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IMPLICATION ALGEBRAS ARE EQUIVALENT TO THE DUAL IMPLICATIVE BCK-ALGEBRAS

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Abstract. In this paper by considering the notion of implicative BCK-algebras, we show that the implication algebras are equivalent to the dual implicative BCK-algebras.

1. Introduction

Several algebras with one binary and one nulary operations were introduced to set up an algebraic counterpart of implication reduct of classical or non-classical propositional logics. For the classical logic, it is the so called implication algebra introduced by J. C. Abbott[1] in 1967. The study of BCK-algebras was initiated by Y. Imai and K. Iséki[3] in 1966 as a generalization of the concept of set-theoretic difference and propositional calcului. Now we follow [1,2] and we show that the implication algebras are equivalent to the dual implicative BCK-algebras.

2. Preliminaries

Definition 2.1. [1] An *implication algebra* is a set X with a binary operation "$*$" which satisfies the following axioms:
(I1) $(x * y) * x = x$,
(I2) $(x * y) * y = (y * x) * x$,
(I3) $x * (y * z) = y * (x * z)$.
for all $x, y, z \in X$.

Lemma 2.2. [2] *In any implication algebra $(X, *)$, the following identities hold;*
(i) $x * (x * y) = x * y$,
(ii) $x * x = y * y$,
(iii) There exists a unique element 1 in X such that for all $x \in X$,
  (a) $x * x = 1$, $1 * x = x$ *and* $x * 1 = 1$,
  (b) if $x * y = 1$ *and* $y * x = 1$ *then* $x = y$.
for all $x, y \in X$.

Definition 2.3. [3] A *BCK-algebra* is a set X with a binary operation "$\circ$" and constant "0" which satisfies the following axioms:
(BCK1) $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = 0$,
(BCK2) $(x \circ (x \circ y)) \circ y = 0$,
(BCK3) $x \circ x = 0$,
(BCK4) $x \circ y = y \circ x = 0$ imply $x = y$,
(BCK5) $0 \circ x = 0$.
for all $x, y, z \in X$.

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Lemma 2.4. [3] *In any BCK-algebra* $(X, \circ, 0)$*, the following identities hold:*
(i) $x \circ 0 = x$,
(ii) $(x \circ y) \circ z = (x \circ z) \circ y$.
for all $x, y, z \in X$.

Definition 2.5. [4] Let $(X, \circ, 0)$ be a BCK-algebra. Then,
(i) X is called *implicative* if $x \circ (y \circ x) = x$, for all $x, y \in X$,
(ii) X is called *commutative* if $x \circ (x \circ y) = y \circ (y \circ x)$, for all $x, y \in X$.

Theorem 2.6. [4] *Any implicative BCK-algebra is a commutative BCK-algebra.*

3. Implication algebras are equivalent to the dual implicative BCK-algebras

Definition 3.1. Let $(X, \circ, 0)$ be a BCK-algebra and binary operation "$*$" on X is defined as follows:

$$x * y = y \circ x$$

Then $(X, *, 1)$ is called *dual BCK-algebra*. In fact, the axioms of that are as follows:
(DBCK1) $(y * z) * ((z * x) * (y * x)) = 1$,
(DBCK2) $y * ((y * x) * x) = 1$,
(DBCK3) $x * x = 1$,
(DBCK4) $y * x = x * y = 1$ imply $x = y$,
(DBCK5) $x * 1 = 1$.
for all $x, y, z \in X$.

Lemma 3.2. *Let* $(X, *)$ *be an implication algebra . Then,* $x * (y * x) = 1$*, for all* $x, y \in X$.

Proof. Let $x, y \in X$. Then, by (I3) and Lemma 2.2(iii)(a),

$$x * (y * x) = y * (x * x) = y * 1 = 1$$

□

Theorem 3.3. *Any implication algebra is a dual BCK-algebra.*

Proof. Let $(X, *)$ be an implication algebra . Then the axioms of $(DBCK3), (DBCK4)$ and $(DBCK5)$ come from Lemma 2.2(iii)(a) and (b). So, it is enough to prove that the axioms of $(DBCK1)$ and $(DBCK2)$. Now, let $x, y, z \in X$. Then,

$$\begin{aligned} (y * z) * ((z * x) * (y * x)) &= (y * z) * (y * ((z * x) * x)), \text{ By (I3)} \\ &= (y * z) * (y * ((x * z) * z)), \text{ By (I2)} \\ &= (y * z) * ((x * z) * (y * z)), \text{ By (I3)} \\ &= 1, \text{ by Lemma 3.2} \end{aligned}$$

Hence, $(DBCK1)$ hold. Moreover, by $(I3)$ and Lemma 2.2(iii)(a),

$$y * ((y * x) * x) = (y * x) * (y * x) = 1$$

Hence, $(DBCK2)$ hold. Therefore, $(X, *, 1)$ is a dual BCK-algebra. □

Definition 3.4. Let $(X, *, 1)$ be a dual BCK-algebra. Then X is called a *dual implicative BCK-algebra*, if

$$x = (x * y) * x$$

for all $x, y \in X$. (In fact, it is dual of implicative BCK-algebra $(X, \circ, 0)$, where $x \circ y = y * x$).

Theorem 3.5. *Any dual implicative BCK-algebra is an implication algebra .*

Proof. Let $(X, *, 1)$ be a dual implicative BCK-algebra. Then by hypothesis $(x*y)*x = x$, for any $x, y \in X$ and so we have (I1). Since, $(X, \circ, 0)$, where $x \circ y = x * y$, is an implicative BCK-algebra and so by Theorem 2.6, it is a commutative BCK-algebra, then

$$y \circ (y \circ x) = x \circ (x \circ y)$$

Hence, in dual implicative BCK-algebra $(X, *, 1)$ we have

$$(x * y) * y = (y * x) * x$$

and this implies that (I2). Now, since by Lemma 2.4(ii), in any BCK-algebra $(X, \circ, 0)$ we have $(x \circ y) \circ z = (x \circ z) \circ y$, for any $x, y, z \in X$. Then, in dual implicative BCK-algebra $(X, *, 1)$ we have $z * (y * x) = y * (z * x)$ and this implies that (I3). Therefore, $(X, *)$ is an implication algebra . □

Corollary 3.6. *Implication algebras are equivalent to the dual implicative BCK-algebras.*

Proof. The proof comes from Theorems 3.3 and 3.5. □

References

[1] J. C. Abbott, *Semi-boolean algebras*, Matematichki Vesnik, 4(1967), 177-198.
[2] W. Y. C Chen, J. S Oliveira, *Implication Algebras and the Metropolis-Rota Axioms for Cubic Lattices*, J. Algebra, 171, No.2, 1993
[3] Y. Imai, K. Iseki, On axiom systems of propositional calculi, *XIV Proc. Japan Academy*, **42** (1966), 19-22.
[4] J. Meng, Y. B. Jun, *BCK-algebras*, Kyung Moonsa, Korea, 1994.

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