ON BRANCHES IN POSITIVE IMPLICATIVE BCI-ALGEBRAS WITH CONDITION (S)

Yisheng Huang*

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Abstract. In this paper we show that given a positive implicative BCI-algebra X with condition (S), every branch $V(a)$ of X with respect to the BCI-ordering $\leqslant$ on X forms an upper semilattice $(V(a); \leqslant)$; especially, if $V(a)$ is a finite set, $(V(a); \leqslant)$ forms a lattice; moreover, if $(V(a); \leqslant)$ is a lattice, it must be distributive. We also obtain some interesting identities on $V(a)$.

K. Iséki and S. Tanaka in [7] discussed more systematically positive implicative BCK-algebras. The author in [3] considered the relations between lattices and positive implicative BCK-algebras with condition (S).

In order to generalize the positive implicativity from BCK-algebras to BCI-algebras, J. Meng and X. L. Xin in [9] introduced positive implicative BCI-algebras, M. A. Chaudhry in [1] introduced weakly positive implicative BCI-algebras. Based on [1], S. M. Wei and Y. B. Jun in [10] investigated a series of properties of weakly positive implicative BCI-algebras. Based on [9], the author in [4] showed that positive implicative BCI-algebras are equivalent to weakly positive implicative BCI-algebras, and obtained some further properties of theirs.

In this paper we will continue our discussion of [3], [4] and [10]. We will first consider the relations between lattices and the branches of a positive implicative BCI-algebra with condition (S), and next give several interesting identities on such a branch.

0 Preliminaries For the notations and elementary properties of BCK and BCI-algebras, we refer the reader to [7], [6] and [8]. And we will use some familiar notions and properties of lattices without explanation.

Recall that given a *BCI-algebra* $(X; *, 0)$, the following identities hold:

$$x * x = 0,\ x * 0 = x \ \text{ and } \ (x * y) * x = 0 * y,$$
$$(x * y) * z = (x * z) * y, \tag{0.1}$$
$$0 * (x * y) = (0 * x) * (0 * y). \tag{0.2}$$

And X with respect to its *BCI-ordering* $\leqslant$ forms a partially ordered set $(X; \leqslant)$ satisfying the following quasi-identities:

$$(x * y) * (x * z) \leqslant z * y, \tag{0.3}$$
$$(x * z) * (y * z) \leqslant x * y, \tag{0.4}$$
$$x * (x * y) \leqslant y, \tag{0.5}$$

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where the binary relation $\leqslant$ on X is defined as follows: $x \leqslant y$ if and only if $x * y = 0$. Moreover, the following assertions are valid: for any x, y, $z \in X$,

$$x \leqslant y \text{ implies } z * y \leqslant z * x, \tag{0.6}$$
$$x \leqslant y \text{ implies } x * z \leqslant y * z, \tag{0.7}$$
$$x * y \leqslant z \text{ implies } x * z \leqslant y. \tag{0.8}$$

A *branch* $V(a)$ of a BCI-algebra X is such set $\{x \in X \mid x \geqslant a\}$ in which a is a *minimal element* of X in the sense that $x \leqslant a$ implies $x = a$ for all $x \in X$. It has been known (see, e.g., [8], §1.3) that the collection $\{V(a) \mid a \in L(X)\}$ of branches of X forms a partition of X, that is, $X = \bigcup_{a \in L(X)} V(a)$ and $V(a) \cap V(b) = \varnothing$ whenever $a \neq b$, where $L(X)$ is the set of the entire minimal elements of X. And the following assertions are true:

$$x \in V(a) \text{ implies } 0 * x = 0 * a, \tag{0.9}$$
$$x \in V(a) \text{ and } y \in V(b) \text{ imply } x * y \in V(a * b), \tag{0.10}$$
$$x \leqslant y \text{ implies that } x \text{ and } y \text{ are in the same branch of } X. \tag{0.11}$$

It has been known (see, e.g., [8], §2.8) that a BCI-algebra X is with *condition (S)* if and only if there is a binary operation $\circ$ on X such that $(X; \circ, 0)$ is a commutative monoid satisfying the identity

$$x * (y \circ z) = (x * y) * z. \tag{0.12}$$

Moreover, if X is with condition (S), the following hold: for any x, y, $z \in X$,

$$(x \circ y) * x \leqslant y, \tag{0.13}$$
$$x * y \leqslant z \text{ if and only if } x \leqslant y \circ z. \tag{0.14}$$

