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REMOTAL SETS IN VECTOR VALUED FUNCTION SPACES*

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Abstract. Let X be a Banach space and E a bounded set in X. For $x \in X$, we set $M(x,E) = \sup\{\|x-y\| : y \in E\}$. The set E is called remotal if for any $x \in X$ there exists $z \in E$ such that $M(x,E) = \|x-z\|$. In this paper, we prove:

(i) $M(f,\, L^1(I,E)) = \int\limits_I M(f(t),\, E)dt$, for $f \in L^1(I,\, X)$.

(ii) If E is closed and span(E) is a finite dimensional subspace of X, then $L^1(I,\, E)$ is remotal in $L^1(I,\, X)$. Some other results are presented.

0.Introduction Let X be a Banach space and E a bounded set in X. For $x \in X$, set $M(x,E) = \sup\{\|x-y\| : y \in E\}$. The set E is called remotal if for any $x \in X$ there exists $z \in E$ such that $M(x,E) = \|x-z\|$. The point z is called the farthest point of E from x. The concept of remotal sets in Banach spaces goes back to the sixties, [1], [6], and [10]. The study of remotal sets is little more difficult and less developed than that of proximinal sets. While best approximation has applications in many branches of mathematics, remotal sets and farthest points have applications in the study of geometry of Banach spaces, [2], [3], [9], and the survey article [7]. Remotal sets in vector valued continuous functions was considered in [4]. Remotal sets in the space of Bochner integrable functions have never been considered. The object of this paper is to study the remotality of $L^1(I,E)$ in $L^1(I,X)$ in connection with remotality of E in X. We prove two main results:

(i) $M(f, L^1(I,E)) = \int\limits_I M(f(t),E)dt$ for any remotal set E in X.

(ii) If E is a closed bounded set in X such that $span(E)$ is a finite dimensional subspace of X, then $L^1(I,E)$ is remotal in $L^1(I,X)$. Some other results are presented.

Throughout this paper, I denotes the unit interval with the Lebesgue measure. For $1 \le p < \infty$, and X a Banach space, $L^p(I,X)$ denotes the Banach space of Bochner integrable functions(equivalence classes) on I with values in X. For $f \in L^p(I,X)$, $\|f\|_p = (\int \|f(t)\|^p)^{\frac{1}{p}}$. For $E \subset X$, we set $L^p(I,E) = \{f \in L^p(I,X) : f(t) \in E \ a.e\}$.

I. Distance Formulae

Let X be a Banach space and G be a remotal subset of X. In this section we prove a distance formulae similar to the one for best approximation, [8].

Theorem 1.1. Let X be a Banach space and G be a remotal subset of X. Then for each $f \in L^1(I,X)$

$$\sup_{g \in L^1(I,G)} \|f-g\| = \int \sup_{a \in G} \|f(s)-a\|\, ds.$$

Proof. Let $f \in L^1(I,X)$ and $u \in L^1(I,G)$. Then

$$\|f-u\| = \int\limits_I \|f(s)-u\ (s)\|\, ds \le \int \sup_{a\in G} \|f(s)-a\|\, ds.$$

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This implies that :

$$\sup_{u\in L^1(I,G)} \|f-u\ \| \leq \int \sup_{a\in G}\|f(s)-a\|\,ds. \tag{1}$$

Now, since simple functions are dense in $L^1(I,X)$ [5], then given $\epsilon>0$, there exists φ, simple function in $L^1(I,X)$, such that $\|f-\varphi\|<\epsilon$. Write $\varphi=\sum_{i=1}^{n}\chi_{A_i}\ y_i$, where χ_{A_i} is the characteristic function of the set $A_i\subseteq I$, and $y_i\in X$. We may assume that $\sum_{i=1}^{n}\chi_{A_i}=1$ and $\mu(A_i)>0$ for all i. Further, $\|y_i\|\ \mu(A_i)<\infty$ for $1\leq i\leq n$.

