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A TWO-VS.-TWO SILENT DUEL WITH EQUAL ACCURACY FUNCTIONS UNDER ARBITRARY MOTION

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Abstract. In this paper we examine a two-person zero-sum timing game with the following structure: Each of players I and II has a gun with two bullets and they fight a duel. Both guns are silent so that neither player can determine whether his opponent has fired the bullets or not. Player I is at the place 0 when the duel begins and he can move as he likes and player II is always at the place 1. Accuracy functions, which denote the probability of hitting the opponent when each player fires his bullet, are identical for both players. If player I hits player II without being hit himself first, then the payoff is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same moment or both survive, the payoff is 0.

The objective of this paper is to obtain the game value and the optimal strategies for the timing game.

1 Introduction A duel under arbitrary motion is a two-person zero-sum timing game with the following structure: Each of two competitors, denoted by player I and player II, has a gun and he can fire his bullets aiming at his opponent. At the moment when the duel begins these two players are one distance apart on a line and each player can move on the line as he likes. The maximum speed of player I is v_1, the maximum speed of player II is v_2 and we assume $v_1 > v_2 \geqq 0$. Without loss of generality, we can suppose $v_1 = 1$ and $v_2 = 0$, and hence, player II is motionless. Thus we assume that player II is at the place 1 all the time and player I is at the place 0 at the moment when the duel begins and he can move toward player II, he can move away from player II, and he can stay in one place. If player I or player II fires his bullet when player I is at a place x, he hits his opponent with probability $p(x)$ or $q(x)$, respectively. The functions $p(x)$ and $q(x)$ are called accuracy functions for players I and II, respectively, and they are continuous and strictly increasing on $[0, 1]$ with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$. The duel ends when at least one player is hit or both players fire all of their bullets; otherwise it continues indefinitely. The gun is said to be silent if the shot of the owner is not heard by his opponent and the gun is said to be noisy if the shot of the owner is heard by his opponent as soon as the owner of the gun fires the bullet. Thus if a player has a silent gun, then his opponent can not determine whether the owner of the gun has fired or not. On the other hand, if a player has a noisy gun, then his opponent can determine whether the owner has fired or not. If each player has a silent gun, the duel is said to be silent and if each player has a noisy gun, the duel is said to be noisy. If player I hits player II without being hit himself first, then the payoff of the duel is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same moment or both survive, the payoff is 0. The objective of player I is to maximize the expected payoff and the objective of player II is to minimize it.

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Trybula [9, 10] solved a silent duel with arbitrary accuracy functions under arbitrary motion. In his model, each player has a silent gun with one bullet and accuracy functions $p(x)$ and $q(x)$ increase with a continuous second derivative each. Trybula [6-8] also solved noisy duels under arbitrary motion. Furthermore Trybula [11] solved an m-versus-n silent duel with arbitrary accuracy functions under arbitrary motion. In the model, player I has a silent gun with m bullets and he has to fire all his bullets simultaneously, whereas player II has a silent gun with n bullets and he can fire each of his bullets at different moments.

The author [2, 3] dealt with a noisy-versus-silent duel under arbitrary motion in which player I has a noisy gun with one bullet and player II has a silent gun with one bullet and the accuracy functions are arbitrary. He [4] also solved a one-noisy-versus-two-silent duel with arbitrary accuracy functions under arbitrary motion. He [5] further solved a two-versus-one silent duel with equal accuracy functions under arbitrary motion.

Further researches on duels under arbitrary motion have been done by Trybula [12, 13] and general researches on games of timing are summarized by Karlin [1].

In this paper, we examine a two-versus-two silent duel with equal accuracy functions under arbitrary motion. In the duel, each player has a silent gun with two bullets and he can fire his bullets at different moments, whereas in the duel by Trybula [11], player I has to fire his bullets simultaneously.

2 Problem In this paper, we examine a two-versus-two silent duel with equal accuracy functions under arbitrary motion. Player I has a silent gun with two bullets and he is at the place 0 at the moment when the duel begins and he can move as he likes. On the other hand, player II has a silent gun with two bullets and he is always at the place 1. The accuracy functions $p(x)$ and $q(x)$ are identical for both players so that, without loss of generality, we suppose $p(x) = q(x) = x$ for all x over $[0, 1]$. If player I hits player II without being hit himself first, then the payoff of the duel is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0. The objective of player I is to maximize the expected payoff and the objective of player II is to minimize it. We denote the game mentioned above by G^*. Note that, in the paper by Trybula [10], player I has to fire all his bullets simultanuously, whereas in our model player I can fire his bullets at different moments.

Before examining the game G^*, we deal with the following auxiliary game G. In G, each of the players I and II has a silent gun with two bullets and both players' accuracy functions are identical and thus we assume $p(x) = q(x) = x$ for all x in [0, 1]. Player I is at the place 0 when the duel begins and player II is at the place 1 all the time. We suppose, in game G, payer I can move toward player II but he can not move away from player II, whereas in game G^* player can move as he likes. Further we assume that the payoff of G is as follows:

(i) If player I hits player II before player II hits player I, then the payoff is +1.

(ii) If player II hits player I before player I has fired both his bullets, then the payoff is -1.

(iii) If player I misses his two bullets before player II has fired both his bullets, then the payoff is 0.

(iv) If both players hit each other at the same time or they miss all their bullets, then the payoff is 0.

Suppose that player I has fired both his bullets and missed them and that player II has missed his first bullet and he still has his second bullet or player II still has his two bullets.

In this case, the payoff in game G is 0 nevertheless player II misses or hits his opponent, whereas the payoff in game G^* is 0 or -1 depending on whether player II misses (or he does not fire) his bullets or he hits his opponent.