A BCI-algebra X is called *positive implicative* if it satisfies the identity

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x)));$$

it is called *weakly positive implicative* if it satisfies the identity

$$(x * y) * z = ((x * z) * z) * (y * z). \tag{0.15}$$

It is known (see, [4], Theorem 2) that a BCI-algebra is positive implicative if and only if it is weakly positive implicative. Thus, if X is positive implicative, (0.15) is valid. Replacing y by 0 and z by y in (0.15), the following holds: for any x, $y \in X$,

$$x * y = ((x * y) * y) * (0 * y). \tag{0.16}$$

Moreover, if y is in the branch $V(b)$ of X, by (0.16) and (0.9), we obtain

$$x * y = ((x * y) * y) * (0 * b). \tag{0.17}$$

Proposition 0.1. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X. Then the following is true: for any $x \in V(a)$,*

$$x = (x * a) * (0 * a), \tag{0.18}$$

or equivalently,

$$x = (x * (0 * a)) * a. \tag{0.19}$$

Proof. For any $x \in V(a)$, we have $(x * a) * (0 * a) \leqslant x$ by (0.4). Denote

$$u = (x * a) * (0 * a).$$

Then $u \leqslant x$. So, by (0.11) and (0.9), we obtain $u \in V(a)$ and $0 * u = 0 * a$. Also, by (0.4) and (0.5), the following holds:

$$(x * (0 * a)) * ((x * a) * (0 * a)) \leqslant x * (x * a) \leqslant a.$$

Since $u = (x * a) * (0 * a)$ and $0 * u = 0 * a$, it follows $(x * (0 * u)) * u \leqslant a$. Then the face that a is a minimal element of X gives $(x * (0 * u)) * u = a$. So, by $u \in V(a)$ (i.e., $a \leqslant u$), we derive

$$((x * (0 * u)) * u) * u = a * u = 0.$$

Hence $((x*u)*u)*(0*u) = 0$ by (0.1). Thus (0.16) implies $x*u = 0$, i.e., $x \leqslant u$. In addition, $u \leqslant x$. Therefore $x = u$. We have shown that $x = (x*a)*(0*a)$, in other words, $x = (x*(0*a))*a$ by (0.1). ☐

1 Relations between lattices and branches Let's begin our discussion with various relations between lattices and the branches of a positive implicative BCI-algebra with condition (S).

Theorem 1.1. *Let X be a positive implicative BCI-algebra with condition (S). Then every branch $V(a)$ of X with respect to the BCI-ordering $\leqslant$ on X forms an upper semilattice $(V(a); \leqslant)$ with $x \vee y = (x \circ y)*a$ for any $x, y \in V(a)$.*

Proof. For any $x, y \in V(a)$, by (0.12) and (0.9), we have

$$x*(x\circ y) = (x*x)*y = 0*y = 0*a.$$

Then (0.14) and the commutativity of $\circ$ give

$$x \leqslant (x\circ y)\circ(0*a) = (0*a)\circ(x\circ y).$$

So, (0.7) and (0.13) imply

$$x*(0*a) \leqslant ((0*a)\circ(x\circ y))*(0*a) \leqslant x\circ y.$$

Using (0.7) once more, it follows $(x*(0*a))*a \leqslant (x\circ y)*a$. Hence $x \leqslant (x\circ y)*a$ by (0.19). Similarly, $y \leqslant (x\circ y)*a$. It is easy to see from (0.11) that $(x\circ y)*a \in V(a)$. Therefore $(x\circ y)*a$ is an upper bound of x and y. Next, let $u \in V(a)$ be any upper bound of x and y. Then $x \leqslant u$ and $y \leqslant u$. By $x \leqslant u$ and (0.6), we obtain $(x\circ y)*u \leqslant (x\circ y)*x$. By (0.13) and $y \leqslant u$, the following holds: $(x\circ y)*x \leqslant y \leqslant u$. Comparison gives $(x\circ y)*u \leqslant u$, i.e., $((x\circ y)*u)*u = 0$. So,

$$\begin{aligned}((x\circ y)*a)*u &= ((((x\circ y)*a)*u)*u)*(0*a) \quad \text{[by (0.17)]}\\ &= ((((x\circ y)*u)*u)*a)*(0*a) \quad \text{[by (0.1)]}\\ &= (0*a)*(0*a) = 0.\end{aligned}$$