Since $y_i\in X$, then for a given $\epsilon>0$, we can select $g_i\in G$ such that :

$$\|y_i-g_i\| > \sup_{a\in G}\|y_i-a\| - \frac{\epsilon}{(n\mu(A_i))}.$$

Let $w=\sum_{i=1}^{n}\chi_{A_i}\ g_i$. Clearly $w\in L^1(I,G)$. Now, since

$$\begin{aligned}\|\varphi-w\| &\leq \|f-\varphi\|+\|f-w\|\\ &\leq \epsilon+\|f-w\|\,,\end{aligned}$$

then,

$$\begin{aligned}\|f-w\|+\epsilon &\geq \|\varphi-w\|\\ &= \int_I \|\varphi(s)-w(s)\|\,ds.\\ &\geq \sum_{i=1}^{n}\int_{A_i}\|y_i-g_i\ \|\,ds\\ &= \sum_{i=1}^{n}\|y_i-g_i\ \|\,\mu(A_i)\\ &\geq \sum_{i=1}^{n}\left(\sup_{a\in G}\|y_i-a\| - \frac{\epsilon}{(n\mu(A_i))}\right)\mu(A_i)\\ &= \sum_{i=1}^{n}\sup_{a\in G}\|y_i-a\|\,\mu(A_i)-\epsilon.\end{aligned}$$

Thus

$$\begin{aligned}\|f-w\|+2\epsilon &\geq \sum_{i=1}^{n}\int_{A_i}\sup_{a\in G}\|y_i-a\|\,ds\\ &= \int_I\sum_{i=1}^{n}\sup_{a\in G}\|y_i-a\|\,\chi_{A_i}ds\\ &= \int_I \sup_{a\in G}\|\varphi(s)-a\|\,ds\\ &\geq \int_I \sup_{a\in G}\|\varphi(s)-a+f(s)-f(s)\|\,ds\end{aligned}$$

Since

$$\|\varphi(s) - a + f(s) - f(s)\| \geq |\|f(s) - a\| - \|\varphi(s) - f(s)\||,$$

then,

$$\begin{aligned}
\|f - w\| + 2\epsilon &\geq \int_I \sup_{a\in G} |\|f(s) - a\| - \|\varphi(s) - f(s)\|| \, ds \\
&\geq \int_I \left|\sup_{a\in G} \|f(s) - a\| - \|\varphi(s) - f(s)\|\right| ds \\
&\geq \left|\int_I \sup_{a\in G} \|f(s) - a\| - \|\varphi(s) - f(s)\| \, ds\right| \\
&= \left|\int_I \sup_{a\in G} \|f(s) - a\| \, ds - \int_I \|\varphi(s) - f(s)\| \, ds\right| \\
&\geq \left|\int_I \sup_{a\in G} \|f(s) - a\| \, ds - \epsilon\right|.
\end{aligned}$$

Consequently,

$$\|f - w\| + 3\epsilon \geq \int_I \sup_{a\in G} \|f(s) - a\| \, ds.$$

Since ϵ was arbitrary, it follows that

$$\|f - w\| \geq \int_I \sup_{a\in G} \|f(s) - a\| \, ds. \tag{2}$$

Equations (1) and (2) give the result. ∎

Corollary 1.2. Let G be a remotal subset of a Banach space X . Then $g \in L^1(I, G)$ is a farthest point in $L^1(I, G)$ from $f \in L^1(I, X)$ if and only if for almost all $t \in I$, $g(t)$ is a farthest point in G from $f(t)$ in X.

Proof. Let g be a farthest element in $L^1(I, G)$ from $f \in L^1(I, X)$. Then

$$\|f - g\| = \sup_{u\in L^1(I,G)} \|f - u\|.$$

By Theorem 1.1 we have:

$$\|f - g\|_1 = \int_I \|f(s) - g(s)\| \, ds = \sup_{u\in L^1(I,G)} \|f - u\| = \int_I \sup_{a\in G} \|f(s) - a\| \, ds.$$

So, $\|f(s) - g\,(s)\| = \sup_{a\in G} \|f(s) - a\|$ for $a.e.\, t \in I$, and $g(s)$ is a farthest point in G from $f(t) \in X$ $a.e.$ $t \in I$. ∎

II Remotal sets in Vector Valued Functions

In this section we study the following question: For which remotal sets E in X one has $L^1(I,E)$ remotal in $L^1(I,X)$?.

Theorem 2.1. Let E be a remotal set in a Banach space X . If E is a finite set, then $L^1(I,E)$ is remotal in $L^1(I,X)$.

Proof. Let $f \in L^1(I,X)$. For $f(t) \in X$, let e_t be the farthest point from $f(t)$ in E. Then

$$\|f(t) - e_t\| \geq \|f(t) - h(t)\|$$

for every $h \in L^1(I,E)$. The function $g(t) = e_t$ is in $L^1(I,E)$, since $g(t)$ is a simple function. Now,

$$\|f(t) - g(t)\| \geq \|f(t) - h(t)\|.$$

This implies

$$\int_0^1 \|f(t) - g(t)\| \, dt \geq \int_0^1 \|f(t) - h(t)\| \, dt.$$

So $\|f - g\| \geq \|f - h\|$. Hence, $L^1(I,E)$ is remotal in $L^1(I,X)$. ■

Theorem 2.2. Let E be a finite remotal set in a Banach space X. Then $l^1(E)$ is remotal in $l^1(X)$ if and only if $E = \{0\}$.