We assume that player I fires his first and second bullets when he is at the places x and y, respectively ($0 \leqq x \leqq y \leqq 1$), and that player II fires his first and second bullets at the moments when player I is at the places u and v ($0 \leqq u \leqq v \leqq 1$). In this case, we denote by $M(x, y, u, v)$ the expected payoff of the game G. The function $M(x, y, u, v)$, called the payoff kernel of the game G, is of the form

$$M(x,y,u,v) = \begin{cases} x + (1-x)y, & 0 \leqq x \leqq y < u \leqq v \leqq 1 \\ x, & 0 \leqq x < y = u < v \leqq 1 \\ x(1-x), & 0 \leqq x = y = u < v \leqq 1 \\ 0, & 0 \leqq x = y = u = v \leqq 1 \\ x - (1-x)(1-u)u, & 0 \leqq x < y = u = v \leqq 1 \\ x - (1-x)u + (1-x)(1-u)y, & 0 \leqq x < u < y < v \leqq 1 \\ x - (1-x)u, & 0 \leqq x < u < y = v \leqq 1 \\ (1-x)^2 y, & 0 \leqq x = u < y < v \leqq 1 \\ 0, & 0 \leqq x = u < y = v \leqq 1 \\ x - (1-x)u - (1-x)(1-u)v + (1-x)(1-u)(1-v)y, & 0 \leqq x < u \leqq v < y \leqq 1 \\ -u + (1-u)x + (1-u)(1-x)y, & 0 \leqq u < x \leqq y < v \leqq 1 \\ -u + (1-u)x - (1-u)(1-x)v + (1-u)(1-x)(1-v)y, & 0 \leqq u < x < v < y \leqq 1 \\ -u + (1-u)(1-x)x, & 0 \leqq u < x = y = v \leqq 1 \\ -u + (1-u)(1-x)^2 y, & 0 \leqq u < x = v < y \leqq 1 \\ -u - (1-u)v + (1-u)(1-v)x + (1-u)(1-v)(1-x)y, & 0 \leqq u \leqq v < x \leqq y \leqq 1. \end{cases}$$

For the game G, we shall search for an optimal strategy for player I with the following structure:
(i) Player I fires both his bullets simultaneously with probability $1 - \alpha$ and he fires his bullets at different moments with probability α.
(ii) If player I fires his two bullets simultaneously, he fires his bullets at a place in $[a, b]$ according to the conditional distribution with a density function $f(x)$ under the condition that he fires both his bullets simultaneously.
(iii) If player I fires his bullets at different moments, then he fires his first bullet when he is at a place x in $[b, c]$ according to the conditional distribution with a density function $f_1(x)$ and he fires his second bullet at a place y in $[c, 1]$, independently of the place where he has fired his first bullet, according to the distribution with a density part $f_2(y)$ over $[c, 1]$ and a mass part β on 1 under the condition that player I fires his bullets at different moments, where

$$\int_a^b f(x)\, dx = \int_b^c f_1(x)\, dx = \int_c^1 f_2(y)\, dy + \beta = 1$$

and

$$0 < a < b < c < 1.$$

We denote such a strategy of player I by $\{\alpha, f(x), f_1(x), f_2(y), \beta\}$. Further we shall search for an optimal strategy for player II with the following structure:
(i) Player II fires his first bullet when player I is at a place u in $[a, s]$ according to the distribution with a density function $g_1(u)$ over $[a, s]$.
(ii) Player II fires his second bullet when player I is at a place v in $[s, 1]$ according to the distribution with a density function $g_2(v)$ over $[s, 1]$, independently of the place where he

has fired his first bullet, where

$$\int_a^s g_1(u)\,dz = \int_s^1 g_2(v)\,dv = 1$$

and

$$0 < a < s < 1.$$

We denote such a strategy of player II by $\{g_1(u), g_2(v)\}$.

3 Preliminary lemmas In this section we prove two preliminary lemmas which will be used in the following sections.

It is seen that the equation

$$\log\frac{1+x}{2x} = \frac{1-4x^2+4x^3-5x^4}{4x^2(1-x)^2} \tag{1}$$

has a unique root in the interval $(0,1)$. We denote by c the unique root in $(0,1)$ of the equation (1). We set

$$b = \frac{1-c}{1+c}\ . \tag{2}$$

The values of b and c are approximately 0.3106 and 0.5261, respectively. It is also shown that the equation

$$\frac{1-x}{x(1+2x-x^2)} - \int_x^b \frac{dt}{t^2(1+2t-t^2)} = \frac{(1-2b-b^2)(1+b^2)}{4b^2(1-b)^2} \tag{3}$$

has a unique root in the interval $(0,\, b)$. We denote by s the unique root of the equation (3). Further the equation

$$\frac{1-x}{x(1+2x-x^2)} - \int_x^s \frac{dt}{t^2(1+2t-t^2)} = \frac{1}{s(1+2s-s^2)} \tag{4}$$

has a unique root in $(0,\, s)$, and we denote by a the root of the equation (4). The values of a and s are nearly equal to 0.0929 and 0.2025, respectively.

Lemma 1. *Set*

$$\alpha = \frac{a(1+2a-a^2)(1-2b-b^2)(1+b^2)}{4(1-a)(1-s)(1-b)^2b^2}\ (= 0.1747),$$

$$f(x) = \begin{cases} \dfrac{k_1}{x^2(1+2x-x^2)}\,, & \text{if } \ a \leqq x \leqq s \\ \dfrac{k_2}{x^2(1+2x-x^2)}\,, & \text{if } \ s < x \leqq b, \end{cases}$$

$$f_1(x) = \frac{k_3}{x^3}\,, \qquad b \leqq x \leqq c,$$

$$f_2(y) = \frac{k_4}{y^2(1+y)}\,, \qquad c \leqq y \leqq 1,$$

$$\beta = \frac{k_4}{2}(= 0.4859),$$

where

$$k_1 = \frac{a(1+2a-a^2)}{(1-\alpha)(1-a)}\,,$$

$$k_2 = \frac{a(1+2a-a^2)}{(1-\alpha)(1-a)(1-s)}\,,$$

$$k_3 = \frac{2b^2(1-b)^2}{(1+b^2)(1-2b-b^2)}$$

and

$$k_4 = \frac{4c^2(1-c)^2}{(3c^2-2c+1)(c^2+2c-1)}\,.$$

Then the following statements hold:

(i) $\displaystyle\int_a^b f(x)\,dx = 1.$

(ii) $\displaystyle\int_b^c f_1(x)\,dx = 1.$

(iii) $\displaystyle\int_c^1 f_2(y)\,dy + \beta = 1.$

(iv) $\displaystyle\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy + \beta\right\} = \frac{k_2(1-\alpha)}{\alpha b}.$

