A SIMPLE NUMBER THEORETIC PROBLEM II

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Abstract. In this note, we shall consider a number theoretic problem finitely determined.

Diophantine equations is a treasure house which has been giving amateurs or professionals in mathematics a lot of problems. One can see many Diophantine equations whose solutions are theoretically or experimentally known (see [3], [4], [5]). Let n be any positive integer. We denote by $DS(n)$ the sum of digits of n. By Mohanty and Kumar [2], Moret Blanc reported in 1879 that the only integers which satisfy $DS(n^3) = n$ are the numbers $1, 8, 17, 26, 27$. And, Iseki and Nakakura [1] revisited and discussed all the positive integral solutions of the equations $DS(n^2) = n$, $DS(n^3) = n$, and $DS(n^4) = n$. In this paper, we consider all positive integral solutions of the following equations with a variable n:

$$DS(n^p) = n \quad (1 \leqq p,\, p \in \mathbf{N}) \tag{1}$$

and

$$DS(n^p + n^q) = n \quad (1 \leqq q < p,\, p, q \in \mathbf{N}) \tag{2}$$

These problems are continuations of the problem treated by Iseki and Nakakura[1].

For a given positive integer p and a positive number c with $c \geqq 1$, let us consider a function $f(x) = \dfrac{x - c}{\log_{10} 9x}$ on $[1, \infty)$. Then, $f(x)$ is monotone increasing because

$$f'(x) = \frac{\log_{10} 9x - \dfrac{x - c}{x \log_e 10}}{(\log_{10} 9x)^2} \geqq 0 \quad x \in [1, \infty).$$

Since $f(1) \leqq 0$ and $\lim_{x \to \infty} f(x) = +\infty$, we define a positive integer $k(p, c)$ as

$$k(p, c) = \max \left\{ \ell \;\middle|\; \ell \in \mathbf{N}, \frac{\ell - c}{\log_{10} 9\ell} < p \right\}.$$

Now we prepare a proposition effetively to find all positive integral solutions of (1) and (2).

Proposition 1. *Let p and q be any postive integers with $p < q$. Then the following statements hold:*

(P1) *Any positive integral solution n of (1) satisfies $n \leqq 9k(p, 1)$.*

(P2) *Any positive integral solution n of (2) satisfies $n \leqq 9k(p, c), c = 1 + \log_{10} 10/9$.*

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Proof. We only show (P2). One can prove (P1) analogously. Let n be any positive integral solution of (1) and suppose that $10^{k-1} \leqq n^p + n^q < 10^k$. Since $DS(n^p + n^q) = n$, we have $n \leqq 9k$. Noting that

$$10^{k-1} \leqq (9k)^p + (9k)^q = (9k)^p(1 + (9k)^{q-p}) \leqq (9k)^p \frac{10}{9},$$

we obtain $k - 1 \leqq p \log_{10} 9k + \log_{10} 10/9$. Since k satisfies $p \geqq \dfrac{k - 1 - \log_{10} 10/9}{\log_{10} 9k}$, we have $n \leqq 9k(p, c), c = 1 + \log_{10} 10/9$. □

By Proposition, we can find all positive integral solutions of (1) and (2). We only give solutions for some specific p and q.

p	solution n of $DS(n^p) = n$ except trivial case 1												
2	9												
3	8	17	18	26	27								
4	7	22	25	28	36								
5	28	35	36	46									
6	18	45	54	64									
7	18	27	31	34	43	53	58	68					
8	46	54	63										
9	54	71	81										
10	82	85	94	97	106	117							
11	98	107	108										
12	108												
13	20	40	86	103	104	106	107	126	134	135	146		
14	91	118	127	135	154								
15	107	134	136	152	154	172	199						
16	133	142	163	169	181	187							
17	80	143	171	216									
18	172	181											
19	80	90	155	157	171	173	181	189	207				
20	90	181	207										
21	90	199	225										
22	90	169	193	217	225	234	256						
23	234	244	271										
24	252	262	288										
25	140	211	221	236	256	257	261	277	295	296	298	299	337
26	306	307	316	324									
27	305	307											
28	90	160	265	292	301	328							
29	305	314	325	332	341								
30	396												
40	250	441	468	486	495	502							
50	685												
60	694	784	792	793									
70	540	882	909										
80	1044	1071	1134	1144									
90	1306	1422											
100	1363	1378	1408	1414	1489								