Hence $(x\circ y)*a \leqslant u$. We have shown that $(x\circ y)*a$ is the least upper bound of x and y. Therefore $(V(a); \leqslant)$ is an upper semilattice with $x \vee y = (x\circ y)*a$. ☐

It is known that the zero element is the only minimal element of a BCK-algebra.

Corollary 1.2 ([5], Theorem 1). *If X is a positive implicative BCK-algebra with condition (S), then $(X; \leqslant)$ forms an upper semilattice with $x \vee y = x\circ y$ for any $x, y \in X$.*

It is interesting that if the branch $V(a)$ in Theorem 1.1 is a finite set, we have a nice result as follows.

Proposition 1.3. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). If $V(a)$ is a finite set, then $(V(a); \leqslant)$ forms a lattice.*

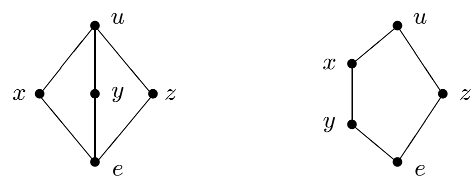
Proof. From Theorem 1.1, $(V(a); \leqslant)$ is an upper semilattice, and we only need to prove that $(V(a); \leqslant)$ is a lower semilattice. For any $x, y \in V(a)$, let Ω denote the set consisting of the whole lower bounds of x and y. Then Ω is nonempty by $a \in \Omega$. It is easily seen from (0.11) that $\Omega \subseteq V(a)$. Now, since $V(a)$ is a finite set, so is Ω. There is no harm in assuming $\Omega = \{b_1, b_2, \cdots, b_n\}$. Put $b = b_1 \vee b_2 \vee \cdots \vee b_n$. It is not difficult to verify that b is just the greatest lower bound of x and y. Therefore $(V(a); \leqslant)$ is a lower semilattice. ☐

However, if $V(a)$ is an infinite set, Proposition 1.3 is false. In fact, a counter example has been given in Example 3 of [3]. That is because every BCK-algebra X is a BCI-algebra with the condition $V(0) = X$.

In the following let's turn to consider the distributivity of $(V(a); \leqslant)$ if $(V(a); \leqslant)$ is a lattice.

Theorem 1.4. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). If $(V(a); \leqslant)$ is a lattice, it must be distributive.*

Proof. From lattice theory, a lattice is distributive if and only if it contains neither a rhombus sublattice nor a pentagon sublattice (see, e.g., [2]). Now, if our assertion is not true, the lattice $(V(a); \leqslant)$ contains either a rhombus sublattice or a pentagon sublattice whose Hasse diagrams are respectively assumed as follows.



As to the first diagram, it is easy to see from Theorem 1.1 that

$$u = x \vee y = (x \circ y) * a.$$

Then (0.4) and (0.13) together give

$$u * (x * a) = ((x \circ y) * a) * (x * a) \leqslant (x \circ y) * x \leqslant y.$$

In a similar fashion we can prove $u * (x * a) \leqslant z$. So, $u * (x * a) \leqslant y \wedge z$. Observing our diagram, we have $y \wedge z = e$. Hence $u * (x * a) \leqslant e$. Thus $u * e \leqslant x * a$ by (0.8). Thereby (0.7) implies that $(u * e) * x \leqslant (x * a) * x$, namely, $(u * x) * e \leqslant 0 * a$. It follows from (0.12) that $u * (x \circ e) \leqslant 0 * a$. Therefore $u * (0 * a) \leqslant x \circ e$ by (0.8). Now, using (0.7) once more, we obtain

$$(u * (0 * a)) * a \leqslant (x \circ e) * a,$$

which means from (0.19) and Theorem 1.1 that $u \leqslant x \vee e$. Note that $e \leqslant x$, we have $x \vee e = x$. Hence $u \leqslant x$, a contradiction with $u > x$.