(v) For all u in $[a, b]$,

$$u\int_u^b (1+2x-x^2)f(x)\,dx \leqq \frac{k_2\{b-u+b(u-s)\}}{b}.$$

(vi) For all u and v such that $a \leqq u \leqq v \leqq b$,

$$u\int_u^b (1+2x-x^2)f(x)\,dx + (1-u)v\int_v^b (1+2x-x^2)f(x)\,dx \leqq k_2\left\{2-s+\frac{uv-u-v}{b}\right\}.$$

Proof. (i) Since s and a are the roots of the equations (3) and (4), respectively, we have

$$\int_s^b \frac{dx}{x^2(1+2x-x^2)} = \frac{1-s}{s(1+2s-s^2)} - \frac{(1-2b-b^2)(1+b^2)}{4b^2(1-b)^2}$$

and

$$\int_a^s \frac{dx}{x^2(1+2x-x^2)} = \frac{1-a}{a(1+2a-a^2)} - \frac{1}{s(1+2s-s^2)}\,.$$

Thus we get

$$\int_a^s \frac{dx}{x^2(1+2x-x^2)} + \frac{1}{1-s}\int_s^b \frac{dx}{x^2(1+2x-x^2)}$$

$$= \frac{1-a}{a(1+2a-a^2)}\left\{1 - \frac{a(1+2a-a^2)(1-2b-b^2)(1+b^2)}{4(1-a)(1-s)(1-b)^2b^2}\right\} = \frac{(1-\alpha)(1-a)}{a(1+2a-a^2)}\,,$$

i.e.,

$$\int_a^b f(x)\,dx = 1.$$

(ii) From (2), it follows that

$$\int_b^c f_1(x)\,dx = \frac{k_3(c^2-b^2)}{2b^2c^2} = 1.$$

(iii) We get

$$\int_c^1 f_2(y)\,dy + \beta = k_4\left\{\log\frac{2c}{1+c} - \frac{1}{2} + \frac{1}{c}\right\}.$$

Since c is the root of the equation (1), we have

$$\int_c^1 f_2(y)\,dy + \beta = \frac{k_4(3c^2-2c+1)(c^2+2c-1)}{4c^2(1-c)^2} = 1.$$

(iv) It follows, from (2), that

$$k_4 = \frac{4b^2(1-b)^2}{(1-2b+3b^2)(1-2b-b^2)} \tag{5}$$

and thus

$$\int_c^1 yf_2(y)dy + \beta = \int_c^1 (y+1)f_2(y)dy + 2\beta - 1 = \frac{4b^2(1-b)(1+b)}{(1-2b+3b^2)(1-2b-b^2)} - 1.$$

Therefore we have

$$\begin{aligned}&\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy + \beta\right\}\\ &= \frac{2k_3(c-b)}{bc} + \frac{4b^2(1-b)(1+b)}{(1-2b+3b^2)(1-2b-b^2)}\left\{1 - \frac{k_3(c-b)}{bc}\right\}\\ &= \frac{4b(1-b)^2}{(1+b^2)(1-2b-b^2)} = \frac{k_2(1-\alpha)}{\alpha b}.\end{aligned}$$

Thus we obtain the desired result.
(v) We directly get

$$u\int_u^b (1+2x-x^2)f(x)\,dx = \frac{k_2\{b-u+b(u-s)\}}{b} \tag{6}$$

for all u in $[a,s]$ and

$$u\int_u^b (1+2x-x^2)f(x)\,dx = \frac{k_2(b-u)}{b} \tag{7}$$

for all u in $[s,b]$. Thus we have the result.
(vi) By (6) and (7), we get the result.

Lemma 2. *Set*

$$g_1(u) = \frac{2(1+2a-a^2)(1-u)}{u(1+2u-u^2)^2}\,, \qquad a \leqq u \leqq s,$$

$$g_2(v) = \begin{cases} \dfrac{2(1+2s-s^2)(1-v)}{v(1+2v-v^2)^2}\,, & \text{if } s \leqq v \leqq b \\ \dfrac{l_1}{v^3}\,, & \text{if } b < v \leqq c \\ \dfrac{l_2}{v(1+v)^2}\,, & \text{if } c < v \leqq 1, \end{cases}$$

where

$$l_1 = \frac{b(1-b)(1+2s-s^2)}{2(1+2b-b^2)}$$

and

$$l_2 = \frac{(1+b)(1+2s-s^2)}{1+2b-b^2}\,.$$

Then the following statements hold:

(i) $\int_a^s g_1(u)\,du = 1.$

(ii) $\int_s^1 g_2(v)\,dv = 1\,.$

(iii) *For all x in* $[b,1]$,

$$1 - \int_s^x v g_2(v)\,dz \leqq \min\left\{\frac{l_2}{1+x}, \qquad \frac{1+2s-s^2}{2(1+2b-b^2)}\left\{1+b+\frac{b(1-b)}{x}\right\}\right\}.$$

Proof. (i) Since a is the root of the equation (4), we get

$$\int_a^s \frac{2(1-u)}{u(1+2u-u^2)^2}\,du = \frac{1}{a(1+2a-a^2)} - \frac{1}{s(1+2s-s^2)} - \int_a^s \frac{du}{u^2(1+2u-u^2)}$$

$$= \frac{1}{a(1+2a-a^2)} - \frac{1-a}{a(1+2a-a^2)} = \frac{1}{1+2a-a^2}\,.$$

Thus we have the desired result.
(ii) As s is the root of the equation (3), we get

$$\int_s^b \frac{2(1+2s-s^2)(1-v)}{v(1-2v-v^2)^2}\,dv = \frac{1}{s} - \frac{1+2s-s^2}{b(1+2b-b^2)} - \int_s^b \frac{1+2s-s^2}{v^2(1+2v-v^2)}\,dv \tag{8}$$

$$= 1 - \frac{1+2s-s^2}{b(1+2b-b^2)} + \frac{(1+2s-s^2)(1-2b-b^2)(1+b^2)}{4b^2(1-b)^2}\,.$$

Further we have

$$\int_b^c \frac{l_1}{v^3}\,dv = \frac{(1+2s-s^2)(1-2b-b^2)(1+b^2)}{4b(1-b)(1+2b-b^2)}\,. \tag{9}$$

In addition, as c is the root of the equation (1), we get

$$\int_c^1 \frac{l_2}{v(1+v)^2}\,dv = l_2\left\{\log\frac{1+c}{2c}+\frac{1}{2}-\frac{1}{1+c}\right\} = l_2\left\{\frac{1-4c^2+4c^3-5c^4}{4c^2(1-c)^2}+\frac{1}{2}-\frac{1}{1+c}\right\}.$$

Thus, from (2), it follows that

$$(10)\qquad \int_c^1 \frac{l_2}{v(1+v)^2}\,dv = \frac{(1+2s-s^2)(1+b)(-2b^5+b^4+6b^3-4b^2+4b-1)}{4b^2(1-b)^2(1+2b-b^2)}.$$

By (8) to (10), we obtain the desired result.

