ON BM-ALGEBRAS

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Abstract. In this paper we introduce the notion of a BM-algebra which is a specialization of B-algebras. We show that the class of BM-algebras is a proper subclass of B-algebras and show that a BM-algebra is equivalent to a 0-commutative B-algebra. Moreover, we prove that a class of Coxeter algebras is a proper subclass of BM-algebras.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4,5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([10]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras ([11, 12]), i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$, which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([8]) introduced a new notion, called an BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$. A. Walendziak obtained the another equivalent axioms for B-algebra ([13]). H. S. Kim, Y. H. Kim and J. Neggers ([7]) introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. In this paper we introduce the notion of a BM-algebras which is a specialization of B-algebras. We prove that the class of BM-algebras is a proper subclass of B-algebras and also show that a BM-algebra is equivalent to a 0-commutative B-algebra. Moreover, we prove that a class of Coxeter algebras is a proper subclass of BM-algebras. And we investigate several relations between BM-algebras and (pre-) Coxeter algebras.

2. BM-algebras.

A BM-*algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(A1) $x * 0 = x$,

(A2) $(z * x) * (z * y) = y * x$,

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for any $x, y, z \in X$.

Example 2.1. Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X; *, 0)$ is a BM-algebra.

It is easy to calculate the number of BM-algebras on a set X with $|X| = 3, 4$.

Proposition 2.2. *Let X be a set such that $|X| = 3$ and let $\Gamma(X)$ be the collection of all BM-algebras defined on X. Then $|\Gamma(X)| = 1$.*

Proposition 2.3. *Let X be a set such that $|X| = 4$ and let $\Gamma(X)$ be the collection of all BM-algebras defined on X. Then $|\Gamma(X)| = 2$.*

Lemma 2.4. *Let $(X; *, 0)$ be a BM-algebra. Then*

(i) $x * x = 0$,

(ii) $0 * (0 * x) = x$,

(iii) $0 * (x * y) = y * x$,

(iv) $(x * z) * (y * z) = x * y$,

(v) $x * y = 0$ *if and only if* $y * x = 0$,

for any $x, y, z \in X$.

Proof. (i). Substituting $x = 0$ and $y = 0$ in (A2), we obtain

$$(z * 0) * (z * 0) = 0 * 0$$

Applying (A1) we obtain $z * z = 0$ for all $z \in X$.

(ii). Substituting $z = 0$ and $x = 0$ in (A2), we obtain

$$(0 * 0) * (0 * y) = y * 0.$$

Applying (A1) we have

$$0 * (0 * y) = y$$

for all $y \in X$.

(iii). Using (A2) with $z = x$ we have

$$(x * x) * (x * y) = y * x$$

Hence, by applying (i), we obtain

$$0 * (x * y) = y * x$$

for any $x, y \in X$.

(iv). For any $x, y, z \in X$, we have

$$\begin{aligned}(x*z)*(y*z) &= (0*(z*x))*(0*(z*y)) && [\text{(iii)}]\\ &= (z*y)*(z*x) && [\text{(A2)}]\\ &= x*y && [\text{(A2)}]\end{aligned}$$

(v). It follows immediately from (iii) and (A1).

Note that there is no non-trivial BM-algebra which is also a BCK-algebra, since $x = 0*(0*x) = 0*0 = 0$ for any $x \in X$.

A B-*algebra* ([11]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B1) $x*x = 0$,

(A1) $x*0 = x$,

(B3) $(x*y)*z = x*(z*(0*y))$,

for any $x, y, z \in X$.

Recently, A. Walendiziak obtained an equivalent axiomatizations for B-algebras ([13]), and he proved that the congruence lattice of any B-algebra is isomorphic to the lattice of its normal subalgebras ([14]).

Theorem 2.5. ([13]) *$(X;*,0)$ is a B-algebra if and only if satisfies the axioms:*

(B1) $x*x = 0$,

(C2) $0*(0*x) = x$,

(C3) $(x*z)*(y*z) = x*y$,

for all $x, y, z \in X$.

¿From (i), (ii) and (iv) of Lemma 2.4, we have the following theorem.

