4^5(K+1)^3 K_{\underline{k_0}}$

$$4^2(K+1)^2\nu_{k_0}K_{\underline{k_0}} = \frac{1}{4} \text{ and } \nu_{k_0}K_{k_0} \ge 4^2.$$

Hence from (6.27) and (6.29) we obtain

$$|p(t, \pm\nu_{k_0}\sigma)| \ge \frac{1}{3}|A_{k_0}(t)|(\nu_{k_0}\sigma)^{m-k_0} > 0$$

when$|t| \le \min\{\frac{1}{2}, \frac{K}{8DK_{k_0}^m}, t_0, C{K_{k_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Here we used $A_{k_0}(t) \ne 0$ when $|t| \le C{K_{k_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$, which follows from (6.24).

Thus , when $k_0 > 0$, we see from (6.3), (6.22) and (6) that the assertion of Lemma 6.2 is valid by choosing $\min\{C_0, \frac{1}{2}, \frac{K}{8DK_{k_0}^m}, t_0, C{K_{k_0}}^{-\max\{1,\frac{m}{r_0+\gamma}\}}\}$ as the constant C.

Finally consider the case where $k_0 = 0$. In this case, we have $a_h = 0$ for $h = 1, \dots, m$. Hence $|A_1(t)| \le D|t|$ when $|t| \le t_0$. Furthermore Lemma 3.4 shows that $a_h^{(j)} = 0$ when $h > j\max\{1, \frac{h}{r_0+\gamma}\}$. Then we see that $a_h^{(j)} = 0$ when $h \ge 1$ and $j < \min\{h, r_0+\gamma\}$. For $A_h(t)$ with $h \ge 2$, Taylor's formula (3.7) with $k = \min\{h-1, r_0\}$ and $\alpha = \min\{h, r_0+\gamma\} - k$ implies

$$|A_h(t)| \le D|t|^{\min\{h, r_0+\gamma\}}$$

when $|t| \le t_0$. Hence, when $|\tau| \le 1$ and $|t| \le t_0$,

$$\begin{aligned} &\sum_{h=1}^{m} |A_h(t)||\tau|^{m-h} \\ &\quad \le D|\tau|^m\Big(\sum_{1\le h\le \min\{r_0,m\}} (|t||\tau|^{-1})^h + (m - \min\{m, r_0\})|t|^{\min\{m, r_0+\gamma\}}|\tau|^{-m}\Big). \end{aligned} \tag{6.30}$$

Then when $|\tau| \le 1$ and $|t| \le \min\{\frac{1}{5(D+1)}, \frac{1}{4m(D+1)}, t_0\}|\tau|^{\max\{1,\frac{m}{r_0+\gamma}\}}$, we have $|p(t,\tau) - \tau^m| \le \frac{1}{2}|\tau|^m$. Therefore for any $\sigma \in (0, 1]$ we see that $p(t, \sigma) \ne 0$ and $p(t, -\sigma) \ne 0$ when $|t| \le \min\{\frac{1}{5(D+1)}, \frac{1}{4m(D+1)}, t_0\}\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}$. Then in the case where $k_0 = 0$ also, the assertion of Lemma 6.2 is valid. The proof is completed. □

Next we give the proof of Lemma 6.2 using only the result of Theorem 1.1.

Second proof of Lemma 6.2. For $\sigma \in (0,1]$, set

$$I_j = (-\frac{j\sigma}{m+1}, -\frac{(j-1)\sigma}{m+1}) \cup (\frac{(j-1)\sigma}{m+1}, \frac{j\sigma}{m+1})$$

for $j = 1, \dots, m+1$. Then since the degree of $p(0,\tau)$ is m, there exists a $j_0 \in \{1, \dots, m+1\}$ so that we have no root of $p(0,\tau)$ in the set I_{j_0}. Then we see by setting $\tau_0 = \frac{(2j_0-1)\sigma}{2m+2}$ that

$$p(0,\tau) \neq 0, \quad |\tau - \tau_0| < \frac{\sigma}{2m+2}. \tag{6.31}$$

and

$$p(0,\tau) \neq 0, \quad |\tau + \tau_0| < \frac{\sigma}{2m+2}.$$

Set $\tau_- = \tau_0 - \frac{\sigma}{4m+4}$, $\tau_+ = \tau_0 + \frac{\sigma}{4m+4}$, $\xi_- = \tau_- + i\tan(\frac{\pi}{4(m+1)})\frac{\sigma}{4m+4}$ and $\xi_+ = \tau_+ + i\tan(\frac{\pi}{4(m+1)})\frac{\sigma}{4m+4}$.