(iii) For all x in $[b,c]$, we have

$$1-\int_s^x vg_2(v)\,dv = \frac{1+2s-s^2}{2(1+2b-b^2)}\left\{1+b+\frac{b(1-b)}{x}\right\},$$

and hence,

$$1-\int_s^x vg_2(v)\,dv \leqq \frac{(1+2s-s^2)(1+b)}{(1+2b-b^2)(1+x)} = \frac{l_2}{1+x}.$$

Further for all x in $[c,1]$, we get

$$1-\int_s^x vg_2(v)\,dv = \frac{1+2s-s^2}{1+2b-b^2}\left\{1-\frac{b(1-b)}{2}\left(\frac{1}{b}-\frac{1}{c}\right)-\frac{1+b}{1+c}+\frac{1+b}{1+x}\right\} = \frac{l_2}{1+x}$$

and hence

$$1-\int_s^x vg_2(v)\,dv \leqq \frac{1+2s-s^2}{2(1+2b-b^2)}\left\{1+b+\frac{b(1-b)}{x}\right\}.$$

Thus we obtain the desired result.

4 A strategy for player I in G In the following sections, we denote by $V_1(u,v)$ the expected payoff of the game G when player I applies the strategy $\{\alpha, f(x), f_1(x), f_2(y), \beta\}$ given in Lemma 1 and player II fires his bullets when player I is at the points u and v $(0 \leqq u \leqq v \leqq 1)$. Similarly, we denote by $V_2(x,y)$ the expected payoff of the game G when player II applies the strategy $\{g_1(u), g_2(v)\}$ given in Lemma 2 and player I fires his first and second bullets when he is at the points x and y $(0 \leqq x \leqq y \leqq 1)$, respectively.

Lemma 3. *For all u and v such that $a \leqq u \leqq v < 1$, $V_1(u,v) \geqq 2a-a^2$ $(=0.1772)$.*

Proof. For all u and v with $a \leqq u \leqq v \leqq b$, we have

$$\begin{aligned}
V_1(u,v) \;=\;& (1-\alpha)\int_a^u (2x-x^2)f(x)\,dx + (1-\alpha)\int_u^v \{-u+(1-u)(2x-x^2)\}f(x)\,dx \\
&+(1-\alpha)\int_v^b \{-u-(1-u)v+(1-u)(1-v)(2x-x^2)\}f(x)\,dx \\
&+\alpha\int_b^c\int_c^1 \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)y\}f_1(x)f_2(y)\,dydx \\
&+\alpha\beta\int_b^c \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)\}f_1(x)\,dx \\
=\;& -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx \\
&-(1-\alpha)\left\{u\int_u^b (1+2x-x^2)f(x)\,dx + (1-u)v\int_v^b (1+2x-x^2)f(x)\,dx\right\} \\
&+\alpha(1-u)(1-v)\left[\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy+\beta\right\}\right].
\end{aligned}$$

Thus, from (iv) and (vi) in Lemma 1, it follows that

$$\begin{aligned} V_1(u,v) &\geqq -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx+\frac{k_2(1-\alpha)(1-2b+bs)}{b} \\ &= -1+\frac{k_1(1-\alpha)(1-a)}{a}=2a-a^2. \end{aligned}$$

For every u and v such that $a \leqq u \leqq b \leqq v \leqq c$, we get

$$\begin{aligned} V_1(u,v)) &= (1-\alpha)\int_a^u (2x-x^2)f(x)\,dx+(1-\alpha)\int_u^b \{-u+(1-u)(2x-x^2)\}f(x)\,dx \\ &\quad +\alpha\int_b^v\int_c^1 \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_b^v \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)\,dx \\ &\quad +\alpha\int_v^c\int_c^1 \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_v^c \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)\}f_1(x)\,dx \\ &= -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx-(1-\alpha)u\int_u^b (1+2x-x^2)f(x)\,dx \\ &\quad +\alpha(1-u)(1-v)\left[\int_b^c (1+x)f_1(x)\,dx+\int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)dy+\beta\right\}\right] \\ &\quad +2\alpha(1-u)v\int_b^v xf_1(x)\,dx. \end{aligned}$$

Thus by (iv), (v) in Lemma 1 and

$$2\alpha k_3=(1-\alpha)k_2 \tag{11}$$

we have

$$V_1(u,v) \geqq -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx+\frac{k_2(1-\alpha)(1-2b+bs)}{b}=2a-a^2.$$

For all u in $[a,b]$ and v in $[c,1)$, we get

$$\begin{aligned} V_1(u,v) &= (1-\alpha)\int_a^u (2x-x^2)f(x)\,dx+(1-\alpha)\int_u^b \{-u+(1-u)(2x-x^2)\}f(x)\,dx \\ &\quad +\alpha\int_b^c\int_c^v \{-u+(1-u)x+(1-u)(1-x)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\int_b^c\int_v^1 \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_b^c \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)\,dx \\ &= -1+(1-\alpha)\int_a^b (1+2x-x^2)\,f(x)dx-(1-\alpha)u\int_u^b (1+2x-x^2)f(x)\,dx \\ &\quad +\alpha(1-u)\left[\int_b^c (1+x)f_1(x)\,dx+\int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy+\beta\right\}\right] \\ &\quad -\alpha(1-u)v\int_b^c (1-x)f_1(x)\,dx\left\{\int_v^1 (1+y)f_2(y)\,dy+2\beta\right\}. \end{aligned}$$