Theorem 2.6. *Every BM-algebra is a B-algebra.*

The converse of Theorem 2.6 does not hold in general. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X;*,0)$ is a B-algebra, but not a BM-algebra, since $(5*1)*(5*4) = 4 \neq 5 = 4*1$.

Proposition 2.7. *If $(X;*,0)$ is a BM-algebra, then*

$$(x*y)*z = (x*z)*y$$

for any $x, y, z \in X$.

Proof. By Theorem 2.6 and Lemma 2.4-(iii),

$$\begin{aligned}
(x*y)*z &= [(z*y)*(z*x)]*z && [\text{(A2)}]\\
&= (z*y)*[z*(0*(z*x))] && [\text{(B3)}]\\
&= [0*(z*x)]*y && [\text{(A2)}]\\
&= (x*z)*y && [\text{Lemma 2.4-(iii)}]
\end{aligned}$$

Lemma 2.8. ([13]) *If $(X;*,0)$ is a B-algebra, then $0*(x*y)=y*x$ for any $x,y \in X$.*

Definition 2.9. ([1]) A B-algebra $(X;*,0)$ is said to be 0-*commutative* if $x*(0*y)=y*(0*x)$ for any $x,y \in X$.

Theorem 2.10. *If $(X;*,0)$ is a 0-commutative B-algebra, then it is a BM-algebra.*

Proof. Since $(X;*,0)$ is a B-algebra, $x*0=x$ for all $x \in X$, i.e., (A1) holds. We show that (A2) holds in X.

$$\begin{aligned}
(z*x)*(z*y) &= (0*(x*z))*(0*(y*z)) && [\text{Lemma 2.8}]\\
&= (y*z)*[0*(0*(x*z))] && [\text{0-commutative}]\\
&= (y*z)*(x*z) && [\text{(C2)}]\\
&= y*z && [\text{(C3)}]
\end{aligned}$$

Thus $(X;*,0)$ is a BM-algebra.

Corollary 2.11. *If $(X;*,0)$ is a B-algebra with $x*y=y*x$ for any $x,y \in X$, then it is a BM-algebra.*

Proof. Since $x*y=y*x$ for any $x,y \in X$, we obtain $x*(0*y)=x*(y*0)=x*y=y*x=y*(x*0)=y*(0*x)$ for any $x,y \in X$. Thus $(X;*,0)$ is a 0-commutative B-algebra. Hence $(X;*,0)$ is a BM-algebra by Theorem 2.10.

Proposition 2.12. ([13]) *An algebra $(X;*,0)$ is a 0-commutative B-algebra if and only if it satisfies the following axioms:*

(B1) $x*x=0$,

(D2) $y*(y*x)=x$,

(C3) $(x*z)*(y*z)=x*y$,

for any $x,y,z \in X$.

Theorem 2.13. *If $(X;*,0)$ is a BM-algebra, then it is a 0-commutative B-algebra.*

Proof. Let X be a BM-algebra. Then, by Theorem 2.6, it is a B-algebra. From Theorem 2.5, we deduce that it satisfies (B1) and (C3). Substituting $x=0$ in (A2) we obtain

$$(z*0)*(z*y)=y*0$$

Applying (A1) we have

$$z*(z*y)=y$$

for any $y,z \in X$. Thus (B1), (D2) and (C3) hold in $(X;*,0)$. Hence, by Proposition 2.12, it is a 0-commutative B-algebra.

¿From Theorem 2.10 and Theorem 2.13, we have the following result.

Corollary 2.14. *An algebra* $(X;*,0)$ *is a* 0*-commutative* B*-algebra if and only if it is a* BM*-algebra.*

3. BM-algebras and (pre-) Coxeter algebras.

H. S. Kim, Y. H. Kim and J. Neggers introduced and investigated a class of (pre-) Coxeter algebras. A *Coxeter algebra* ([7]) is a non-empty set with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(B1) $x * x = 0$,

(A1) $x * 0 = x$,

(E3) $(x * y) * z = x * (y * z)$,

for any $x, y, z \in X$.

It is known that a Coxeter algebra is a special type of abelian groups (see [7]).

Proposition 3.1. ([7]) *If* $(X;*,0)$ *is a Coxeter algebra, then*

(i) $0 * x = x$,

(ii) $x * y = y * x$,

for any $x, y \in X$.