Let Γ be the line segment joining ξ_- and ξ_+ in the complex plane with the orientation from ξ_+ to ξ_-. Then for each $\lambda \in \mathbb{R}$ we see that $\arg(\tau - \lambda)$ increases when τ moves from ξ_+ to ξ_- along Γ. We denote this total change of argument by $\Delta(\lambda)$ that is a positive number. Then we have

$$\Delta(\lambda) \begin{cases} \leq \frac{\pi}{4m+4} & \lambda \geq \tau_0 + \frac{\sigma}{2m+2} \text{ or } \lambda \leq \tau_0 - \frac{\sigma}{2m+2} \\ = \frac{(2m+1)\pi}{2m+2} & \lambda = \tau_0. \end{cases}$$

Let

$$T(t) = \Im \int_\Gamma \frac{\partial_\tau p(t,\tau)}{p(t,\tau)}\, d\tau..$$

Then if $p(t,\tau_0) = 0$, we have $T(t) \geq \frac{(2m+1)\pi}{2m+2}$. On the other hand we see from (6.31) that $T(0) \leq \frac{m\pi}{4m+4}$. Hence if $p(t,\tau_0) = 0$, then we have

$$T(t) - T(0) \geq \frac{(3m+2)\pi}{4m+4}. \tag{6.32}$$

Since

$$\partial_t T(t) = \Im \int_\Gamma \Big(\frac{\partial_t\partial_\tau p(t,\tau)}{\partial_\tau p(t,\tau)}\frac{\partial_\tau p(t,\tau)}{p(t,\tau)} - \frac{\partial_t p(t,\tau)}{p(t,\tau)}\frac{\partial_\tau p(t,\tau)}{p(t,\tau)}\Big)\, d\tau$$

From hyperbolicity of $p(t,\tau)$, we have

$$\Big|\frac{\partial_\tau p(t,\tau)}{p(t,\tau)}\Big| \leq \frac{m}{\tan(\frac{\pi}{4(m+1)})\frac{\sigma}{4m+4}} \quad \text{for } \tau \in \Gamma.$$

According to Theorem 1.1 applied to $p(t,\tau)$ and $\partial_\tau p(t,\tau)$, we have

$$\Big|\frac{\partial_t p(t,\tau)}{p(t,\tau)}\Big| \leq \frac{D}{\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}} \text{ and } \Big|\frac{\partial_t\partial_\tau p(t,\tau)}{\partial_\tau p(t,\tau)}\Big| \leq \frac{D}{\sigma^{\max\{1,\frac{m-1}{r_0+\gamma}\}}}.$$

Since the length of Γ is $\frac{\sigma}{2m+2}$, we have $|\partial_t T(t)| \le D/(\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}})$. Hence when $|t| \le \frac{\pi\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}}{4(m+1)D}$,

$$|T(t) - T(0)| \le \frac{\pi}{4(m+1)}.$$

Then we see from (6.32) that

$$p(t, \tau_0) \ne 0, \quad |t| \le \frac{\pi\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}}{4(m+1)D}.$$

Similar argument can be applied to $\tau = -\tau_0$. Then we have

$$p(t, \pm\tau_0) \ne 0, \quad |t| \le \frac{\pi\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}}{4(m+1)D}.$$

Then for any $\sigma \in (0, 1]$ there exists $j_0 \in \{1, \dots, m+1\}$ such that we have

$$p(t, \pm\frac{2j_0+1}{2m+2}\sigma) \ne 0, \quad |t| \le \frac{\pi\sigma^{\max\{1,\frac{m}{r_0+\gamma}\}}}{4(m+1)D}.$$

The proof of Lemma 6.2 is completed. □

7 Proof of Theorem 1.1 based on Theorem 1.4 In this section, we prove Theorem 1.1 assuming that Theorem 1.4 is valid. That is to say, assuming the Hölder continuity of roots, we show the estimate (1.1). From the argument of the section 4, we see that it is sufficient to show Lemma 4.1 under the assumption that Theorem 1.4 is valid.