Consequently, by (iv) and (v) in Lemma 1, we have

$$\begin{aligned} V_1(u,v) &\geqq -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx - \frac{k_2(1-\alpha)\{b-u+b(u-s)\}}{b} \\ &\quad + \frac{k_2(1-\alpha)(1-u)}{b} - \alpha k_4(1-u)\int_b^c (1-x)f_1(x)\,dx \\ &= -1+(1-\alpha)\int_a^b (1+2x-x^2)f_(x)\,dx + \frac{k_2(1-\alpha)(1-2b+bs)}{b} = 2a-a^2. \end{aligned}$$

For every u and v such that $b \leqq u \leqq v \leqq c$, we get

$$\begin{aligned} V_1(u,v) &= (1-\alpha)\int_a^b (2x-x^2)f_(x)dx \\ &\quad +\alpha\int_b^u\int_c^1 \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_b^u \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)\}f_1(x)\,dx \\ &\quad +\alpha\int_u^v\int_c^1 \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_u^v \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)\,dx \\ &\quad +\alpha\int_v^c\int_c^1 \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_v^c \{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)\}f_1(x)\,dx. \end{aligned}$$

Thus

$$\begin{aligned} V_1(u,v) &= -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx \\ &\quad +\alpha(1-u)(1-v)\left[\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy+\beta\right\}\right] \\ &\quad +2\alpha\left\{u\int_b^u xf_1(x)\,dx + (1-u)v\int_b^v xf_1(x)\,dx\right\}, \end{aligned}$$

and hence by (11) and (iv) in Lemma 1, we get

$$\frac{\partial V_1(u,v)}{\partial u} = 2\alpha k_3 > 0$$

and thus

$$V_1(u,v) \geqq V_1(b,v) \geqq 2a-a^2$$

for all u and v with $b \leqq u \leqq v \leqq c$. Further, for all u and v with $b \leqq u \leqq c \leqq v < 1$, we get

$$\begin{aligned} V_1(u,v) &= (1-\alpha)\int_a^b (2x-x^2)f(x)\,dx + \alpha\int_b^u\int_c^v \{x-(1-x)u+(1-x)(1-u)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\int_b^u\int_v^1 \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_b^u \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)\}f_1(x)\,dx \\ &\quad +\alpha\int_u^c\int_c^v \{-u+(1-u)x+(1-u)(1-x)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\int_u^c\int_v^1 \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_u^c \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)\,dx \\ &= -1+(1-\alpha)\int_a^b (1+2x-x^2)f(x)\,dx \\ &\quad +\alpha(1-u)\left[\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy+\beta\right\}\right] \\ &\quad +2\alpha u\int_b^u xf_1(x)\,dx - \alpha(1-u)v\int_b^c (1-x)f_1(x)\,dx\left\{\int_v^1 (1+y)f_2(y)\,dy+2\beta\right\}. \end{aligned}$$

Thus we have, by (iv) in Lemma 1 and (11),

$$\frac{\partial V_1(u,v)}{\partial u} = \alpha k_4\int_b^c (1-x)f_1(x)\,dx > 0.$$

Consequently we get

$$V_1(u,v) \geqq V_1(b,v) \geqq 2a-a^2$$

for all u and v such that $b \leqq u \leqq c \leqq v < 1$. For all u and v with $c \leqq u \leqq v < 1$, we have

$$\begin{aligned} V_1(u,v) &= (1-\alpha)\int_a^b (2x-x^2)f(x)\,dx + \alpha\int_b^c\int_c^u \{x+(1-x)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\int_b^c\int_u^v \{x-(1-x)u+(1-x)(1-u)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\int_b^c\int_v^1 \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)y\}f_1(x)f_2(y)\,dydx \\ &\quad +\alpha\beta\int_b^c \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)\}f_1(x)\,dx. \end{aligned}$$

Accordingly we get

$$\begin{aligned} V_1(u,v) &= -1+(1-\alpha)\int_a^b (1+2x-x^2)f_1(x)\,dx \\ &\quad +\alpha\left[\int_b^c (1+x)f_1(x)\,dx + \int_b^c (1-x)f_1(x)\,dx\left\{\int_c^1 yf_2(y)\,dy+\beta\right\}\right] \\ &\quad -\alpha\int_b^c (1-x)f_1(x)\,dx\left\{u\int_u^1 (1+y)f_2(y)\,dy + v(1-u)\int_v^1 (1+y)f_2(y)\,dy\right\} \\ &\quad -2\alpha\beta\{u+(1-u)v\}\int_b^c (1-x)f_1(x)\,dx. \end{aligned}$$

It is seen that
$$\frac{\partial V_1(u,v)}{\partial u} = \alpha k_4 \int_b^c (1-x) f_1(x)dx > 0$$
for all u and v with $c \leqq u \leqq v < 1$, and hence
$$V_1(u,v) \geqq V_1(c,v) \geqq 2a - a^2.$$
Therefore we obtain
$$V_1(u,v) \geqq 2a - a^2$$
for all u and v such that $a \leqq u \leqq v \leqq< 1$. This completes our proof.

5 A strategy for player II in G

Lemma 4. *For all x and y such that $a \leqq x \leqq y \leqq 1$, $V_2(x,y) \leqq 2a - a^2$ $(= 0.1772)$.*

Proof. We get, for all x in $[a, s]$,

$$\begin{aligned} V_2(x,x) &= \int_a^x \{-u + (1-u)(2x - x^2)\} g_1(u)\, du + \int_x^s (2x - x^2) g_1(u)\, du \\ &= -1 + (1 + 2x - x^2)\left\{1 - \int_a^x u g_1(u)\, du\right\} = 2a - a^2 \end{aligned}$$

and for all x in $[s, b]$

$$\begin{aligned} V_2(x,x) &= \int_a^s \int_s^x \{-u - (1-u)v + (1-u)(1-v)(2x - x^2)\} g_1(u) g_2(v)\, dvdu \\ &\quad + \int_a^s \int_x^1 \{-u + (1-u)(2x - x^2)\} g_1(u) g_2(v)\, dvdu \\ &= -1 + (1 + 2x - x^2) \int_a^s (1-u) g_1(u)\, du \left\{1 - \int_s^x v g_2(v)\, dv\right\} = 2a - a^2. \end{aligned}$$