p	q	solution n of $DS(n^p + n^q) = n$							
1	2	3	6	9	12				
1	3	3	6	12	15	18	21	24	
1	4	6	15	21	27	33	36		
1	5	21	33						
1	6	18	33	39	45				
1	7	18	27	33	39	42	48	63	78
1	8	24	39	54	69				
1	9	30	51	81	84				
1	10	30	75	87	93	102	117		
1	20	165	192						
2	21	207							
2	22	234							
2	23	225							
2	24	243							
2	25	261	279						
2	26	140	180	284	293	306	324		
2	27	none							
2	28	315	324						
2	29	180							
2	30	396							
26	91	1215							
26	92	1274	1278	1301	1364				
26	93	none							
26	94	none							
26	95	1278	1377						
26	96	1170							
26	97	1404							
26	98	1341	1373	1391	1436				
26	99	1170							
26	100	1413	1449	1476					
30	100	1130	1386	1472					
38	58	540	702	738	749	758	765	801	
40	90	1233	1238	1314					
70	71	1008							
70	72	540	932	959					
70	73	864							
70	74	1028							
70	75	918							
70	76	927	1026	1035					
70	77	1107							
70	78	680	1017	1026	1044	1049	1058	1098	
70	79	954							
70	80	720	1053	1089	1143				
74	76	983	999	1026	1028	1037	1044		
88	99	1413	1431	1494					
92	100	830	1332	1397	1433	1478			
99	100	1323	1422						

For example, $DS(n^{100}) = n$. Range of n is $n \leq 9 \times 350 = 3150$. In fuct, we discoverd the following five numbers using a computer.

1363^{100} = 2815694830343602687514118800351900390160927160105070643398467636147297
9358660181739591513955012701514141009863340083608788949636723746834761818305569
9372556585737643467160512769709407632592051353267392512862600143108701723581143
2057385089089679542131568833494945686880003812851795666859587298101911842160359
3114001

1378^{100} = 8412435688813725654225020035454375219351933040132468234938243208843224
3041470696088350462334382025186267163670246436206724455318657786488741769506104
2063336080149949614536516930116911274543833925255653828487735148538230093904026
9346942143502863616490034332734531749317956811361985978778528858808968008651995
9781376

1408^{100} = 7248789371401965807551954760268824659670228666986885181691706689654710
8057069525606581221838301928124772206405845851150858651287424520740236742460207
1070108921567614407489506523776565930545038990628874797210009137860822342344334
5075123555532835323711672983934473131573962019823991587356796278129575903306220
05477376

1414^{100} = 1109024003632305658467979069271663097822697973382487125623660048199131
0687805720766673761261719321663007513666969123708291767166261534377800600654247
3666233147216560438011547394178426055082979750398487190486997202403047155311377
3150663875754160182631734334364080352283171866650393533988774848986658065819396
919525376

1489^{100} = 1947465511104095129200185685811806761771474539312945150347820371969159
3973808385345524068020676660602896969605784987582389938586294297978177020004368
8865666271980080275941228958332132777322069482207651726576748098711809799154256
5349772948954238237536117472180483077282159698332249672048264279563745530138849
63900296001

Remark 1. *A solution* $k = 8$ *of* $DS(k^3) = k$, *and a solution* $k = 7$ *of* $DS(k^4) = k$ *aren't shown in Iseki and Nakakura* [1].

Remark 2. *The calculation was done by* UBASIC ver. 8.8f *developed by Dr. Y. Kida, a program available at the URL* `http://www.rkmath.rikkyo.ac.jp/~kida/ubasic.htm`. *By sending the author an e-mail, the readers could know solutions of* (1) *and* (2) *for* p *and* q *which are not contained in the above tables.*

Remark 3. *Generalizing the equations stated above, we could analogously consider problems of type* $DS(n^{p_1} + n^{p_2} + \cdots + n^{p_k}) = n$ *with positive integers* $p_1 > p_2 > \cdots > p_k \geqq 1$.

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