As to the second diagram, we have $(y \circ z) * a = y \vee z = u$ by Theorem 1.1. Then

$$((x * a) * a) * ((y \circ z) * a) = ((x * a) * a) * u = ((x * u) * a) * a. \tag{1.1}$$

By (0.15), the left side of (1.1) is equal to $(x * (y \circ z)) * a$; by $x \leqslant u$, the right side to $(0 * a) * a$. So, $(x * (y \circ z)) * a = (0 * a) * a$. Hence

$$((x * (y \circ z)) * a) * (0 * a) = ((0 * a) * a) * (0 * a) = 0 * a.$$

Also, by (0.1) and (0.18), the following holds:

$$((x * (y \circ z)) * a) * (0 * a) = ((x * a) * (0 * a)) * (y \circ z) = x * (y \circ z).$$

Comparison gives $x * (y \circ z) = 0 * a$. Thus $(x * y) * z = 0 * a$ by (0.12). Thereby (0.8) implies $(x * y) * (0 * a) \leqslant z$. On the other hand, by (0.9) and (0.4), we have

$$(x * y) * (0 * a) = (x * y) * (0 * y) \leqslant x.$$

Then $(x * y) * (0 * a) \leqslant z \wedge x$. Because of $z \wedge x = e$, it follows $(x * y) * (0 * a) \leqslant e$, that is, $(x * (0 * a)) * y \leqslant e$. Thus $x * (0 * a) \leqslant y \circ e$ by (0.14). Hence (0.7) implies

$$(x * (0 * a)) * a \leqslant (y \circ e) * a,$$

which means from (0.19) and Theorem 1.1 that $x \leqslant y \vee e$. Note that $e \leqslant y$, we have $y \vee e = y$. Therefore $x \leqslant y$, a contradiction with $x > y$.

Summarizing the above arguments, the lattice $(V(a); \leqslant)$ is distributive. □

Corollary 1.5. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). If $V(a)$ is a finite set, then $(V(a); \leqslant)$ is a distributive lattice.*

Corollary 1.6 ([3], Theorem 3). *Let X be a positive implicative BCK-algebra with condition (S). If $(X; \leqslant)$ is a lattice, it must be distributive.*

2 Several identities on a branch We now consider several identities on a branch of a positive implicative BCI-algebra with condition (S), which are similar to those on a positive implicative BCK-algebra with condition (S).

Proposition 2.1. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). Then the following are valid:*
(1) $x = (x \circ x) * a$ *for any* $x \in V(a)$*;*
(2) $x \leqslant y$ *implies* $y = (x \circ y) * a$ *for any* $x, y \in V(a)$*;*
(3) $(x \circ y) * (x \circ z) = (y * (x \circ z)) * (0 * a)$ *for any* $x \in V(a)$ *and* $y, z \in X$*.*

Proof. (1) and (2) are two immediate results of Theorem 1.1, and we only need to show (3). Assume that x is any element in $V(a)$, and y, z in X. By (0.13), we have $(x \circ y) * x \leqslant y$. Using (0.7) two times, we obtain $((x \circ y) * x) * x \leqslant y * x$ and

$$(((x \circ y) * x) * x) * (0 * a) \leqslant (y * x) * (0 * a).$$

Then (0.17) implies
$$(x \circ y) * x \leqslant (y * x) * (0 * a). \tag{2.1}$$

Using (0.7) once more and applying (0.1), it follows

$$((x \circ y) * x) * z \leqslant ((y * x) * z) * (0 * a),$$

which means from (0.12) that

$$(x \circ y) * (x \circ z) \leqslant (y * (x \circ z)) * (0 * a). \tag{2.2}$$

Next, by (0.12) and (0.9), one has

$$y * (x \circ y) = (y * x) * y = 0 * x = 0 * a.$$

Then (0.4) gives

$$(y * (x \circ z)) * ((x \circ y) * (x \circ z)) \leqslant y * (x \circ y) = 0 * a.$$

So, (0.8) implies

$$(y * (x \circ z)) * (0 * a) \leqslant (x \circ y) * (x \circ z). \tag{2.3}$$

Combining (2.2) with (2.3), it yields $(x \circ y) * (x \circ z) = (y * (x \circ z)) * (0 * a)$. □