Lemma 3.2. *Let* $(X;*,0)$ *be a Coxeter algebra. Then*

$$(y * x) * y = x$$

for any $x, y \in X$.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
x &= 0 * x && \text{[Proposition 3.1-(i)]} \\
&= [(y * x) * (y * x)] * x && \text{[(B1)]} \\
&= (y * x) * [(y * x) * x] && \text{[(E3)]} \\
&= (y * x) * [(y * (x * x)] && \text{[(E3)]} \\
&= (y * x) * (y * 0) && \text{[(B1)]} \\
&= (y * x) * y, && \text{[(A1)]}
\end{aligned}$$

proving the lemma.

Theorem 3.3. *Every Coxeter algebra is a* BM*-algebra.*

Proof. It is enough to show that the axiom (A2) holds in Coxeter algebra $(X;*,0)$. For any $x, y, z \in X$, we have

$$\begin{aligned}
(z * x) * (z * y) &= (z * x) * (y * z) && \text{[Proposition 3.1-(ii)]} \\
&= [(z * x) * y] * z && \text{[(E3)]} \\
&= [z * (x * y)] * z && \text{[(E3)]} \\
&= x * y && \text{[Lemma 3.2]} \\
&= y * x, && \text{[Proposition 3.1-(ii)]}
\end{aligned}$$

proving that $(X;*,0)$ is a BM-algebra.

The converse of Theorem 3.3 does not hold in general. The BM-algebra $(X;*,0)$ given by Example 2.1 is not a Coxeter algebra, since $(0*0)*1 = 2 \neq 1 = 0*(0*1)$.

¿From Corollary 2.14 and Theorem 3.3, we have the following result.

Theorem 3.4. *Every Coxeter algebra is a* 0*-commutative* B*-algebra.*

Theorem 3.5. *If* $(X;*,0)$ *is a* BM*-algebra with* $0*x = x, \forall x \in X$*, then it is a Coxeter algebra.*

Proof. It is enough to show (E3). By applying Theorem 2.13, we have, for any $x, y, z \in X$,

$$\begin{aligned} (x*y)*z &= (x*z)*y && \text{[Proposition 2.7]} \\ &= x*[y*(0*z)] && \text{[(B3)]} \\ &= x*(y*z), \end{aligned}$$

completing the proof.

¿From Proposition 3.1-(i), Theorem 3.3 and Theorem 3.5, we have the following result.

Corollary 3.6. *An algebra* $(X;*,0)$ *is a Coxeter algebra if and only if it is a* BM*-algebra with* $0*x = x$ *for all* $x \in X$*.*

An algebra $(X;*,0)$ is called a *pre-Coxeter algebra* ([7]) if it satisfies the axioms: (B1); (A1); (F3) if $x*y = 0 = y*x$, then $x = y$; (F4) $x*y = y*x$, for any $x, y \in X$.

Theorem 3.7. *Every* BM*-algebra* X *with* $0*x = x, \forall x \in X$*, is a pre-Coxeter algebra.*

Proof. We show that the axioms (F3) and (F4) hold in X. Assume $x*y = 0 = y*x$ where $x, y \in X$. Then $x = x*0 = (x*0)*(x*y) = y*0 = y$. It follows from Proposition 3.1-(ii) and Theorem 3.5 that $x*y = y*x$ for any $x, y \in X$. This completes the proof.

In general, a pre-Coxeter algebra need not be a BM-algbra.

Example 3.8. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	3
2	2	3	0	1
3	3	3	1	0

Then $(X;*,0)$ is a pre-Coxeter, but not a BM-algebra, since $(1*0)*(1*2) = 3 \neq 2 = 2*0$.

By Theorem 2.6, Corollary 2.14 and Theorem 3.3, we have the following relation:

$$\begin{aligned} &\text{The class of Coxeter algebras} \subset \text{The class of 0-commutative } B\text{-algebras} \\ &= \text{The class of } BM\text{-algebras} \subset \text{The class of } B\text{-algebras} \\ &\subset \text{The class of } BG\text{-algebras} \subset \text{The class of } BH\text{-algebras}. \end{aligned}$$

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