Let $\lambda_l(t)$ $(l = 1, \dots, m)$ be roots of a polynomial $p(t, \tau) = \tau^m + \sum_{h=1}^m A_h(t)\tau^{m-h}$ satisfying the assumptions of Lemma 4.1. Since the multiplicity of roots is at most m and $A_h(t) \in C^{r_0,\gamma}([-T, T])$, we see from Theorem 1.4 that roots $\lambda_l(t)$ $(l = 1, \dots, m)$ are locally Hölder continuous of order $\min\{1, \frac{r_0+\gamma}{m}\}$ on $(-T, T)$. Let $\delta = \min\{1, \frac{r_0+\gamma}{m}\}$, $S \in (\frac{T}{2}, T)$ and $d = \frac{T-S}{2}$. Then we have

$$|\lambda_l(t) - \lambda_l(s)| \le C|t - s|^{\delta} \tag{7.1}$$

for $l = 1, \dots, m$ and $t, s \in [-S - d, S + d]$. Let $\chi(s)$ be a real-valued C^∞ function on $\mathbb{R}$ satisfying

$$\chi(s) = 0 \quad |s| \ge 1$$

and

$$\int s^k \chi(s)\, ds = \begin{cases} 1 & k = 0 \\ 0 & k = 1, \dots, r_0. \end{cases} \tag{7.2}$$

According to Remark 4.1, we have only to show the estimate (4.1) when $|\Re\tau| \le D$ and $0 < |\Im\tau| \le 1$ with some constant D. We set

$$\rho = C_0|\Im\tau|^{-\delta^{-1}}$$

with some constant $C_0 \ge 1/d$ that will be chosen later.

Let for $t \in [-S, S]$

$$\tilde{p}(t,\tau) = \int \rho\chi(\rho(t-s))p(s,\tau)\,ds$$

and

$$\tilde{\lambda}_l(t) = \int \rho\chi(\rho(t-s))\lambda_l(s)\,ds \quad l = 1, \dots, m.$$

We note that $t \in [-S, S]$ and $\chi(\rho(t-s)) \neq 0$ imply $|s| \leq S + d$. Then we get from (7.1)

$$|\tilde{\lambda}_l(t) - \lambda_l(t)| \leq C\rho^{-\delta} \tag{7.3}$$

and

$$\left|\frac{d^j\tilde{\lambda}_l(t)}{dt^j}\right| \leq C\rho^{-\delta+j} \quad j \geq 1 \tag{7.4}$$

when $t \in [-S, S]$.

Since $A_h(t) \in C^{r_0,\gamma}([-T, T])$, we see from (7.2) that

$$\tilde{A}_h(t) = \int \rho\chi(\rho(t-s))A_h(s)\,ds$$

satisfies

$$\left|\frac{d^j\tilde{A}_h(t)}{dt^j} - \frac{d^jA_h(t)}{dt^j}\right| \leq C\rho^{-(r_0+\gamma)+j} \qquad j = 0, \dots, r_0$$

when $t \in [-S, S]$. Then we see that

$$\tilde{p}(t,\tau) = \tau^m + \sum_{h=1}^{m} \tilde{A}_h(t)\tau^{m-h}$$

and

$$\left|\partial_t^j p(t,\tau) - \partial_t^j \tilde{p}(t,\tau))\right| \leq C\rho^{-(r_0+\gamma)+j} \qquad j = 0, \dots, r_0$$

when $t \in [-S, S]$, $|\Re\tau| \leq D$ and $|\Im\tau| \leq 1$. Since $|p(t,\tau)| \geq |\Im\tau|^m$ and $\delta^{-1}(r_0+\gamma) \geq m$, we see that

$$\left|\partial_t^j p(t,\tau) - \partial_t^j \tilde{p}(t,\tau))\right| \leq C|p(t,\tau)|\rho^j \qquad j = 0, \dots, r_0. \tag{7.5}$$

Let

$$\tilde{\tilde{p}}(t,\tau) = \prod_{l=1}^{m}(\tau - \tilde{\lambda}_l(t)). \tag{7.6}$$