Theorem 2.2. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). Then for any $x, y \in V(a)$ and any $z \in X$, the least upper bound $(x * z) \vee (y * z)$ of $x * z$ and $y * z$ exists, and $(x * z) \vee (y * z) = (x \vee y) * z$.*

Proof. For any $x, y \in V(a)$ and any $z \in X$, there is no harm in assuming $z \in V(b)$, then $x * z \in V(a * b)$ and $y * z \in V(a * b)$ by (0.10). So, by Theorem 1.1, the least upper bound $(x * z) \vee (y * z)$ of $x * z$ and $y * z$ exists. It is easy to see from (0.7) that $(x \vee y) * z$ is an upper bound of $x * z$ and $y * z$. Then

$$(x * z) \vee (y * z) \leqslant (x \vee y) * z. \tag{2.4}$$

It remains to show that the opposite inequality of (2.4) holds. Denote

$$t = (x \vee y) * z \quad \text{and} \quad u = (x * z) \vee (y * z).$$

Then we have $u \leqslant t$ by (2.4), and we only need to show $t \leqslant u$. We first assert that the following are valid:

$$t = (t * z) * (0 * b), \tag{2.5}$$
$$t = (t * (0 * (a * b))) * (a * b), \tag{2.6}$$
$$t = ((x \circ y) * a) * z, \tag{2.7}$$
$$u = ((x * z) \circ (y * z)) * (a * b). \tag{2.8}$$

In fact, by (0.17), we have

$$t = (x \vee y) * z = (((x \vee y) * z) * z) * (0 * b) = (t * z) * (0 * b),$$

(2.5) holding. Because $t \in V(a * b)$, (2.6) is a direct result of (0.19). Finally, (2.7) and (2.8) can be seen from Theorem 1.1, as asserted. Now, combining (2.6) with (2.8) and noticing (0.4), we obtain

$$t * u \leqslant (t * (0 * (a * b))) * ((x * z) \circ (y * z)). \tag{2.9}$$

By (0.1) and (0.12), (2.9) is equivalent to

$$t * u \leqslant ((t * (x * z)) * (y * z)) * (0 * (a * b)). \tag{2.10}$$

Also, by (0.13), one has $(x \circ y) * x \leqslant y$, then $((x \circ y) * x) * z \leqslant y * z$ by (0.7). So,

$$(((x \circ y) * x) * z) * (y * z) = 0. \tag{2.11}$$

Right $*$ multiplying both sides of (2.11) by a and applying (0.1), one obtains

$$((((x \circ y) * a) * z) * x) * (y * z) = 0 * a.$$

Hence (2.7) gives

$$(t * x) * (y * z) = 0 * a. \tag{2.12}$$

Moreover, by (0.4), we have $(t * z) * (x * z) \leqslant t * x$. Then (0.7) implies

$$((t * z) * (x * z)) * (0 * b) \leqslant (t * x) * (0 * b).$$

That is,

$$((t * z) * (0 * b)) * (x * z) \leqslant (t * x) * (0 * b).$$

So, by (2.5), we obtain $t * (x * z) \leqslant (t * x) * (0 * b)$. Hence

$$\begin{aligned} (t * (x * z)) * (y * z) &\leqslant ((t * x) * (0 * b)) * (y * z) && [\text{by } (0.7)] \\ &= ((t * x) * (y * z)) * (0 * b) && [\text{by } (0.1)] \\ &= (0 * a) * (0 * b) && [\text{by } (2.12)] \\ &= 0 * (a * b). && [\text{by } (0.2)] \end{aligned}$$

From this, we derive

$$((t * (x * z)) * (y * z)) * (0 * (a * b)) = 0. \tag{2.13}$$

Comparing (2.10) with (2.13), it yields $t * u \leqslant 0$, in other words, $t * u = 0$ by 0 being a minimal element of X. Consequently, $t \leqslant u$. The proof is complete. □

Theorem 2.3. *Let $V(a)$ be a branch of a positive implicative BCI-algebra X with condition (S). Then the following hold: for any x, y, $z \in V(a)$,*

(1) $x = (x * (x * y)) \vee ((x * y) * (0 * a))$;

(2) $x \vee y = x \vee ((y * x) * (0 * a))$;

(3) $(x \vee y) * x = y * x$ *and* $(x \vee y) * y = x * y$;

(4) $z * (x \vee y) = (z * x) * (z * y)$.