Further for all x and y such that $a \leqq x \leqq y \leqq s$, we have

$$\begin{aligned} V_2(x,y) &= \int_a^x \{-u + (1-u)x + (1-u)(1-x)y\} g_1(u)\, du \\ &\quad + \int_x^y \{x - (1-x)u + (1-x)(1-u)y\} g_1(u)\, du + \int_y^s \{x + (1-x)y\} g_1(u)\, du \\ &= -1 + (1-x)(1+y)\left\{1 - \int_a^y u g_1(u)\, du\right\} + 2x\left\{1 - \int_a^x u g_1(u) du\right\} \\ &= -1 + (1 + 2a - a^2)\left\{\frac{(1-x)(1+y)}{1 + 2y - y^2} + \frac{2x}{1 + 2x - x^2}\right\}. \end{aligned}$$

It is seen that $V_2(x,y)$ is decreasing in y for every x such that $a \leqq x \leqq y \leqq s$, and hence
$$V_2(x,y) \leqq V_2(x,x) = 2a - a^2$$
for all x and y with $a \leqq x \leqq y \leqq s$. For all x and y such that $s \leqq x \leqq y \leqq 1$, we get

$$\begin{aligned} V_2(x,y) &= \int_a^s\int_s^x\{-u-(1-u)v+(1-u)(1-v)x+(1-u)(1-v)(1-x)y\}\,g_1(u)g_2(v)\,dvdu \\ &\quad+\int_a^s\int_x^y\{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\}g_1(u)g_2(v)\,dvdu \\ &\quad+\int_a^s\int_y^1\{-u+(1-u)x+(1-u)(1-x)y\}g_1(u)g_2(v)\,dvdu \\ &= -1+X(x,y)\int_a^s(1-u)g_1(u)\,du, \end{aligned}$$

where

$$\begin{aligned} X(x,y) &= \int_s^x\{1-v+(1-v)x+(1-v)(1-x)y\}g_2(v)\,dv \\ &\quad+\int_x^y\{1+x-(1-x)v+(1-x)(1-v)y\}g_2(v)\,dv+\int_y^1\{1+x+(1-x)y\}g_2(v)\,dv \\ &= (1-x)(1+y)\left\{1-\int_s^y vg_2(v)\,dv\right\}+2x\left\{1-\int_s^x vg_2(v)\,dv\right\}. \end{aligned}$$

Thus, by (iii) in Lemma 2, we have

$$X(x,y) \leqq (1-x)l_2+\frac{1+2s-s^2}{1+2b-b^2}\{(1+b)x+b(1-b)\}=1+2s-s^2$$

for all x and y such that $b \leqq x \leqq y \leqq 1$. We get, for every x and y with $s \leqq x \leqq y \leqq b$,

$$X(x,y)=(1+2s-s^2)\left\{\frac{(1-x)(1+y)}{1+2y-y^2}+\frac{2x}{1+2x-x^2}\right\}$$

and for all x in $[s,b]$ and y in $[c,1]$

$$X(x,y)=(1+2s-s^2)\left\{\frac{(1-x)(1+b)}{1+2b-b^2}+\frac{2x}{1+2x-x^2}\right\}.$$

In both cases above we have

$$X(x,y) \leqq 1+2s-s^2.$$

Further, for any x and y such that $s \leqq x \leqq b \leqq y \leqq c$, we get

$$X(x,y)=(1+2s-s^2)\left[\frac{(1-x)(1+y)}{2(1+2b-b^2)}\left\{1+b+\frac{b(1-b)}{y}\right\}+\frac{2x}{1+2x-x^2}\right]$$

and hence

$$\frac{\partial^2 X(x,y)}{\partial y^2}>0.$$

Thus we get

$$X(x,y) \leqq 1+2s-s^2.$$

Accordingly we obtain

$$V_2(x,y) \leqq -1+(1+2s-s^2)\left\{1-\int_a^s ug_1(u)\,du\right\}=2a-a^2$$

for all x and y such that $s \leqq x \leqq y \leqq 1$. For every x and y such that $a \leqq x \leqq s \leqq y \leqq 1$, we have

$$\begin{aligned} V_2(x,y) &= \int_a^x \int_s^y \{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)y\} g_1(u)g_2(v)\,dvdu \\ &+ \int_a^x \int_y^1 \{-u+(1-u)x+(1-u)(1-x)y\} g_1(u)g_2(v)\,dvdu \\ &+ \int_x^s \int_s^y \{x-(1-x)u-(1-x)(1-u)v+(1-x)(1-u)(1-v)y\} g_1(u)g_2(v)\,dvdu \\ &+ \int_x^s \int_y^1 \{x-(1-x)u+(1-x)(1-u)y\} g_1(u)g_2(v)\,dvdu \end{aligned}$$

and thus

$$\begin{aligned} (12)\quad V_2(x,y) &= -1+2x\left\{1-\int_a^x ug_1(u)\,du\right\} \\ &+(1-x)(1+y)\int_a^s (1-u)g_1(u)\,du\left\{1-\int_s^y vg_2(v)\,dv\right\} \\ &= -1+(1+2a-a^2)\left[\frac{2x}{1+2x-x^2}+\frac{(1-x)(1+y)}{1+2s-s^2}\left\{1-\int_s^y vg_2(v)\,dv\right\}\right]. \end{aligned}$$

Hence for all x and y such that $a \leqq x \leqq s \leqq y \leqq b$ we get

$$V_2(x,y) = -1+(1+2a-a^2)\left\{\frac{2x}{1+2x-x^2}+\frac{(1-x)(1+y)}{1+2y-y^2}\right\}$$

and thus $V_2(x,y)$ is decreasing in y over $[s,b]$ for all x in $[a,s]$. Accordingly we have

$$V_2(x,y) \leqq V_2(x,s) \leqq 2a-a^2$$

for all x and y such that $a \leqq x \leqq s \leqq y \leqq b$. For any x in $[a,s]$ and y in $[c,1]$, by (12) we get

$$V_2(x,y) = -1+(1+2a-a^2)\left\{\frac{2x}{1+2x-x^2}+\frac{(1-x)(1+b)}{1+2b-b^2}\right\} \leqq 2a-a^2.$$

Further for all x in $[a,s]$ and y in $[b,c]$, we have

$$V_2(x,y) = -1+(1+2a-a^2)\left[\frac{(1-x)(1+y)}{2(1+2b-b^2)}\left\{1+b+\frac{b(1-b)}{y}\right\}+\frac{2x}{1+2x-x^2}\right].$$

It is seen that $V_2(x,y)$ is a convex function of y over $[b,c]$ for every x in $[a,s]$. Therefore we have

$$V_2(x,y) \leqq 2a-a^2$$

for every x in $[a,s]$ and y in $[b,c]$. Thus we obtain

$$V_2(x,y) \leqq 2a-a^2$$

for all x and y such that with $a \leqq x \leqq y \leqq 1$. This completes our proof.