Proof. If $E = \{0\}$, then $l^1\,(E) = \{0\}$, the zero sequence, and hence $l^1(E)$ is remotal in $l^1(X)$.

Conversely, suppose that $l^1(E)$ is remotal in $l^1(X)$. Let $f = (0,0,0,...,0,...) \in l^1(X)$ and $e \in F(0,E)$. If $e \neq 0 \in E$, then $(e,e,e,....e,0,0,0,0,.....) \in l^1\,(E)$,where e appears in the first $n-$coordinates. But, in such a case we have:

$$\|(,e,e,e,...,e,0,0,...) - f\| = n\,\|e\|.$$

This implies that $d(f,l^1(E)) = \infty$ and $l^1(E)$ is not remotal. Consequently, e must equal to zero, and $E = \{0\}$. ■

Now we state and prove one of the main results of this paper.

Theorem 2.3. Let X be a Banach space, and E be a closed convex bounded subset in X. If $\overline{span(E)} = Y$ is finite dimensional, then $L^1(I,E)$ is remotal in $L^1(I,X)$.

Proof. Let $\{x_1,x_2,x_3,...,x_m\}$ be a basis of Y such that $\{x_1,x_2,x_3,...,x_m\} \subseteq E$. Then every function $g \in L^1(I,E)$ has the form

$$g = \sum_{k=1}^{m} g_k \otimes x_k.$$

where $g_k \in L^1(I)$.

Now, let f be any element in $L^1(I,X)$, and (S_n) be a sequence of simple functions in $L^1(I,X)$ such that $\|f - S_n\| \to 0$. Since E is a compact set, (being closed and bounded in a finite dimensional space Y), E is remotal. If

$$S_n = \sum_{k=1}^{l} 1_{E_{kn}} \otimes y_k,$$

then

$$\hat{S_n} = \sum_{k=1}^{l} 1_{E_{kn}} \otimes \hat{y_k}$$

is the farthest point of S_n in $L^1(I, E)$, where $\hat{y_k}$ is the farthest point of y_k in E.

Let $y \in E$ such that $\|y\| = \max\{\|z\| : z \in E\}$. Such y exists since E is compact. Now,

$$\left\| \hat{S_n}(t) \right\| \leq \|y\|$$

for every $t \in I$, since $\hat{S_n}(t) \in E$. Thus

$$\int_K \left\| \hat{S_n}(t) \right\| dt \leq \int_K \|y\| \, dt$$

for any measurable set $K \subseteq I$. But the sequence $(1 \otimes y)$, (the constant sequence) is uniformly integrable. Thus $\left\{ \hat{S_n} \right\}$ is uniformly integrable. Being bounded, the Dunford compactness Theorem [5], implies that $\left\{ \hat{S_n} \right\}$ is relatively weakly compact. Thus we can assume that $\left(\hat{S_n} \right)$ (or a subsequence) is weakly convergent. Let $\hat{S_n} \xrightarrow{w} \hat{f}$. We claim that $\hat{f}$ is the farthest point to f. Indeed : each $\hat{S_n}$ has the form :

$$\hat{S_n} = \sum_{k=1}^{m} f_{kn} \otimes x_k.$$

Let $g \in L^\infty(I),\ x^* \in X^*$. Weak convergence of $\hat{S_n}$ implies :

$$\left\langle \hat{S_n}, g \otimes x^* \right\rangle = \sum_{k=1}^{m} \langle f_{kn}, g \rangle \otimes \langle x_k, x^* \rangle .$$

Choose x^* such that

$$x^*(x_k) = \begin{cases} 1 & k = 1 \\ 0 & 1 < k \leq m \end{cases} .$$

Then

$$\left\langle \hat{S_n}, g \otimes x^* \right\rangle = \langle f_{1n}, g \rangle .$$