Proof. (1) For any x, $y \in V(a)$, we have $x * y \in V(a * a) = V(0)$ by (0.10). Then $0 \leqslant x * y$. So, by (0.6) and (0.11), we obtain

$$x * (x * y) \leqslant x \quad \text{and} \quad x * (x * y) \in V(a).$$

Also, by (0.4), one has $(x * y) * (0 * y) \leqslant x$. So, by (0.9) and (0.11), one obtains

$$(x * y) * (0 * a) \leqslant x \quad \text{and} \quad (x * y) * (0 * a) \in V(a). \tag{2.14}$$

Since $(V(a); \leqslant)$ is an upper semilattice, it follows

$$(x * (x * y)) \vee ((x * y) * (0 * a)) \leqslant x. \tag{2.15}$$

Next, by (0.3), we have

$$(x * (0 * a)) * (x * (x * y)) \leqslant (x * y) * (0 * a).$$

Then (0.14) gives

$$x * (0 * a) \leqslant (x * (x * y)) \circ ((x * y) * (0 * a)).$$

So, by (0.7), we obtain

$$(x * (0 * a)) * a \leqslant ((x * (x * y)) \circ ((x * y) * (0 * a))) * a.$$

Hence (0.19) and Theorem 1.1 imply

$$x \leqslant (x * (x * y)) \vee ((x * y) * (0 * a)). \tag{2.16}$$

Comparing (2.15) with (2.16), it yields $x = (x * (x * y)) \vee ((x * y) * (0 * a))$.

(2) Following the proof of (2.14), one has

$$(y * x) * (0 * a) \leqslant y \quad \text{and} \quad (y * x) * (0 * a) \in V(a).$$

Since $(V(a); \leqslant)$ is an upper semilattice and x, $y \in V(a)$, it follows

$$x \vee ((y * x) * (0 * a)) \leqslant x \vee y. \tag{2.17}$$

Next, following the proof of (2.1), we have $(x \circ y) * x \leqslant (y * x) * (0 * a)$. Then (0.14) implies $x \circ y \leqslant x \circ ((y * x) * (0 * a))$. So, by (0.7), we derive

$$(x \circ y) * a \leqslant (x \circ ((y * x) * (0 * a))) * a.$$

Therefore $x \vee y \leqslant x \vee ((y * x) * (0 * a))$ by Theorem 1.1. Comparison with (2.17) gives $x \vee y = x \vee ((y * x) * (0 * a))$.

(3) It is a direct result of Theorem 2.2.

(4) By (0.19), we have $z = (z * (0 * a)) * a$; by (2) and Theorem 1.1, we obtain

$$x \vee y = x \vee ((y * x) * (0 * a)) = (x \circ ((y * x) * (0 * a))) * a.$$

Then

$$\begin{aligned} z * (x \vee y) &= ((z * (0 * a)) * a) * ((x \circ ((y * x) * (0 * a))) * a) \\ &\leqslant (z * (0 * a)) * (x \circ ((y * x) * (0 * a))) && [\text{by } (0.4)] \\ &= ((z * (0 * a)) * x) * ((y * x) * (0 * a)) && [\text{by } (0.12)] \\ &= ((z * x) * (0 * a)) * ((y * x) * (0 * a)) && [\text{by } (0.1)] \\ &\leqslant (z * x) * (y * x). && [\text{by } (0.4)] \end{aligned}$$

That is,

$$z * (x \vee y) \leqslant (z * x) * (y * x). \tag{2.18}$$

Next, by (0.3) and (3), one has

$$(z * x) * (z * (x \vee y)) \leqslant (x \vee y) * x = y * x.$$

So, (0.8) implies

$$(z * x) * (y * x) \leqslant z * (x \vee y). \tag{2.19}$$

Combining (2.18) with (2.19), it follows $z * (x \vee y) = (z * x) * (y * x)$. □

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Department of Mathematics, Sanming College, Sanming, Fujian 365004, P. R. China
E-mail: smcaihy@126.com