6 A Theorem

Theorem 1. *For the game G, the strategy $\{\alpha, f(x), f_1(x), f_2(y), \beta\}$ given in Lemma 1 is optimal for player I, and the strategy $\{g_1(u), g_2(v)\}$ given in Lemma 2 is optimal for player II. Moreover, the game value of the game G is $2a - a^2$ $(= 0.1772)$.*

Proof. It suffices to show that

$$V_1(u, v) \geqq 2a - a^2$$

for all u and v such that $0 \leqq u \leqq v \leqq 1$ and

$$V_2(x, y) \leqq 2a - a^2$$

for all x and y such that $0 \leqq x \leqq y \leqq 1$. From Lemma 3, we have

$$V_1(u, v) \geqq 2a - a^2$$

for every u and v with $a \leqq u \leqq v < 1$. It is seen that $V_1(u, v)$ decreases in u over $[0, a]$ for all v in $[a, 1)$, and hence

$$V_1(u, v) \geqq V_1(a, v) \geqq 2a - a^2.$$

Further, for all u and v such that $0 \leqq u \leqq v \leqq a$, it is seen that $V_1(u, v)$ decreases in v, and thus

$$V_1(u, v) \geqq V_1(u, a) \geqq 2a - a^2.$$

We can verify that

$$V_1(u, 1) \geqq \lim_{v \to 1-0} V_1(u, v) \geqq 2a - a^2$$

for all u in $[0, 1)$ and

$$V_1(1, 1) \geqq 2a - a^2.$$

Therefore we obtain

$$V_1(u, v) \geqq 2a - a^2$$

for all u and v such that $0 \leqq u \leqq v \leqq 1$. Now by Lemm 4

$$V_2(x, y) \leqq 2a - a^2$$

for all x and y such that $a \leqq x \leqq y \leqq 1$. For all x and y such that $0 \leqq x \leqq y \leqq a$, we have

$$V_2(x, y) = x + (1 - x)y \leqq 2a - a^2.$$

In addition, we get

$$\begin{aligned} V_2(x, y) &= x + (1 - x) \int_a^y \{-u + (1 - u)y\} g_1(u)\, du + (1 - x) \int_y^s y g_1(u)\, du \\ &= x + (1 - x)y - (1 - x)(1 + y) \int_a^y u g_1(u) du \end{aligned}$$

for all x in $[0, a]$ and y in $[a, s]$, and

$$\begin{aligned} V_2(x, y) &= x + (1 - x) \int_a^s \int_s^y \{-u - (1 - u)v + (1 - u)(1 - v)y\} g_1(u) g_2(v)\, dv du \\ &\quad + (1 - x) \int_a^s \int_y^1 \{-u + (1 - u)y\} g_1(u) g_2(v)\, dv du \\ &= 2x - 1 + (1 - x)(1 + y) \int_a^s (1 - u) g_1(u)\, du \Big\{ 1 - \int_s^y v g_2(v)\, dv \Big\} \end{aligned}$$

for all x in $[0,a]$ and y in $[s,1]$. In both cases above, $V_2(x,y)$ is increasing in x and hence

$$V_2(x,y) \leqq V_2(a,y) \leqq 2a - a^2$$

for all x and y such that $0 \leqq x \leqq a$ and $x \leqq y \leqq 1$. Therefore we obtain

$$v_2(x,y) \leqq 2a - a^2$$

for all x and y such that $0 \leqq x \leqq y \leqq 1$. This completes our proof.

7 Optimal Strategies In this section, we examine the game G^* defined at the beginning of section 2. For any ε in $(0,1)$, we define N as the smallest natural number that is larger than $1/\varepsilon$. For the N, we define constants a_i $(i = 0,1,2,\cdots,n_1+1)$ and c_i $(i = 1,2,\cdots,n_2+1)$ as follows:

$$a_0 = a,$$

$$\int_{a_{i-1}}^{a_i} f(x)\,dx = \frac{1}{(1-\alpha)N}\,, \qquad i = 1,2,\cdots,n_1,$$

$$a_{n_1+1} = b,$$

$$c_0 = c,$$

$$\int_{c_{i-1}}^{c_i} f_2(y)\,dy = \frac{1}{\alpha N}, \qquad i = 1,2,\cdots,n_2,$$

$$c_{n_2+1} = 1,$$

where

$$\int_a^b f_1(x)\,dx > \int_a^{a_{n_1}} f_1(x)\,dx \geqq \int_a^b f_1(x)\,dx - \frac{1}{(1-\alpha)N}$$

and

$$\int_c^1 f_2(y)\,dy > \int_c^{c_{n_2}} f_2(y)\,dy \geqq \int_c^1 f_2(y)\,dy - \frac{1}{\alpha N}\,.$$

Now we define the strategy $\{\alpha, f(x), f_1(x), f_2(y), \beta\}^{\varepsilon}$ of player I in the game G^* as follows:

(i) Player I fires both of his bullets simultaneously with probability $1-\alpha$ and he fires his bullets at different moments with probability α.

(ii) Player I moves back and forth at first between 0 and a_1, then between 0 and a_2, $\cdots$, and then between 0 and a_{n_1+1} regardless of whether he fires his bullets simultaneously or not. If he fires his bullets simultaneously, then he fires both of his bullets, at the i-th step $(i = 1,2,\cdots,n_1+1)$, only if he is between a_{i-1} and a_i and goes forward, according to the distribution with the conditional probability density $f(x)$ under the condition that he fires both his bullets simultaneously. After he has fired both of his bullets at the i-th step, he reaches the point a_i, escapes to 0 and never approaches player II.