So (f_{1n}) converges weakly, to f_1 say. Similarly $f_{2n} \xrightarrow{w} f_2, ...,\ f_{mn} \xrightarrow{w} f_m$, with $f_1, f_2, ..., f_m \in L^1(I)$, since $L^1(E)$ is weakly sequentially complete [5]. Now, since $f_{1n} \xrightarrow{w} f_1$, there exists a subsequence that converges to f_1 point wise (a.e), say f_{1n_k}. Indeed : if there is no subsequence of f_{1n} that converges to f_1 a.e, then for every subsequence f_{1n_k} and for every $\epsilon > 0$ there exists $Q \subseteq I$, $\mu(Q) > 0$ such that $|f_{1n_k}(t) - f_1(t)| > \epsilon$, for every $t \in Q$ and for every $n \in \mathcal{N}$. Let

$$Q_{nk} = \{t \in Q : f_{1n_k}(t) - f_1(t) > 0\}$$

and

$$Q_{nk}^c = P_{nk} = \{t \in Q : f_{1n_k}(t) - f_1(t) < 0\}.$$

If $\bigcap Q_{nk} = \phi$, then,

$$Q_{nk} \cap Q_{nk}^c = \phi.$$

So

$$\begin{aligned} \bigcup Q_{nk} \cap Q_{nk}^c &= \phi \\ \left(\bigcup Q_{nk}\right) \bigcap \left(\bigcup Q_{nk}^c\right) &= \phi \\ \left(\left(\bigcup Q_{nk}\right) \bigcap \left(\bigcup Q_{nk}^c\right)\right)^c &= Q \\ \left(\bigcup Q_{nk}\right)^c \bigcup \left(\bigcup Q_{nk}^c\right)^c &= Q \\ \left(\bigcap Q_{nk}^c\right) \bigcup \left(\bigcap Q_{nk}\right) &= Q. \end{aligned}$$

So, if $\bigcap Q_{nk} = \phi$, we get $\bigcap Q_{nk}^c = Q$. Hence, with no loss of generality we can assume that there exists $P \subseteq Q$, $\mu(P) > 0$ and $f_{nk}(t) - f(t) > \epsilon$ on P for every n_k.

Thus

$$\int_P (f_{nk} - f_1)\, dt \geq \epsilon\mu(P) > 0.$$

But weak convergence implies

$$\int_P (f_{nk} - f_1)\, dt \longrightarrow 0.$$

This is a contradiction. Thus (f_{1n}), has a subsequence (f_{1n_k}) that converges to f_1 point wise. Similarly f_{2n_k} has a subsequence that converges point wise say $f_{2n_{kj}}$, and so on. We get

f_{1n} has a subsequence g_{1n} that converges point wise to $f_1(t)$
f_{2n} has a subsequence g_{2n} that converges point wise to $f_2(t)$
.
.
.
f_{mn} has a subsequence g_{mn} converges point wise to $f_m(t)$.

So we have,

$$\begin{aligned} \lim_{n\to\infty} g_{1n}(t) &= f_1(t) \\ \lim_{n\to\infty} g_{2n}(t) &= f_2(t) \\ &\;\;. \\ &\;\;. \\ \lim_{n\to\infty} g_{mn}(t) &= f_m(t) \end{aligned}$$

But again, these are sequentially bounded functions and can't exceed $h(t) = \|y\|$. Hence

$$|g_{in}(t) - f_i| \leq 2\|y\|.$$

Hence, the Labesgue Dominated Convergence Theorem gives

$$\|g_{in} - f_i\|_1 \longrightarrow 0.$$

Thus

$$\begin{aligned} \overset{\wedge}{S_{nk}} &= \sum_{i=1}^{m} g_{in} \otimes x_i. \\ \left\| \overset{\wedge}{S_{nk}} - \hat{f} \right\| &= \left\| \sum_{i=1}^{m} g_{in} \otimes x_i - \sum_{i=1}^{m} f_i \otimes x_i \right\| \\ &\leq \sum_{i=1}^{m} \|g_{in} - f_i\| \, \|x_i\| \longrightarrow 0. \end{aligned}$$

Now,

$$\begin{aligned} \left\| f - \hat{f} \right\| &= \lim \left\| S_{nk} - \overset{\wedge}{S_{nk}} \right\| \\ &\geq \lim \|S_{nk} - h\| \\ &= \|f - h\| \end{aligned}$$

for every $h \in L^1(I, E)$. Hence $L^1(I, E)$ is remotal in $L^1(I, X)$. ■

We end this section with the following question:

Question 1. Is Theorem 2.3 true if $\overline{span(E)}$ is assumed to be reflexive?.

III. Remotal sets:

Let X be any Banach space, and Y be a closed subspace of X. The unit ball of X is $B[X] = \{x : \|x\| \leq 1\}$. For a set $E \subset Y$ and $x \in X$, let

$d(E) = \sup\{\|x - y\| : x, y \in E\}$.