Then we have

$$p(s,\tau) = \tilde{\tilde{p}}(t,\tau) + \sum_{\Theta \subsetneq \{1,\dots,m\}} C_\Theta(s,t)\tilde{\tilde{p}}_\Theta(t,\tau) \tag{7.7}$$

where

$$C_\Theta(s,t) = \prod_{l\in\{1,\dots,m\}\setminus\Theta} (\tilde\lambda_l(t) - \lambda_l(s))$$

and

$$\tilde{\tilde p}_\Theta(t,\tau) = \prod_{l\in\Theta} (\tau - \tilde\lambda_l(t)).$$

Noting that (7.1) and (7.3) imply

$$|C_\Theta(s,t)| \le C(|t-s|^\delta + \rho^{-\delta})^{m-|\Theta|} \tag{7.8}$$

and that

$$|\tilde{\tilde p}_\Theta(t,\tau)| \le \frac{|\tilde{\tilde p}(t,\tau)|}{|\Im\tau|^{m-|\Theta|}}, \tag{7.9}$$

we see that

$$|p(t,\tau) - \tilde{\tilde p}(t,\tau)| \le C|\tilde{\tilde p}(t,\tau)|\frac{\rho^{-\delta}}{|\Im\tau|}(1+\frac{\rho^{-\delta}}{|\Im\tau|})^{m-1}.$$

We see from $\rho^{-\delta} = C_0^{-\delta}|\Im\tau|$ that with a large C_0 we have

$$|p(t,\tau) - \tilde{\tilde p}(t,\tau)| \le \frac{1}{2}|\tilde{\tilde p}(t,\tau)|. \tag{7.10}$$

By the definition of $\tilde p(t,\tau)$, we see from (7.7) that

$$\tilde p(t,\tau) = \tilde{\tilde p}(t,\tau) + \sum_{\Theta\subsetneq\{1,\dots,m\}} \tilde C_\Theta(t)\tilde{\tilde p}_\Theta(t,\tau) \tag{7.11}$$

where

$$\tilde C_\Theta(t) = \int \rho\chi(\rho(t-s))C_\Theta(s,t)\,ds.$$

We obtain from (7.4) and (7.8)

$$\left|\frac{d^j\tilde C_\Theta(t)}{dt^j}\right| \le C\rho^{j-\delta(m-|\Theta|)}. \tag{7.12}$$

From the definition of $\tilde{\tilde p}(t,\tau)$ and (7.4) we see that for $j\ge 1$

$$\partial_t^j\tilde{\tilde p}(t,\tau) = \sum_{\Theta\subsetneq\{1,\dots,m\}} D_{\Theta,j}(t)\tilde{\tilde p}_\Theta(t,\tau)$$

where

$$|D_{\Theta,j}(t)| \le C\rho^{j-\delta(m-|\Theta|)},$$

from which and (7.9) we have

$$|\partial_t^j \tilde{\tilde{p}}(t,\tau)| \le C\rho^j |\tilde{\tilde{p}}(t,\tau)|. \tag{7.13}$$

Similarly we have

$$|\partial_t^j \tilde{\tilde{p}}_\Theta(t,\tau)| \le C\rho^j |\tilde{\tilde{p}}_\Theta(t,\tau)|. \tag{7.14}$$

Noting that (7.9) implies $\rho^{-\delta(m-\Theta)}|\tilde{\tilde{p}}_\Theta(t,\tau)| \le C|\tilde{\tilde{p}}(t,\tau)|$, we see that it follows from (7.11), (7.12), (7.13) and (7.14) that

$$|\partial_t^j \tilde{p}(t,\tau)| \le C\rho^j |\tilde{\tilde{p}}(t,\tau)|.$$

The estimate above, (7.5) and (7.10) imply the desired estimates

$$\left|\frac{\partial_t^j p(t,\tau)}{p(t,\tau)}\right| \le C|\Im\tau|^{-j\delta^{-1}}$$

when $0 \le j \le r_0$, $|\Re\tau| \le D$, $0 < |\Im\tau| \le 1$ and $|t| \le S$. Then Lemma 4.1 is proven under the assumption that Theorem 1.4 is valid.

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