(iii) When player I has not fired his bullets in $[a,b]$, he further moves back and forth between 0 and c and he fires his first bullet between b and c, only if he goes forward, according to the conditional distriburion with the conditional probability density $f_1(x)$ under the condition that he fires his bullets at different moments. Furthermore player I moves back and forth between 0 and c_1, then between 0 and c_2, $\cdots$, and then between 0 and c_{n_2+1}. When he moves back and forth between 0 and c_i, he fires his second bullet only if he is between c_{i-1} and c_i and goes forward, according to the conditional distribution with the density part $f_2(y)$ and the mass part β at 1, independently of the point where he has fired his first bullet. If he has fired his second bullet between c_{i-1} and c_i, he reaches the point c_i, escapes to 0 and never approaches player II.

Theorem 2. *For the game G^*, the strategy $\{\alpha, f(x), f_1(x), f_2(y), \beta\}^\varepsilon$ is ε-optimal for player* I*, and the strategy $\{g_1(u), g_2(v)\}$ given in Lemma* 2 *is optimal for player* II. *Moreover, the game value of G^* is $2a - a^2$.*

Proof. It is seen that if player I fires at a point, then he has to fire when he is at the point for the first time. Similarly, if player II fires when player I is at a point, then player II has to fire when player I is at the point for the first time. Thus, in what follows, we assume that player I fires at points when he is at these points for the first tme, and player II fires his bullets when player I is at the points for the first time. Now we denote by $V_1^*(u,v)$ the expected payoff of the game G^* when player I applies the strategy $\{\alpha, f(x), f_1(x), f_2(y), \beta\}^\varepsilon$ and player II fires his bullets when player I is at the points u and v $(0 \leqq u \leqq v \leqq 1)$. Similarly, we denote by $V_2^*(x,y)$ the expected payoff of the game G^* when player II applies the strategy $\{g_1(u), g_2(v)\}$ and player I fires his bullets when he is at the points x and y $(0 \leqq x \leqq y \leqq 1)$. For all u in $(a_i, a_{i+1}]$ $(i = 0, 1, 2, \cdots, n_1)$ and v in $(c_j, c_{j+1}]$ $(j = 0, 1, 2, \cdots, N_2)$, we get

$$\begin{aligned}
V_1^*(u,v) &= (1-\alpha)\int_a^{a_i}(2x-x^2)f(x)\,dx + (1-\alpha)\int_{a_i}^{u}\{2x-x^2-(1-x)^2u\}f(x)\,dx \\
&\quad +(1-\alpha)\int_u^b\{-u+(1-u)(2x-x^2)\}f(x)\,dx \\
&\quad +\alpha\int_b^c\int_c^{c_j}\{-u+(1-u)x+(1-u)(1-x)y\}f_1(x)f_2(y)\,dydx \\
&\quad +\alpha\int_b^c\int_{c_j}^{v}\{-u+(1-u)x+(1-u)(1-x)y-(1-u)(1-x)(1-y)v\}f_1(x)f_2(y)\,dydx \\
&\quad +\alpha\int_b^c\int_v^1\{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)f_2(y)\,dydx \\
&\quad +\alpha\beta\int_b^c\{-u+(1-u)x-(1-u)(1-x)v+(1-u)(1-x)(1-v)\}f_1(x)f_2(y)\,dydx \\
&= V_1(u,v)-(1-\alpha)u\int_{a_i}^{u}(1-x)^2f_1(x)\,dx \\
&\quad -\alpha(1-u)v\int_b^c\int_{c_j}^{v}(1-x)(1-y)f_1(x)f_2(y)\,dydx \\
&\geqq V_1(u,v)-\frac{1}{N} > 2a-a^2-\varepsilon.
\end{aligned}$$

Similarly we can show that

$$V_1^*(u,v) \geqq V_1(u,v) - \frac{1}{N} > 2a - a^2 - \varepsilon$$

for every u and v with $0 \leqq u \leqq v \leqq 1$. It is clear that

$$V_2^*(x,y) = V_2(x,y) \leqq 2a - a^2$$

for all x and y such that $0 \leqq x \leqq y \leqq 1$. We suppose that player I fires his first bullet when he is at a place x in $(0,1]$, and escapes to 0 and stays in 0 forever. We denote such a strategy by $(x,*)$ and we denote by $V_2^*(x,*)$ the expected payoff when players I and II apply the strategies $(x,*)$ and $\{g_1(u), g_2(v)\}$, respectively. Then we get

$$V_2^*(x,*) = x \leqq 2a - a^2$$

for every x in $[0, a)$ and

$$\begin{aligned} V_2^*(x,*) &= \int_a^x \{-u+(1-u)x\}g_1(u)\,du + \int_x^s xg_1(u)\,du \\ &= x-(1+x)\int_a^x ug_1(u)\,du \leqq a \leqq 2a-a^2 \end{aligned}$$

for every x in $[a,\, s]$. Further we have

$$\begin{aligned} V_2^*(x,*) &= \int_a^s \int_s^x \{-u-(1-u)v+(1-u)(1-v)x\}g_1(u)g_2(v)\,dvdu \\ &\quad + \int_a^s \int_x^1 \{-u+(1-u)x\}g_1(u)g_2(v)\,dvdu \\ &\leqq a \leqq 2a-a^2 \end{aligned}$$

for all x in $[s, 1]$. Thus we can conclude that, if player II applies the strategy $\{g_1(u), g_2(v)\}$, the expected payoff is at most $2a-a^2$ whatever strategy player I may apply. This completes our proof.

In this paper, we have assumed that player I can fire his two bullets at different moments and we have figured out that the game value is $2a-a^2 = 0.1772$. As is mentioned in section 1, Trybula [10] solved an m-versus-n silent duel with arbitrary accuracy functions under arbitrary motion. In Trybula's model, player I has to fire his m bullets simultaneously, whereas player II can fire his n bullets at different moments. If we set $m = 2$, $n = 2$ and $p(x) = q(x) = x$ in Trybula's model, then the game value is $2\hat{a} - \hat{a}^2 (= 0.1618)$, where $\hat{a} = 0.0845$ is the unique root in $(0,\, \hat{b})$ of the equation

$$\int_x^{\hat{b}} \frac{2-2t}{t(1+2t-t^2)^2}\,dt = \frac{1}{1+2x-x^2}$$

and $\hat{b} = 0.1779$ is the unique root in $(0,\, 1)$ of the equation

$$\int_x^1 \frac{2-2t}{t(1+2t-t^2)^2}\,dt = \frac{1}{1+2x-x^2}\ .$$

Thus the game value of our model is larger than the game value of the Trybula's model as might be expected.

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