If E is a closed and convex set in a reflexive Banach space X, then for $x \in X$, $d(x, E) = \|x - y\|$, for some $y \in E$. This is not the case in general for $M(x, E)$ [3].

Lemma 3.1 For any Banach space X, $B[X]$ is remotal.

Proof. Let $x \in X$. If $x = 0$, then every point of y such that $\|y\| = 1$ is the farthest point to $x = 0$. If $\|x\| \geq 1$, then $z = \frac{x}{\|x\|}$ is the best approximant of x in $B[X]$. Consequently, $-z$ is the farthest point from x in $B[X]$. Indeed:

$\|x - (-z)\| = \left\| x + \frac{x}{\|x\|} \right\| = 1 + \|x\|$.

But, for any point $y \in B[X]$ we have:

$\|x - y\| \leq \|y - z\| + \|x - z\| \leq 2 + \|x - z\| = 2 + (\|x\| - 1) = 1 + \|x\| = \|x - (-z)\|$.

If If $x \neq 0$ and $\|x\| < 1$, let $z = -\frac{x}{\|x\|}$. Then $\|x - z\| = 1 + \|x\|$. If y is any element in $B[X]$, then

$\|x - y\| \leq \|x\| + \|y\| \leq \|x\| + 1 = \|x - z\|$. Hence $B[X]$ is remotal. ■

Question 2. If Y is a reflexive subspace, must $B[Y]$ be remotal in X?

Theorem 3.2. Let E be a closed convex set in a Banach space X and $x \in X$. Set $F(x, E) = \{e \in E : \|x - e\| \geq \|x - y\| \text{ for all } y \in E\}$. Then $F(x, E)$ is an extremal set in E.

Proof. Let $z, y \in E$ and $w = tz + (1-t)y \in F(x,E)$. We claim that $z, y \in F(x,E)$.
Now

$$\|x-w\| \leq t\|x-z\| + (1-t)\|x-y\|.$$

If $\|x-z\| < \|x-w\|$ (or $\|x-y\| < \|x-w\|$), then $\|x-w\| < \|x-w\|$, which can't be true. So

$$\|x-z\| = \|x-y\| = \|x-w\|.$$

Hence $z, y \in F(x,E)$ and $F(x,E)$ is an extremal set in E. Note that $F(x,E)$ is closed in E. ■

Corollary 3.3. If E is any closed bounded uniqely remotal set in X^*, then

$$F(x,E) \cap Ext(E) \neq \phi.$$

Proof. Since E is w^*-closed, by Aloglu Theorem and the Krein Milman Theorem, $E = \overline{Ext(E)}^{w^*}$. By Theorem 1.2 above, $F(x,E)$ is an extremal subset consisting of one point (E being uniquely remotal). Hence $F(x,E)$ is an extreme point. ■

Theorem 3.4. Let E be any closed bounded uniqely remotal set in a metric linear space X. If $x \in X$ and $e \in E$ such that $d(x,e) = \sup_{y\in E} d(x,y)$, then e is an extreme point of E.

Proof. Suppose not. Then, there exist points $s, w \in E,\ e \neq w \neq s$ such that $e = \frac{1}{2}(w+s)$. But

$$d(x,e) = d(x, \frac{1}{2}(w+s)) \leq \frac{1}{2}d(x,w) + \frac{1}{2}d(x,s).$$

Now either

$$d(x,e) = d(x,w) = d(x,s)$$

(and since E is uniquely remotal we have $e = w = s$), or $d(x,w),$ or $d(x,s)$ is greater than $d(x,e)$. But, this is a contradiction. Thus e is an extreme point of E. ■

Theorem 3.5. Let E is any bounded remotal set in a Banach space X. Then E can't be open.

Proof. Suppose E is an open remotal subset of X and let $x \in X$. Then there exists $e \in E$ such that $\|x-e\| = \sup_{y\in E}\|x-y\|$. Since E is open, there exists a number $r > 0$ such that $\{y \in X : \|y-e\| < r\} \subseteq E$. Put $u = e - \frac{r}{2\|x-e\|}(x-e)$. Then $\|u-e\| = \frac{r}{2} < r$, and hence $u \in E$. But

$$\begin{aligned}
\|x-u\| &= \left\| x - e + \frac{r}{2\|x-e\|}(x-e) \right\| \\
&= \left\| (x-e)\left(1 + \frac{r}{2\|x-e\|}\right) \right\| \\
&= \left(1 + \frac{r}{2\|x-e\|}\right)\|(x-e)\| > \|(x-e)\|.
\end{aligned}$$

This is a contradiction. So E can not be open. ■

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